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ENTIRE FUNCTIONS AND THEIR DERIVATIVES SHARE TWO FINITE SETS

Chao Meng

Abstract. In this paper, we study the uniqueness of entire functions and prove two theorems which improve the result given by Fang [M.L. Fang, Entire functions and their derivatives share two finite sets, Bull. Malaysian Math. Sci. Soc. 24 (2001), 7–16].

1. Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane C. If for some $a \in C \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities). We assume that the reader is familiar with the notations of Nevanlinna theory that can be found, for instance, in [3] or [6].

Let S be a set of distinct elements of $C \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $\bigcup_{a \in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM. Let m be a positive integer or infinity and $a \in C \cup \{\infty\}$. We denote by $E_{m)}(a, f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. For a set S of distinct elements of C we define $E_{m)}(S, f) = \bigcup_{a \in S} E_{m)}(a, f)$. If for some $a \in C \cup \{\infty\}$, $E_{\infty)}(a, f) = E_{\infty)}(a, g)$, we say that f and g share the value a CM. We can define $\overline{E}_{m)}(a, f)$ and $\overline{E}_{m)}(S, f)$ similarly.

In 1977, Gross [2] posed the following question.

Question. Can one find two finite sets $S_j (j = 1, 2)$ such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical?

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Yi [7] gave a positive answer to the question. He proved

THEOREM A. [7] *Let f and g be two nonconstant entire functions, $n \geq 5$ a positive integer, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a\}$, where $a \neq 0$ is a constant satisfying $a^{2n} \neq 1$. If $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.*

In 2001, Fang [1] investigated the question and proved the following theorems

THEOREM B. [1] *Let f and g be two nonconstant entire functions, $n \geq 5$, k two positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a, b, c\}$, where a, b, c are nonzero finite distinct constants satisfying $a^2 \neq bc$, $b^2 \neq ac$, $c^2 \neq ab$. If $E_f(S_1) = E_g(S_1)$ and $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, then $f \equiv g$.*

THEOREM C. [1] *Let f and g be two nonconstant entire functions, $n \geq 5$, k two positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a, b\}$, where a, b are two nonzero finite distinct constants. If $E_f(S_1) = E_g(S_1)$ and $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, then one of the following cases must occur: (1) $f \equiv g$; (2) $b = -a$, $f = e^{cz+d}$, $g = te^{-cz-d}$, where c, d, t are three constants satisfying $t^n = 1$ and $(-1)^k tc^{2k} = a^2$; (3) $f = e^{cz+d}$, $g = te^{-cz-d}$, where c, d, t are three constants satisfying $t^n = 1$ and $(-1)^k tc^{2k} = ab$; (4) $b = -a$, $f \equiv -g$.*

THEOREM D. [1] *Let f and g be two nonconstant entire functions, $n \geq 5$, k two positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a\}$, where $a \neq 0, \infty$. If $E_f(S_1) = E_g(S_1)$ and $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, then one of the following cases must occur: (1) $f \equiv g$; (2) $f = e^{cz+d}$, $g = te^{-cz-d}$, where c, d, t are three constants satisfying $t^n = 1$ and $(-1)^k tc^{2k} = a^2$.*

In this paper, we consider the more general sets $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. We prove the following results which improve Theorem B, Theorem C and Theorem D.

THEOREM 1. *Let $n(\geq 5)$, k, m be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. If two nonconstant entire functions f and g satisfy $E_{3)}(S_1, f) = E_{3)}(S_1, g)$, and $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, then one of the following cases must occur: (1) $f = tg$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$, where t is a constant satisfying $t^n = 1$; (2) $f(z) = de^{cz}$, $g(z) = \frac{t}{d}e^{-cz}$, $\{a_1, a_2, \ldots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \ldots, \frac{1}{a_m}\}$, where t, c, d are nonzero constants and $t^n = 1$.*

THEOREM 2. *Let $n(\geq 5)$, k, m be positive integers, and let $S_1 = \{z : z^n = 1\}$, $S_2 = \{a_1, a_2, \ldots, a_m\}$, where $a_1, a_2, \ldots, a_m$ are distinct nonzero constants. If two nonconstant entire functions f and g satisfy $E_{2)}(S_1, f) = E_{2)}(S_1, g)$, and $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, then one of the following cases must occur: (1) $f = tg$, $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$, where t is a constant satisfying $t^n = 1$; (2) $f(z) = de^{cz}$, $g(z) = \frac{t}{d}e^{-cz}$, $\{a_1, a_2, \ldots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \ldots, \frac{1}{a_m}\}$, where t, c, d are nonzero constants and $t^n = 1$.*

2. Some lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right).$$

LEMMA 1. [5] *Let f be a nonconstant meromorphic function, and let a_0, a_1, a_2, ..., a_n be finite complex numbers, $a_n\neq 0$. Then*

$$T(r,a_nf^n+\cdots+a_2f^2+a_1f+a_0)=nT(r,f)+S(r,f).$$

LEMMA 2. [4] *Let F, G be two nonconstant meromorphic functions such that $E_{3)}(1,F)=E_{3)}(1,G)$, then one of the following cases holds: (1) $T(r,F)+T(r,G)\leq 2\{N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)\}+S(r,F)+S(r,G)$; (2) $F\equiv G$; (3) $FG\equiv 1$.*

LEMMA 3. [9] *Let F and G be two nonconstant meromorphic functions and $E_{2)}(1,F)=E_{2)}(1,G)$. If $H\not\equiv 0$, then*

$$\begin{aligned}T(r,F)+T(r,G)\leq 2\left(N_2\left(r,\frac{1}{F}\right)+N_2(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\right)\\+\overline{N}_{(3}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(3}\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G).\end{aligned}$$

LEMMA 4. [8] *Let H be defined as above. If $H\equiv 0$ and*

$$\limsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+\overline{N}(r,F)+\overline{N}(r,G)}{T(r)}<1,\quad r\in I,$$

where I is a set with infinite linear measure and $T(r)=\max\{T(r,F),T(r,G)\}$, then $FG\equiv 1$ or $F\equiv G$.

LEMMA 5. [3] *Let f be a nonconstant meromorphic function, n be a positive integer, and let Ψ be a function of the form $\Psi=f^n+Q$, where Q is a differential polynomial of f with degree $\leq n-1$. If*

$$N(r,f)+N\left(r,\frac{1}{\Psi}\right)=S(r,f),$$

then $\Psi=(f+\alpha)^n$, where α is a meromorphic function with $T(r,\alpha)=S(r,f)$, determined by the term of degree $n-1$ in Q.

3. Proof of Theorem 1

Set $F = f^n$, $G = g^n$. By Lemma 1, we have

$$T(r, F) = nT(r, f) + S(r, f), \qquad T(r, G) = nT(r, g) + S(r, g). \tag{1}$$

From $E_{3)}(S_1, f) = E_{3)}(S_1, g)$, we deduce $E_{3)}(1, F) = E_{3)}(1, G)$. Then F and G satisfy the condition of Lemma 2. We assume Case (1) in Lemma 2 holds, that is,

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\{N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G})\} + S(r, F) + S(r, G) \\ &\leq 4T(r, f) + 4T(r, g) + S(r, f) + S(r, g) \end{aligned} \tag{2}$$

Combining (1) and (2) together we have

$$(n-4)T(r, f) + (n-4)T(r, g) \leq S(r, f) + S(r, g), \tag{3}$$

which contradicts $n \geq 5$. Thus by Lemma 2, we have $FG \equiv 1$ or $F \equiv G$, that is $f = tg$ or $fg = t$ where t is a constant and $t^n = 1$. Next we consider the following two cases:

Case 1. $f = tg$. Then $f^{(k)} = tg^{(k)}$. By $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, we get $\{a_1, a_2, \ldots, a_m\} = t\{a_1, a_2, \ldots, a_m\}$.

Case 2. $fg = t$. Then there exists an entire function h such that $f = e^h$ and $g = te^{-h}$. Therefore

$$f^{(i)} = \alpha_i f, g^{(i)} = \beta_i g, i = 1, 2, \ldots, \tag{4}$$

where $\alpha_1 = h'$, $\beta_1 = -h'$, and α_i, β_i satisfy the following recurrence formulas, respectively.

$$\alpha_{i+1} = \alpha_i' + \alpha_i^2, \beta_{i+1} = \beta_i' + \beta_i^2, i = 1, 2, \ldots \tag{5}$$

Without loss of the generality, we assume that a_1 is not an exceptional value of $f^{(k)}$. Suppose $f^{(k)}(z_0) = a_1$. Then $\frac{t}{a_1}\alpha_k(z_0)\beta_k(z_0) = g^{(k)}(z_0) \in S_2$. Therefore,

$$\prod_{j=1}^{m}(\frac{t}{a_1}\alpha_k(z_0)\beta_k(z_0) - a_j) = 0. \tag{6}$$

Note that $\overline{N}(r, 1/(f^{(k)} - a_1)) \neq S(r, f)$. We get

$$\prod_{j=1}^{m}(\frac{t}{a_1}\alpha_k\beta_k - a_j) = 0, \tag{7}$$

which implies that $\alpha_k\beta_k$ is a nonzero constant. And thus α_k and β_k have no zeros. The recurrence formulas in (5) show that

$$\alpha_k = \alpha_1^k + P(\alpha_1), \beta_k = \beta_1^k + Q(\beta_1), \tag{8}$$

where $P(\alpha_1)$ is a differential polynomial in α_1 of degree $k-1$, and $Q(\beta_1)$ is a differential polynomial in β_1 of degree $k-1$. If α_1 and β_1 are not constants, then by Lemma 5, we have

$$\alpha_k = \left(\alpha_1 + \frac{\gamma_1}{k}\right)^k, \beta_k = \left(\beta_1 + \frac{\gamma_2}{k}\right)^k, \tag{9}$$

where γ_1, γ_2 are small functions of α_1 and β_1, respectively. Note that $\alpha_1 = -\beta_1 = h'$. We conclude that $\alpha_k\beta_k$ can not be constant, which is a contradiction. Hence one of α_1 and β_1 is constant. Thus h is a linear function. Therefore, $f(z) = de^{cz}$ and $g(z) = \frac{t}{d}e^{-cz}$, where c, d are nonzero constants. Now from $E_{f^{(k)}}(S_2) = E_{g^{(k)}}(S_2)$, we get $\{a_1, a_2, \ldots, a_m\} = (-1)^k c^{2k} t\{\frac{1}{a_1}, \ldots, \frac{1}{a_m}\}$, which completes the proof of Theorem 1. ∎

4. Proof of Theorem 2

Set $F = f^n$, $G = g^n$. From $E_{2)}(S_1, f) = E_{2)}(S_1, g)$, we deduce $E_{2)}(1, F) = E_{2)}(1, G)$. By Lemma 1, we have

$$T(r,F) = nT(r,f) + S(r,f), \qquad T(r,G) = nT(r,g) + S(r,g). \tag{10}$$

Assume $H \not\equiv 0$. By Lemma 3, we have

$$\begin{aligned} T(r,F) + T(r,G) \le 2\left(N_2\left(r,\frac{1}{F}\right) + N_2(r,F) + N_2\left(r,\frac{1}{G}\right) + N_2(r,G)\right) \\ + \overline{N}_{(3}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(3}\left(r,\frac{1}{G-1}\right) + S(r,F) + S(r,G). \end{aligned} \tag{11}$$

Obviously we have

$$\begin{aligned} \overline{N}_{(3}\left(r,\frac{1}{F-1}\right) &\le \frac{1}{2}N\left(r,\frac{F}{F'}\right) = \frac{1}{2}N\left(r,\frac{F'}{F}\right) + S(r,f) \\ &\le \frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right) + S(r,f) \le \frac{1}{2}T(r,f) + S(r,f). \end{aligned} \tag{12}$$

Similarly we have

$$\overline{N}_{(3}\left(r,\frac{1}{G-1}\right) \le \frac{1}{2}T(r,g) + S(r,g). \tag{13}$$

Combining (10), (11), (12) and (13) together we have

$$(n-\frac{9}{2})T(r,f) + (n-\frac{9}{2})T(r,g) \le S(r,f) + S(r,g), \tag{14}$$

which contradicts $n \ge 5$. Thus $H \equiv 0$. By Lemma 4, we have $FG \equiv 1$ or $F \equiv G$, that is $f = tg$ or $fg = t$ where t is a constant and $t^n = 1$. Proceeding as in the proof of Theorem 1, we get the conclusion of Theorem 2. This completes the proof of Theorem 2. ∎

5. Some Remarks

From Theorem 2, we know Theorem 1 still holds if we replace $E_{3)}(S_1, f) = E_{3)}(S_1, g)$ by $E_{2)}(S_1, f) = E_{2)}(S_1, g)$. But we do not know whether Theorem 1 and 2 still hold for $n < 5$. We intend to study the question in future work.

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School of Science, Shenyang Aerospace University, Shenyang 110136, People's Republic of China

E-mail: `mengchaosdu@yahoo.com.cn`