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TOTALLY BOUNDED ENDOMORPHISMS ON A TOPOLOGICAL RING

Madjid Mirzavaziri and Omid Zabeti

Abstract. Let X be a topological ring. In this paper, we consider the three classes (*btb*-bounded, *tbtb*-bounded, and *tbb*-bounded) of endomorphisms defined on X and denote these classes by $B_{btb}(X), B_{tbtb}(X)$, and $B_{tbb}(X)$, respectively. We equip them with an appropriate topology and we find some sufficient conditions under which, each class of these endomorphisms is complete.

1. Introduction

In [3], some different notions of boundedness has been introduced for group homomorphisms on a topological ring. In this paper, using the notion of totally boundedness in a topological ring, we present some different aspects for totally bounded group homomorphisms on a topological ring. We endow these classes of totally bounded group homomorphisms to an appropriate uniform convergence topology. We deduce that each class of these group homomorphisms on a locally bounded topological ring, with respect to the assumed topology, forms a topological ring. Also, as the main result, we show that each class of totally bounded group homomorphisms on a unitary topological ring X is complete if and only if so is X. There are many examples of topological rings among which we are interested in ring of scalar valued, ring of bounded linear operators on a topological vector space and ring of continuous real-valued functions on some topological space (where the topology is given by pointwise convergence). Let X be a *topological ring*. Recall that a subset $E \subseteq X$ is said to be *totally bounded* if for each zero neighborhood $V \subseteq X$ there is a finite set $F \subseteq X$ such that $E \subseteq F + V$. Also, $B \subseteq X$ is called *bounded* if for each zero neighborhood $W \subseteq X$ there is a zero neighborhood $U \subseteq X$ with $UB \subseteq W$ and $BU \subseteq W$. For more information about bounded group homomorphisms on a topological ring and the related notions, the reader is referred to [1–5].

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In this paper, every ring is supposed to be associative and every topological ring R is assumed to be locally bounded which means there exists a zero neighborhood $U \subseteq R$ such that for each zero neighborhood $W \subseteq R$ there is a zero neighborhood $V \subseteq R$ with $VU \subseteq W$ and $UV \subseteq W$.

2. Totally bounded group homomorphisms

Definition 2.1. Let X and Y be two topological rings. A group homomorphism $T : X \to Y$ is said to be

(i) *btb*-bounded if for every bounded set $B \subseteq X$, $T(B)$ is totally bounded in Y;

(ii) *tbtb*-bounded if for every totally bounded subset $E \subseteq X$, $T(E)$ is totally bounded in Y;

(iii) *tbb*-bounded if for every totally bounded subset $E \subseteq X$, $T(E)$ is bounded in Y.

Since every totally bounded set is also bounded, then it is easy to see that (i)→(ii)→(iii). But these concepts are not equivalent, in general. In prior to anything, we show this.

Example 2.2. Let X be c_0, the space of all vanishing sequences, with the pointwise product and the uniform norm topology. Consider the identity group homomorphism I on X. Indeed, it is *tbtb*-bounded but it is not *btb*-bounded since the open unit ball $N_1^{(0)}$ is not totally bounded.

Example 2.3. Let X be c_0, with the zero multiplication and the coordinatewise topology. Also, suppose Y is c_0, with the zero multiplication and the uniform norm topology. Consider the identity group homomorphism I from X into Y. It is easy to see that I is *tbb*-bounded but it is not *tbtb*-bounded. For, suppose $(a_n) \subseteq c_0$ is the sequence defined by $a_n = (1, 2, \dots, n, 0, \dots)$. Indeed, (a_n) is totally bounded in X but it is not totally bounded in Y. For, if $\varepsilon > 0$ is arbitrary, there is no finite set $F \subseteq c_0$ with $(a_n) \subseteq F + N_\varepsilon^{(0)}$.

In the rest of the paper, an endomorphism on a topological ring X means an endomorphism of its additive group.

The class of all *btb*-bounded endomorphisms on a topological ring X is denoted by $B_{btb}(X)$ and is equipped with the topology of uniform convergence on bounded sets that means a net (S_α) of *btb*-bounded endomorphisms converges uniformly to zero on a bounded set $B \subseteq X$ if for each zero neighborhood $V \subseteq X$ there is an α_0 such that $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$. The class of all *tbtb*-bounded endomorphisms on X is denoted by $B_{tbtb}(X)$ and is endowed with the topology of uniform convergence on bounded sets, namely, a net (S_α) of *tbtb*-bounded endomorphisms converges uniformly to zero on a bounded set $B \subseteq X$ if for each zero neighborhood $V \subseteq X$ there is an α_0 with $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$. Finally, the class of all *tbb*-bounded endomorphisms on X is denoted by $B_{tbb}(X)$ and is also assigned with the topology of uniform convergence on bounded sets. It is not difficult to see that $B_{btb}(X)$, $B_{tbtb}(X)$, and $B_{tbb}(X)$ are nested subrings of the ring of all endomorphisms on a topological ring X.

3. Main results

In the following proposition, we show that each class of totally bounded endomorphisms, with respect to the topology of uniform convergence on bounded sets, forms a topological ring. The proofs in the following proposition, follow the same line. We give the details for the sake of completeness.

Proposition 3.1.

(i) *The operations of addition and product in $B_{btb}(X)$ are continuous with respect to the topology of uniform convergence on bounded sets;*

(ii) *The operations of addition and product in $B_{tbtb}(X)$ are continuous with respect to the topology of uniform convergence on bounded sets;*

(iii) *The operations of addition and product in $B_{tbb}(X)$ are continuous with respect to the topology of uniform convergence on bounded sets.*

Proof. (i) Suppose two nets (T_α) and (S_α) of *btb*-bounded endomorphisms converge uniformly to zero on bounded sets. Fix a bounded set $B \subseteq X$. Let W be an arbitrary zero neighborhood in X. Find a zero neighborhood V such that $V + V \subseteq W$. There are some α_0 and α_1 with $T_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$ and $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_1$. Take an α_2 such that $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. If $\alpha \geq \alpha_2$ then $(T_\alpha + S_\alpha)(B) \subseteq T_\alpha(B) + S_\alpha(B) \subseteq V + V \subseteq W$. So, the addition is continuous. Now, we show the continuity of the product. Suppose $U \subseteq X$ is a zero neighborhood which is bounded. Since the net (S_α) converges to zero uniformly on B, there is an α_3 such that $E = \cup_{\alpha \geq \alpha_3} S_\alpha(B) \subseteq U$, so that it is bounded. Find an α_4 with $T_\alpha(E) \subseteq W$ for each $\alpha \geq \alpha_4$. Choose an α_5 such that $\alpha_5 \geq \alpha_3$ and $\alpha_5 \geq \alpha_4$. Thus, for all $\alpha \geq \alpha_5$, $T_\alpha S_\alpha(B) \subseteq T_\alpha(E) \subseteq W$, as we wanted.

(ii) Suppose two nets (T_α) and (S_α) of *tbtb*-bounded endomorphisms converge uniformly to zero on bounded sets. Fix a bounded set $B \subseteq X$. Let W be an arbitrary zero neighborhood in X. Find a zero neighborhood V such that $V + V \subseteq W$. There are some α_0 and α_1 with $T_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$ and $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_1$. Take an α_2 such that $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. If $\alpha \geq \alpha_2$ then $(T_\alpha + S_\alpha)(B) \subseteq T_\alpha(B) + S_\alpha(B) \subseteq V + V \subseteq W$. So, the addition is continuous. Now, we show the continuity of the product. Suppose $U \subseteq X$ is a zero neighborhood which is bounded. Since the net (S_α) converges to zero uniformly on B, there is an α_3 such that $E = \cup_{\alpha \geq \alpha_3} S_\alpha(B) \subseteq U$, so that it is bounded. Find an α_4 with $T_\alpha(E) \subseteq W$ for each $\alpha \geq \alpha_4$. Choose an α_5 such that $\alpha_5 \geq \alpha_3$ and $\alpha_5 \geq \alpha_4$. Thus, for all $\alpha \geq \alpha_5$, $T_\alpha S_\alpha(B) \subseteq T_\alpha(E) \subseteq W$.

(iii) Suppose two nets (T_α) and (S_α) of *tbb*-bounded endomorphisms converge uniformly to zero on bounded sets. Fix a bounded set $B \subseteq X$. Let W be an arbitrary zero neighborhood in X. Find a zero neighborhood V such that $V + V \subseteq W$. There are some α_0 and α_1 with $T_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$ and $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_1$. Take an α_2 such that $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. If $\alpha \geq \alpha_2$ then $(T_\alpha + S_\alpha)(B) \subseteq T_\alpha(B) + S_\alpha(B) \subseteq V + V \subseteq W$. So, the addition is continuous. Now, we show the continuity of the product. Suppose $U \subseteq X$ is a zero neighborhood which is bounded. Since the net (S_α) converges to zero uniformly on B, there is an α_3 such that $E = \cup_{\alpha \geq \alpha_3} S_\alpha(B) \subseteq U$, so that it is bounded. Find an α_4 with $T_\alpha(E) \subseteq W$ for each $\alpha \geq \alpha_4$. Choose an α_5 such that $\alpha_5 \geq \alpha_3$ and $\alpha_5 \geq \alpha_4$. Thus, for all $\alpha \geq \alpha_5$, $T_\alpha S_\alpha(B) \subseteq T_\alpha(E) \subseteq W$, as desired. ■

Now, we start our main work with this observation that each class of totally bounded endomorphisms on a topological ring X, with respect to the topology of uniform convergence on bounded sets , is a closed subring of the ring of all endomorphisms on X, as shown by the following propositions.

Proposition 3.2. *Suppose a net (h_α) of btb-bounded endomorphisms converges uniformly on bounded sets to an endomorphism h. Then h is also btb-bounded.*

Proof. Fix a bounded set $B \subseteq X$. Let W be an arbitrary zero neighborhood in X. There is a zero neighborhood V with $V + V \subseteq W$. Find an α_0 such that $(h_\alpha - h)(B) \subseteq V$ for each $\alpha \geq \alpha_0$. Fix an $\alpha \geq \alpha_0$. Take a finite set $F \subseteq X$ with $h_\alpha(B) \subseteq F + V$. Therefore

$$h(B) \subseteq h_\alpha(B) + V \subseteq F + V + V \subseteq F + W.$$

Thus, h is also *btb*-bounded. ■

Proposition 3.3. *Suppose a net (h_α) of tbtb-bounded endomorphisms converges uniformly on bounded sets to an endomorphism h. Then h is also tbtb-bounded.*

Proof. Fix a totally bounded set $E \subseteq X$. Let W be an arbitrary zero neighborhood in X. Choose a zero neighborhood V with $V + V \subseteq W$. Since E is also bounded, there is an α_0 such that $(h_\alpha - h)(E) \subseteq V$ for each $\alpha \geq \alpha_0$. Fix an $\alpha \geq \alpha_0$. Find a finite set $F \subseteq X$ with $h_\alpha(E) \subseteq F + V$. Thus

$$h(E) \subseteq h_\alpha(E) + V \subseteq F + V + V \subseteq F + W.$$

It follows that h is also *tbtb*-bounded. ■

Proposition 3.4. *Suppose a net (h_α) of tbb-bounded endomorphisms converges uniformly on bounded sets to an endomorphism h. Then h is also tbb-bounded.*

Proof. Fix a totally bounded set $E \subseteq X$. Let W be an arbitrary zero neighborhood in X. Take a zero neighborhood V such that $V + VV \subseteq W$. Since E is bounded, there exists an α_0 with $(h_\alpha - h)(E) \subseteq V$ for each $\alpha \geq \alpha_0$. Fix an $\alpha \geq \alpha_0$. Choose a zero neighborhood $V_1 \subseteq V$ such that $V_1 h_\alpha(E) \subseteq V$ and $h_\alpha(E)V_1 \subseteq V$. Therefore

$$V_1 h(E) \subseteq V_1 h_\alpha(E) + V_1 V \subseteq V + VV \subseteq W.$$

Similarly, $h(E)V_1 \subseteq W$. This shows that h is also *tbb*-bounded. ■

In what follows, we show that each class of totally bounded endomorphisms, with respect to the topology of uniform convergence on bounded sets, on a topological ring X with unity, is complete if and only if X is complete.

THEOREM 3.5.

(i) $B_{btb}(X)$ *is complete if and only if so is* X*;*

(ii) $B_{tbtb}(X)$ *is complete if and only if so is* X*;*

(iii) $B_{tbb}(X)$ *is complete if and only if so is* X*.*

Proof. (i) Let X be complete and (T_α) be a Cauchy net of *btb*-bounded endomorphisms on X. Suppose W is an arbitrary zero neighborhood in X. Since every singleton is bounded, for each $x \in X$ there is an α_0 such that $(T_\alpha - T_\beta)(x) \in W$ for each $\alpha \geq \alpha_0$ and for each $\beta \geq \alpha_0$. So, $(T_\alpha(x))$ is a Cauchy net in X, so that it converges. Thus there is an endomorphism T on X with $T(x) = \lim T_\alpha(x)$. Since this convergence is in $B_{btb}(X)$, by Proposition 3.2, T is also *btb*-bounded.

For the converse, let $B_{btb}(X)$ be complete and (x_α) be a Cauchy net in X. For each α, define endomorphism T on X by $T_\alpha(x) = x_\alpha x$. It is easy to see that each T_α is *btb*-bounded. Fix a bounded set $B \subseteq X$. There is a zero neighborhood $V \subseteq X$ such that $VB \subseteq W$. Also, choose an α_1 with $(x_\alpha - x_\beta) \in V$ for each $\alpha \geq \alpha_1$ and for each $\beta \geq \alpha_1$. Thus for each $x \in B$,

$$(T_\alpha - T_\beta)(x) = T_\alpha(x) - T_\beta(x) = (x_\alpha - x_\beta)x \in Vx \subseteq VB \subseteq W.$$

So, $(T_\alpha - T_\beta)(B) \subseteq W$. Therefore, (T_α) is a Cauchy net in $B_{btb}(X)$, so that it converges. It follows that there is a *btb*-bounded endomorphism T such that (T_α) converges to T uniformly on bounded sets. Therefore, $T_\alpha(1) \to T(1)$, so that $x_\alpha \to T(1)$, as we wanted.

(ii) Suppose X is complete and (T_α) is a Cauchy net of *tbtb*-bounded endomorphisms on X. Assume $W \subseteq X$ is an arbitrary zero neighborhood. Since every singleton is bounded, there is an α_0 with $(T_\alpha - T_\beta)(x) \in W$ for each $\alpha \geq \alpha_0$ and each $\beta \geq \alpha_0$. So, $(T_\alpha(x))$ is a Cauchy net in X, so that it converges. Thus, there is an endomorphism T on X such that $T(x) = \lim T_\alpha(x)$. Since this convergence is in $B_{tbtb}(X)$, by Proposition 3.3, T is also *tbtb*-bounded.

For the converse, assume $B_{tbtb}(X)$ is complete and (x_α) is a Cauchy net in X. For each α, define endomorphism T on X by $T_\alpha(x) = x_\alpha x$. It is easy to see that each T_α is *tbtb*-bounded. Fix a bounded set $B \subseteq X$. There is a zero neighborhood $V \subseteq X$ with $VB \subseteq W$. Also, find an α_1 such that $(x_\alpha - x_\beta) \in V$ for all $\alpha \geq \alpha_1$ and for all $\beta \geq \alpha_1$. Therefore, for any $x \in B$, we have

$$(T_\alpha - T_\beta)(x) = T_\alpha(x) - T_\beta(x) = (x_\alpha - x_\beta)x \in Vx \subseteq VB \subseteq W.$$

Thus, $(T_\alpha - T_\beta)(B) \subseteq W$. This means that (T_α) is a Cauchy net in $B_{tbtb}(X)$, so that it converges. So, there is a *tbtb*-bounded endomorphism T such that (T_α) converges to T uniformly on bounded sets. Therefore, $T_\alpha(1) \to T(1)$, so that $x_\alpha \to T(1)$, as desired.

(iii) Let X be complete and (T_α) be a Cauchy net of *tbb*-bounded endomorphisms on it. Suppose W is an arbitrary zero neighborhood in X. Since every singleton is bounded, there is an α_0 such that $(T_\alpha - T_\beta)(x) \in W$ for any $\alpha \geq \alpha_0$ and for any $\beta \geq \alpha_0$. So, $(T_\alpha(x))$ is a Cauchy net in X, so that it converges. Thus, there is an endomorphism T on X with $T(x) = \lim T_\alpha(x)$. Since this convergence is in $B_{tbb}(X)$, by Proposition 3.4, T is also *tbb*-bounded.

For the converse, let $B_{tbb}(X)$ be complete and (x_α) be a Cauchy net in X. For each α, define endomorphism T on X by $T_\alpha(x) = x_\alpha x$. It is easy to see that each T_α is *tbb*-bounded. Fix a bounded set $B \subseteq X$. Find a zero neighborhood $V \subseteq X$ with $VB \subseteq W$. Also, take an α_1 such that $(x_\alpha - x_\beta) \in V$ for each $\alpha \geq \alpha_1$ and for each $\beta \geq \alpha_1$. Thus, for each $x \in B$,

$$(T_\alpha - T_\beta)(x) = T_\alpha(x) - T_\beta(x) = (x_\alpha - x_\beta)x \in Vx \subseteq VB \subseteq W.$$

So, $(T_\alpha - T_\beta)(B) \subseteq W$. It follows that (T_α) is a Cauchy net in $B_{tbb}(X)$, so that it converges. Therefore, there is a *tbb*-bounded endomorphism T such that (T_α) converges to T uniformly on bounded sets. Thus, $T_\alpha(1) \to T(1)$, so that $x_\alpha \to T(1)$, as we claimed. ■

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Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O.Box 1159, Mashhad 91775, Iran

E-mail: mirzavaziri@gmail.com, omid.zabeti@stu-mail.um.ac.ir