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A NOTE ON GENERATING FUNCTIONS OF CESÀRO POLYNOMIALS OF SEVERAL VARIABLES

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Abstract. The present paper deals with certain generating functions of Cesàro polynomials of several variables.

1. Introduction

Let the sequence of functions $\{S_n(x) \mid n = 0, 1, 2, \ldots\}$ be generated by Singal and Srivastava [11]:

$$\sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x) t^n = \frac{f(x,t)}{[g(x,t)]^m} S_m[h(x,t)] \tag{1.1}$$

where m is a nonnegative integer, the $A_{m,n}$ are arbitrary constants and f, g, h are suitable functions of x and t. The importance of a generating function of the form (1.1) in obtaining the bilateral and trilateral generating relations for the functions $S_n(x)$ was realized by several authors.

In particular, the present work is based on the papers due to Agarwal and Manocha [2], Chatterjea [6], Singal and Srivastava [11] and the book written by Srivastava and Manocha [9].

The Pochhammer symbol $(\lambda)_n$ is defined by

$$(\lambda)_n = \begin{cases} 1, & \text{if } n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n = 1, 2, \ldots \end{cases}$$

2. Cesàro polynomials

The Cesàro polynomials are denoted by $g_n^{(m)}(x)$ and is defined as (Chihara [15])

$$g_n^{(m)}(x) = \binom{m+n}{n} {}_2F_1\left[\begin{matrix} -n,\ 1; \\ -m-n; \end{matrix}\ x\right] \tag{2.1}$$

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which can also be written as

$$g_n^{(m)}(x) = \frac{(m+n)!}{m!\ n!} \sum_{r=0}^{n} \frac{(-n)_r\ (1)_r x^r}{r!\ (-m-n)_r}$$

Agarwal and Manocha [2] defined the polynomials $g_n^m(x)$ by the generating relation.

$$\sum_{n=0}^{\infty} g_n^{(m)}(x) t^n = (1-t)^{-m-1}(1-xt)^{-1}. \tag{2.2}$$

which is easy to derive from (2.1).

Starting, as usual, from (2.2) one gets the following formula of the type (1.1) for the polynomials $g_n^m(x)$:

$$\sum_{n=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(m)}(x) t^n = (1-t)^{-m-1-k}(1-xt)^{-1} g_k^{(m)}\left(\frac{x(1-t)}{1-xt}\right)$$

which provided them the basic tool to deduce the following theorem on trilateral generating functions for the polynomials $g_n^m(x)$.

Theorem 1. *Let*

$$Y_{r,\mu}[x,y,t] = \sum_{n=0}^{\infty} a_{n,\mu} g_{rn}^{(m)}(x) g_{n+\mu}(y) t^n$$

be a bilateral generating function. Then the following trilateral generating relation holds:

$$\sum_{n=0}^{\infty} g_n^{(m)}(x) \Omega_n^{r,\mu}(y,z) t^n = (1-t)^{-m-1}(1-xt)^{-1} Y_{r,\mu}\left[\frac{x(1-t)}{1-xt}, y, z\left(\frac{t}{1-t}\right)^r\right].$$

where, as well as throughout this paper,

$$\Omega_n^{r,\mu}(y,z) = \sum_{k=0}^{[n/r]} \binom{n}{rk} a_{k,\mu} g_{k+\mu}(y) z^k.$$

3. Cesàro polynomials of two variables

We define the Cesàro polynomials of two variables $g_n^{(m)}(x,y)$ as follows:

$$g_n^{(m)}(x,y) = \binom{m+n}{n} F\left[\begin{array}{ll} -n:\ 1;1; & \\ & x,y \\ -m-n:\ -;-; & \end{array}\right] \tag{3.1}$$

which can also be written as:

$$g_n^{(m)}(x,y) = \frac{(m+n)!}{m!n!} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(1)_r(1)_s}{(-m-n)_{r+s} r! s!} x^r y^s.$$

The following generating relation holds for (3.1).

Theorem 2.

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,y)t^n \;=\; (1-t)^{-1-m}(1-xt)^{-1}(1-yt)^{-1}. \tag{3.2}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} g_n^{(m)}(x,y)t^n &= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(m+n)!(-n)_{r+s}(1)_r(1)_s}{m!n!(-m-n)_{r+s}r!s!}x^r y^s t^n \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r} \frac{(m+n)!(-1)^{r+s}n!r!s!}{m!n!r!s!(-m-n)_{r+s}(n-r-s)!}x^r y^s t^n \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(m+n+r+s)!(-1)^{r+s}}{m!n!(-m-n-r-s)_{r+s}}x^r y^s t^{n+r+s} \\
&= \sum_{n=0}^{\infty} \tfrac{(m+n)!}{m!n!}t^n \sum_{r=0}^{\infty}(xt)^r \sum_{s=0}^{\infty}(yt)^s \\
&= \sum_{n=0}^{\infty} \tfrac{m!(1+m)_n}{m!n!}t^n \; (1-xt)^{-1}(1-yt)^{-1} \\
&= (1-t)^{-1-m}(1-xt)^{-1}(1-yt)^{-1}. \quad \blacksquare
\end{aligned}$$

Starting, as usual, from (3.2) we get the the following formula of the type (1.1) for the polynomials $g_n^m(x,y)$.

Theorem 3.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(m)}(x,y)t^n \\
&\qquad = (1-t)^{-m-1-k}(1-xt)^{-1}(1-yt)^{-1} g_k^{(m)}\left(\frac{x(1-t)}{1-xt}, \frac{y(1-t)}{1-yt}\right)
\end{aligned} \tag{3.3}$$

Proof.

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(m)}(x,y)t^n v^k &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+k)!}{n!k!} g_{n+k}^{(m)}(x,y)t^n v^k \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} g_n^{(m)}(x,y)t^{n-k}v^k \\
&= \sum_{n=0}^{\infty} \; g_n^{(m)}(x,y)t^n \sum_{k=0}^{n} \frac{(-n)_k}{k!}\left(\frac{-v}{t}\right)^k \\
&= \sum_{n=0}^{\infty} \; g_n^{(m)}(x,y)t^n (1+\frac{v}{t})^n \\
&= \sum_{n=0}^{\infty} \; g_n^{(m)}(x,y)(t+v)^n
\end{aligned}$$

$$= (1-(t+v))^{-1-m}(1-x(t+v))^{-1}(1-y(t+v))^{-1}$$
$$= (1-t)^{-1-m}(1-xt)^{-1}(1-yt)^{-1}(1-\frac{v}{1-t})^{-1-m}(1-\frac{xv}{1-xt})^{-1}$$
$$\times (1-\frac{yv}{1-yt})^{-1}$$
$$= (1-t)^{-1-m}(1-xt)^{-1}(1-yt)^{-1}(1-\frac{v}{1-t})^{-1-m}(1-\frac{xv(1-t)}{(1-xt)(1-t)})^{-1}$$
$$\times (1-\frac{yv(1-t)}{(1-yt)(1-t)})^{-1}$$
$$= (1-t)^{-1-m}(1-xt)^{-1}(1-yt)^{-1}\sum_{n=0}^{\infty} g_k^{(m)}\left(\frac{x(1-t)}{1-xt}, \frac{y(1-t)}{1-yt}\right)\frac{v^k}{(1-t)^k}.$$

Equating the coefficient of v^k we get (3.3), which provides us with the basic tool to deduce the following theorem on mixed trilateral generating functions for the polynomials $g_n^{(m)}(x,y)$. ∎

Theorem 4. *Let*

$$Y_{r,\mu}[x_1,x_2,y,t] = \sum_{n=0}^{\infty} a_{n,\mu} g_{rn}^{(m)}(x_1,x_2) g_{n+r}(y) t^n$$

be a mixed bilateral generating function involving Cesàro polynomials of two variables and another one variable polynomials $g_{n+\mu}(y)$. Then the following mixed trilateral generating relation holds:

$$\sum_{n=0}^{\infty} g_n^{(m)}(x_1,x_2)\Omega_n^{r,\mu}(y,z)t^n$$
$$= (1-t)^{-m-1}(1-x_1t)^{-1}(1-x_2t)^{-1}Y_{r,\mu}\left[\frac{x_1(1-t)}{1-x_1t}, \frac{x_2(1-t)}{1-x_2t}, y, z\left(\frac{t}{1-t}\right)^r\right]. \tag{3.4}$$

Proof.

$$\sum_{n=0}^{\infty} g_n^{(m)}(x_1,x_2)\Omega_n^{r,\mu}(y,z)t^n$$
$$= \sum_{n=0}^{\infty} g_n^{(m)}(x_1,x_2)\left(\sum_{k=0}^{[n/r]}\binom{n}{rk} a_{k,\mu} g_{k+\mu}(y) z^k\right)t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/r]} g_n^{(m)}(x_1,x_2)\frac{(n)!}{(n-rk)!\,(rk)!} a_{k,\mu} g_{k+\mu}(y) z^k t^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} g_{n+rk}^{(m)}(x_1,x_2)\frac{(n+rk)!}{(rk)!\,(n)!} a_{k,\mu} g_{k+\mu}(y) z^k t^{n+rk}$$
$$= \sum_{k=0}^{\infty} a_{k,\mu} g_{k+\mu}(y) z^k t^{rk}\sum_{n=0}^{\infty}\binom{n+rk}{rk} g_{n+rk}^{(m)}(x_1,x_2)t^n$$

$$= \sum_{k=0}^{\infty} a_{k,\mu} g_{k+\mu}(y) z^k t^{rk} (1-t)^{-m-1-rk} (1-x_1 t)^{-1} (1-x_2 t)^{-1}$$
$$\times\, g_{rk}^{(m)} \left(\frac{x_1(1-t)}{1-x_1 t}, \frac{x_2(1-t)}{1-x_2 t} \right)$$
$$= (1-t)^{-m-1} (1-x_1 t)^{-1} (1-x_2 t)^{-1}$$
$$\sum_{k=0}^{\infty} a_{k,\mu} g_{rk}^{(m)} \left(\frac{x_1(1-t)}{1-x_1 t}, \frac{x_2(1-t)}{1-x_2 t} \right) g_{k+\mu}(y) [z(\frac{t}{1-t})^r]^k$$
$$= (1-t)^{-m-1} (1-x_1 t)^{-1} (1-x_2 t)^{-1} Y_{r,\mu} \left[\frac{x_1(1-t)}{1-x_1 t}, \frac{x_2(1-t)}{1-x_2 t}, y, z \left(\frac{t}{1-t} \right)^r \right]$$

which proves (3.4). ■

4. Cesàro polynomials of three variables

We define the Cesàro polynomials of three variables $g_n^{(m)}(x,y,z)$ as follows:

$$g_n^{(m)}(x,y,z) \;=\; \binom{m+n}{n} \; F \left[\begin{matrix} -n :: \; -;-;- : \; 1;1;1; \\ \\ -m-n :: \; -;-;- : \; -;-;-; \end{matrix} \quad x, y, z \right] \tag{4.1}$$

which can also be written as

$$g_n^{(m)}(x,y,z) = \frac{(m+n)!}{m!n!} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{k=0}^{n-r-s} \frac{(-n)_{r+s+k} (1)_r (1)_s (1)_k}{(-m-n)_{r+s+k} r! s! k!} x^r y^s z^k.$$

The following generating relation holds for (4.1)

$$\sum_{n=0}^{\infty} g_n^{(m)}(x,y,z) t^n = (1-t)^{-1-m} (1-xt)^{-1} (1-yt)^{-1} (1-zt)^{-1}. \tag{4.2}$$

Starting, as usual, from (4.2) we get the following formula of the type (1.1) for the polynomials $g_n^{(m)}(x,y,z)$:

$$\sum_{n=0}^{\infty} \binom{n+k}{k} g_{n+k}^{(m)}(x,y,z) t^n = (1-t)^{-m-1-k} (1-xt)^{-1} (1-yt)^{-1}$$
$$\times\, (1-zt)^{-1} g_k^{(m)} \left(\frac{x(1-t)}{1-xt}, \frac{y(1-t)}{1-yt}, \frac{z(1-t)}{1-zt} \right) \tag{4.3}$$

which provides us with the basic tool to deduce the following theorem on mixed trilateral generating functions for the polynomials $g_n^{(m)}(x,y,z)$.

Theorem 5. *Let*

$$Y_{r,\mu}[x_1, x_2, y, t] = \sum_{n=0}^{\infty} a_{n,\mu} g_{rn}^{(m)}(x_1, x_2) g_{n+r}(y) t^n$$

be a mixed bilateral generating function involving Cesàro polynomials of three variables and another one variable polynomials $g_{n+\mu}(y)$. Then the following mixed trilateral generating relation holds:

$$\sum_{n=0}^{\infty} g_n^{(m)}(x_1,x_2,x_3)\Omega_n^{r,\mu}(y,z)t^n = (1-t)^{-m-1}(1-x_1t)^{-1}(1-x_2t)^{-1} \times (1-x_3t)^{-1} Y_{r,\mu}\left[\frac{x_1(1-t)}{1-x_1t}, \frac{x_2(1-t)}{1-x_2t}, \frac{x_3(1-t)}{1-x_3t}, y, z\left(\frac{t}{1-t}\right)^r\right]. \quad (4.4)$$

The proof of (4.2), (4.3) and (4.4) are similar to those (3.2), (3.3) and (3.4) respectively.

Concluding remark. Cesàro polynomials can be extended up to n-variables and analogous results of this paper can be obtained.

REFERENCES

[**1**] A. Edrelyi et al., *Higher Transcendental Functions*, Vol.II McGraw-Hill Book Co.Inc;New York, 1953.

[**2**] A.K. Agarwal, H.L. Manocha, *On some new generating functions*, Mat. Vesnik 4(17)(32) (1980), 395–402. (Also published in Indian J.Pure App. Math. **13** (1982), 1369-1375).

[**3**] A.K. Agarwal et al, *Alternative methods of obtaining differential operator representations*, Indian J. Pure Appl. Math. **13** (1982), 237–246.

[**4**] D.K. Basu, *On sets generated by $A(t)\phi(xt)$*, C. R. Acad. Bulgare Sci. **29** (1976), 1107–1110.

[**5**] E.D. Rainville, *Special Functions*, Chelsea Publishing Company, Bronx,New York, 1971.

[**6**] F. Brafman, *Some generating functions for Laguerre and Hermite polynomials*, Canad. J. Math. **9** (1957), 180–187.

[**7**] H.M. Srivastava, *Some generalization of Carlitz's theorem*, Pacific J. Math. **85** (1979), 471–477.

[**8**] H.M. Srivastava, *Some bilateral generating functions for certain special functions*, I and II, Indag. Math. **42** (1980).

[**9**] H.M. Srivastava, H.L. Manocha, *A treatise on generating functions*, John Wily and Sons (Halsted Press, New York, Ellis Horwood, Chichester).

[**10**] H.M. Srivastava, J.L. Lavoie, *A certain method of obtaining functions*, Indag. Math. **37** (1975), 304–320.

[**11**] J.P. Singhal, H.M. Srivastava, *A class of bilateral generating functions for certain classical polynomials*, Pacific J. Math. **42** (1972), 755–762.

[**12**] L. Carlitz, *A class of generating functions*, SIAM J. Math. Anal. **8** (1977), 518–532.

[**13**] M.J. Gottlib, *Concerning some polynomials orthogonal on a finite or enumerable set of points*, Amer. J. Math. **60** (1938), 453–458.

[**14**] Shy-Der Lin, Shuoh-Jung Liu, Han-Chun Lu, H.M. Srivastava, *Integral representations for the generalized Bedient polynomials and the generalized Cesàro polynomials*, Appl. Math. Comput. **218** (2011), 1330–1341.

[**15**] S.K. Chatterjea, *An extension of a class of bilateral generating functions for certain special functions*, Bull. Inst. Acad. Sinica **5** (1977), 323–331.

[**16**] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach Science Publishers Inc. New York, 1978.

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