МАТЕМАТИЧКИ ВЕСНИК
65, 4 (2013), 547–554
December 2013

оригинални научни рад
research paper

THE UNIVALENCE OF SOME INTEGRAL OPERATORS USING THE BESSEL FUNCTIONS

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Abstract. In this paper we will introduce some new integral operators using the generalized Bessel functions and analytic functions. For this operators we will prove the univalence condition.

1. Preliminaries and definitions

Let $\mathcal{A}$ the class of all functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \tag{1}$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and satisfy the condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the class of univalent functions.

We will consider the generalized Bessel function of the first kind and order ν as the particular solution of the second-order linear homogenous differential equation

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - \nu^2 + (1-b)\nu]\omega(z) = 0.$$

This solution is denoted by $\omega_{\nu,b,c}(z)$ and has the familiar infinite sum representation

$$\omega(z) = \omega_{\nu,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\,\Gamma(\nu + n + \frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+\nu} \quad z \in \mathbb{C}, \tag{2}$$

where Γ is the Euler gamma function. Considering this series we can study the Bessel, modified Bessel and spherical Bessel functions. Geometric properties for this were obtained by Baricz in [4].

The Bessel functions are obtained for $b = c = 1$ in (2) and are defined by (see [4])

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{2n+\nu},$$

2010 AMS Subject Classification: 30C45
Keywords and phrases: Bessel functions; integral operator; univalent; analytic functions.

for $z \in \mathbb{C}$. For $b = -c = 1$ in (2) are obtained the modified Bessel functions and are defined by (see [4])

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad z \in \mathbb{C}.$$

The spherical Bessel functions are obtained for $b-1 = c = 1$ in (2) and are defined by (see [4])

$$K_\nu(z) = \sqrt{\frac{2}{z}} J_{\nu+\frac{1}{2}}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\nu+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+\nu}, \quad z \in \mathbb{C}.$$

In particular cases Bessel functions of the first kind reduces to some elementary functions, like sine and cosine and modified Bessel functions of the first kind are reduced to hyperbolic sine and cosine.

The generalized Bessel function of the first kind were studied by Á. Baricz in [3]. Recently, in 2010 Baricz and Frasin proved in [6] the univalence of some integral operators involving generalized Bessel functions and with Ponnusamy some conditions of starlikeness and convexity of generalized Bessel functions in [7].

We consider the function defined by

$$u_{\nu,b,c}(z) = 2^\nu \Gamma(\nu + \frac{b+1}{2}) z^{\nu/2} \omega_{\nu,b,c}(\sqrt{z})$$

and using the Pochhammer symbol, defined in terms of Euler-Gamma function

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \nu = 0, \lambda \in \mathbb{C} - 0 \\ \lambda(\lambda+1)\dots(\lambda+n-1), & \mu = n, \lambda \in \mathbb{C} \end{cases}$$

we obtain for the function $u_{\nu,b,c}(z)$ the following series representation

$$u_{\nu,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (\kappa)_n n!} z^{n+1} \tag{3}$$

for $\kappa := \nu + \frac{b+1}{2} \notin \mathbb{Z}_0$.

More results about function $u_{\nu,b,c}(z)$ of generalized Bessel function $\omega_{\nu,b,c}(z)$ we find in the papers of Baricz in [4] and [5], where he proves some geometric properties and also some interesting inequalities that involving generalized Bessel functions.

To prove our main results we will use Ahlfors [1] and Becker [2] univalence criterion:

Theorem 1.1. *Let d be a complex number, $|d| \le 1, d \ne -1$. If $f(z) = z + a_2 z^2 + \dots$ is a regular function in $\mathcal{U}$ and*

$$\left| d|z|^2 + (1-|z|^2)\frac{zf''(z)}{f'(z)} \right| \le 1 \tag{4}$$

for all $z \in \mathcal{U}$, then the function f is regular and univalent in $\mathcal{U}$.

Also we will use some lemmas that were proven by E. Deniz, H. Orhan and H. M. Srivastava in [8]. These were obtained by the generalization of the results obtained by Baricz and Ponnusamy in [7].

LEMMA 1.1. *If the parameters $\nu, b \in \mathbb{R}$ and $c \in \mathbb{C}$ are so constrained that*

$$\kappa > \max\left\{0, \frac{|c|-2}{4}\right\},$$

then the function $\varphi_{\nu,b,c} : \mathcal{U} \to \mathbb{C}$,

$$\varphi_{\nu,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (\kappa)_n n!} z^{n+1}$$

satisfies the following inequalities:

$$\left|\varphi'_{\nu,b,c}(z) - \frac{\varphi_{\nu,b,c}(z)}{z}\right| \le \frac{(\kappa+1)\,|c|}{\kappa[4(\kappa+1)-|c|]},\ z \in \mathcal{U},$$

$$\left|\frac{z\varphi'_{\nu,b,c}(z)}{\varphi_{\nu,b,c}(z)} - 1\right| \le \frac{8(\kappa+1)\,|c|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|c| + |c|^2},\ z \in \mathcal{U}, \tag{5}$$

$$\frac{4\kappa(\kappa+1) - (3\kappa+2)\,|c|}{\kappa[4(\kappa+1)-|c|]} \le \left|z\varphi'_{\nu,b,c}(z)\right| \le \frac{4\kappa(\kappa+1) + (\kappa+2)\,|c|}{\kappa[4(\kappa+1)-|c|]},\ z \in \mathcal{U},$$

$$\left|z^2\varphi''_{\nu,b,c}\right| \le \frac{|c|\,4(\kappa+1)+|c|}{2\kappa 4(\kappa+1)-|c|},\ z \in \mathcal{U},$$

and

$$\left|\frac{z\varphi''_{\nu,b,c}(z)}{\varphi'_{\nu,b,c}(z)}\right| \le \frac{4(\kappa+1)\,|s| + |c|^2}{8\kappa(\kappa+1) - 2(3\kappa+2)\,|c|},\ z \in \mathcal{U}. \tag{6}$$

LEMMA 1.2. *If the parameters $\nu, b \in \mathbb{R}$ and $c \in \mathbb{C}$ are so constrained that*

$$k > \max\left\{0, \frac{|c|}{8} - 1\right\}$$

then the function $\varphi_{\nu,b,c} : \mathcal{U} \to \mathbb{C}$,

$$\varphi_{\nu,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (\kappa)_n n!} z^{n+1}$$

satisfies the following inequalities:

$$\left|\varphi'_{\nu,b,c}(z) - \frac{\varphi_{\nu,b,c}(z)}{z}\right| \le \frac{|c|}{4\kappa}\left(\frac{8(\kappa+1)+|c|}{8(\kappa+1)-|c|}\right),\ z \in \mathcal{U},$$

$$\left|\frac{z\varphi'_{\nu,b,c}(z)}{\varphi_{\nu,b,c}(z)} - 1\right| \le \frac{8(\kappa+1)\,|c| + |c|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)\,|c|}, \tag{7}$$

$$\frac{32\kappa(\kappa+1) - 8(3\kappa+2)\,|c| - |c|^2}{4\kappa[8(\kappa+1)-|c|]} \le \left|z\varphi'_{\nu,b,c}(z)\right|$$

$$\le \frac{32\kappa(\kappa+1) + 4(3\kappa+4)\,|c| + |c|^2}{4\kappa[8(\kappa+1)-|c|]},\ z \in \mathcal{U},$$

$$\left|z^2\varphi''_{\nu,b,c}(z)\right| \le \frac{|c|}{2\kappa}\left(\frac{|c|^2}{8(\kappa+1)}\cdot\frac{8(\kappa+2)+|c|}{8(\kappa+2)-|c|}+\frac{8(\kappa+1)+|c|}{8(\kappa-1)-|c|}\right),\ z\in\mathcal{U}$$

and

$$\left|\frac{z\varphi''_{\nu,b,c}(z)}{\varphi'_{\nu,b,c}(z)}\right| \le \frac{|c|}{2}\times$$
$$\times\frac{64(k+1)^2[8(\kappa+2)-|c|]+128(\kappa+1)(\kappa+2)-[8(\kappa+2)+|c|]\,|c|^2}{2(\kappa+1)[8(\kappa+1)-|c|]16(\kappa+1)(2\kappa-|c|)-|c|\,(4\kappa+|c|)}. \quad (8)$$

In this paper, using the generalized Bessel functions, we will introduce the operators:

$$I_{\nu_i}(u,g)(z)=\int_0^z\prod_{i=1}^n\left(\frac{u_{\nu_i,b,c}(t)}{g_i(t)}\right)^{\gamma_i}dt, \quad (9)$$

$$J_{\nu_i}(u,g)(z)=\int_0^z\prod_{i=1}^n\frac{(u_{\nu_i,b,c}(t))^{\gamma_i}}{(g_i(t))^{\sigma_i}}dt \quad (10)$$

$$T_{\nu}(u,g)(z)=\int_0^z(u'_{\nu,b,c}(t)e^{g(t)})^{\alpha}dt, \quad (11)$$

where $u_{\nu_i,b,c}(z)$ are generalized Bessel functions and $g_i(z)\in\mathcal{A}$.

These operators are derived from the operators defined in [9] and [10].

2. Main results

Theorem 2.1. *Let $\nu_i,b,c\in\mathbb{R},\kappa_i>\frac{|c|-2}{4}$ for $\kappa_i=\nu_i+\frac{b+1}{2}$ and the function $u_{\nu_i,b,c}(z)$ be defined by (3) for $i=\overline{1,n}$. If $\kappa=\min\{\kappa_1,\dots,\kappa_n\}$, $\gamma_i\in\mathbb{C}-0$, $d\in\mathbb{C}$, $d\ne-1$, $g_i(z)\in\mathcal{A}$ with $\left|\frac{zg_i'(z)}{g_i(z)}\right|\le M_i$ for $M_i\ge1, i=\overline{1,n}$ and if we have the inequality*

$$|d|+\sum_{i=1}^n|\gamma_i|\left(1+\frac{8(\kappa+1)\,|c|}{32\kappa(\kappa+1)-8(2\kappa+1)\,|c|+|c|^2}+M_i\right)\le1, \quad (12)$$

then the operator $I_{\nu_i}(u,g)(z)$ defined by (9) is in the univalent function class $\mathcal{S}$.

Proof. From the definition of $I_{\nu_i}(u,g)(z)$ we obtain that

$$\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)}=\sum_{i=1}^n\gamma_i\left(\frac{zu'_{\nu_i,b,c}(z)}{u_{\nu_i,b,c}(z)}-\frac{zg_i'(z)}{g_i(z)}\right).$$

It follows that

$$\left|\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)}\right|\le\sum_{i=1}^n|\gamma_i|\left(\left|\frac{zu'_{\nu_i,b,c}(z)}{u_{\nu_i,b,c}(z)}\right|+\left|\frac{zg_i'(z)}{g_i(z)}\right|\right). \quad (13)$$

We will use relation (5) from Lemma 1.1 and we obtain

$$\left|\frac{zu'_{\nu_i,b,c}(z)}{u_{\nu_i,b,c}(z)}\right|\le1+\frac{8(\kappa_i+1)\,|c|}{32\kappa_i(\kappa_i+1)-8(2\kappa_i+1)\,|c|+|c|^2}. \quad (14)$$

From (14) and from the hypothesis that $\left|\frac{zg_i'(z)}{g_i z}\right| \le M_i$ the relation (13) is equivalent with

$$\left|\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)}\right| \le \sum_{i=1}^{n} |\gamma_i| \left(1 + \frac{8(\kappa_i+1)\,|c|}{32\kappa_i(\kappa_i+1) - 8(2\kappa_i+1)\,|c| + |c|^2} + M_i\right).$$

We define the function $G : \left(\frac{|c|-2}{4}, \infty\right) \to \mathbb{R}$, $G(x) = \frac{8(x+1)|c|}{32x(x+1)-8(2x+1)|c|+|c|^2}$. This is a decreasing function, so it follows that

$$\frac{8(\kappa_i+1)\,|c|}{32\kappa_i(\kappa_i+1) - 8(2\kappa_i+1)\,|c| + |c|^2} \le \frac{8(\kappa+1)\,|c|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|c| + |c|^2}.$$

In order to prove the univalence we will use Theorem 1.1. So we have that

$$\left|d|z|^2 + (1-|z|^2)\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)}\right| \le |d| + \sum_{i=1}^{n} |\gamma_i| \left(1 + \frac{8(\kappa+1)\,|c|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|c| + |c|^2} + M_i\right). \quad (15)$$

Using (5) in (15) we obtain that

$$\left|d|z|^2 + (1-|z|^2)\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)}\right| \le 1,$$

so the operator $I_{\nu_i}(u,g)(z)$ is in the univalent function class $\mathcal{S}$. ■

For $\gamma_1 = \cdots = \gamma_n = \gamma$ and $M_1 = \cdots = M_n = M$ in Theorem 2.1 we obtain

COROLLARY 2.1. *Let $\nu_i, b, c \in \mathbb{R}$, $\kappa_i > \frac{|c|-2}{4}$ for $\kappa_i = \nu_i + \frac{b+1}{2}$ and the function $u_{\nu_i,b,c}(z)$ be defined by* (3) *for $i = \overline{1,n}$. If $\kappa = \min\{\kappa_1, \ldots, \kappa_n\}$, $\gamma \in \mathbb{C} - 0$, $d \in \mathbb{C}$, $d \neq -1$, $g_i(z) \in \mathcal{A}$ with $\left|\frac{zg_i'(z)}{g_i(z)}\right| \le M$ for $M \ge 1, i = \overline{1,n}$ and if we have the inequality*

$$|d| + n\,|\gamma| \left(1 + \frac{8(\kappa+1)\,|c|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|c| + |c|^2} + M\right) \le 1,$$

then the operator $I_{\nu_i}(u,g)(z) = \int_0^z \prod_{i=1}^n \left(\frac{u_{\nu_i,b,c}(t)}{g_i(t)}\right)^{\gamma} dt$ is in the univalent function class $\mathcal{S}$.

THEOREM 2.2. *Let $\nu_i, b, c \in \mathbb{R}$, $\kappa_i > \frac{|c|-2}{4}$ for $\kappa_i = \nu_i + \frac{b+1}{2}$ and the function $u_{\nu_i,b,c}(z)$ be defined by* (3) *for $i = \overline{1,n}$. If $\kappa = \min\{\kappa_1, \ldots, \kappa_n\}$, $\gamma_i \in \mathbb{C} - 0$, $d \in \mathbb{C}$, $d \neq -1$, $g_i(z) \in \mathcal{A}$ with $\left|\frac{zg_i'(z)}{g_i(z)}\right| \le M_i$ for $M_i \ge 1, i = \overline{1,n}$ and if we have the inequality*

$$|d| + \sum_{i=1}^{n} |\gamma_i| \left(1 + \frac{8(\kappa+1)\,|c|}{32\kappa(\kappa+1) - 8(2\kappa+1)\,|c| + |c|^2}\right) + \sum_{i=1}^{n} \sigma_i M_i \le 1,$$

then the operator $J_{\nu_i}(u,g)(z)$ defined by (10) *is in the univalent function class $\mathcal{S}$.*

Proof. Similar with the proof of Theorem 2.1. ∎

Theorem 2.3. *Let $\nu_i, b, c \in \mathbb{R}$, $\kappa_i > \frac{|c|}{8} - 1$ for $\kappa_i = \nu_i + \frac{b+1}{2}$ and the function $u_{\nu_i,b,c}(z)$ be defined by* (3) *for $i = \overline{1,n}$. If $\kappa = \min\{\kappa_1, \dots, \kappa_n\}$, $\gamma_i \in \mathbb{C} - 0$, $d \in \mathbb{C}$, $d \neq -1$, $g_i(z) \in \mathcal{A}$ with $\left|\frac{zg_i'(z)}{g_i(z)}\right| \leq M_i$ for $M_i \geq 1, i = \overline{1,n}$ and if we have the inequality*

$$|d| + \sum_{i=1}^{n} |\gamma_i| \left(1 + \frac{8(\kappa+1)|c| + |c|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)|c|} + M_i\right) \leq 1, \tag{16}$$

then the operator $I_{\nu_i}(u,g)(z)$ defined by (9) *is in the univalent function class $\mathcal{S}$.*

Proof. From (9) we obtain that

$$\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{zu'_{\nu_i,b,c}(z)}{u_{\nu_i,b,c}(z)} - \frac{zg_i'(z)}{g_i(z)}\right)$$

and

$$\left|\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)}\right| \leq \sum_{i=1}^{n} |\gamma_i| \left(\left|\frac{zu'_{\nu_i,b,c}(z)}{u_{\nu_i,b,c}(z)}\right| + \left|\frac{zg_i'(z)}{g_i(z)}\right|\right).$$

Now we will use the relation (7) from Lemma 1.2 and we obtain that

$$\left|\frac{zu'_{\nu_i,b,c}(z)}{u_{\nu_i,b,c}(z)}\right| \leq 1 + \frac{8(\kappa_i+1)|c| + |c|^2}{32\kappa_i(\kappa_i+1) - 4(2\kappa_i+3)|c|}.$$

Using this and the hypothesis that $\left|\frac{zg_i'(z)}{g_i(z)}\right| \leq M_i$ it follows that

$$\left|\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)}\right| \leq \sum_{i=1}^{n} |\gamma_i| \left(1 + \frac{8(\kappa_i+1)|c| + |c|^2}{32\kappa_i(\kappa_i+1) - 4(2\kappa_i+3)|c|}\right).$$

We consider the function $H : \left(\frac{|c|}{8} - 1, \infty\right)$, $H(x) = \frac{8(x+1)|c|+|c|}{32x(x+1)-4(2x+3)|c|}$ that is a decreasing function, so

$$\frac{8(\kappa_i+1)|c| + |c|^2}{32\kappa_i(\kappa_i+1) - 4(2\kappa_i+3)|c|} \leq \frac{8(\kappa+1)|c| + |c|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)|c|}.$$

Using Theorem 1.1 it follows that

$$\left|d|z|^2 + (1-|z|^2)\frac{zI''_{\nu_i}(u,g)(z)}{I'_{\nu_i}(u,g)(z)}\right|$$
$$\leq |d| + \sum_{i=1}^{n} |\gamma_i| \left(1 + \frac{8(\kappa+1)|c| + |c|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)|c|} + M_i\right) \leq 1.$$

From the above relation it follows that $I_{\nu_i}(u,g)(z) \in \mathcal{S}$. ∎

Theorem 2.4. *Let $\nu_i, b, c \in \mathbb{R}$, $\kappa_i > \frac{|c|}{8} - 1$ for $\kappa_i = \nu_i + \frac{b+1}{2}$ and the function $u_{\nu_i,b,c}(z)$ be defined by* (3) *for $i = \overline{1,n}$. If $\kappa = \min\{\kappa_1, \dots, \kappa_n\}$, $\gamma_i \in \mathbb{C} - 0$, $d \in \mathbb{C}$,*

$d \neq -1$, $g_i(z) \in \mathcal{A}$ with $\left|\frac{zg_i'(z)}{g_i(z)}\right| \leq M_i$ for $M_i \geq 1, i = \overline{1,n}$ and if we have the inequality

$$|d| + \sum_{i=1}^{n} |\gamma_i| \left(1 + \frac{8(\kappa+1)|c| + |c|^2}{32\kappa(\kappa+1) - 4(2\kappa+3)|c|}\right) + \sum_{i=1}^{n} \sigma_i M_i \leq 1, \tag{17}$$

then the operator $I_{\nu_i}(u,g)(z)$ defined by (9) is in the univalent function class $\mathcal{S}$.

Proof. The proof is similar with the proof of Theorem 2.3. ■

THEOREM 2.5. *Let $\nu, b, c \in \mathbb{R}$, $\kappa > \frac{|c|-2}{4}$ for $\kappa = \nu + \frac{b+1}{2}$ and the function $u_{\nu,b,c}(z)$ be defined by (3). If $\alpha \in \mathbb{C} - 0$, $d \in \mathbb{C} - -1$, $g(z) \in \mathcal{A}$ with $|zg_i'(z)| \leq M$ and if we have the inequality*

$$|d| + |\alpha| \left(\frac{4(\kappa+1)|c| + |c|^2}{8\kappa(\kappa+1) - 2(3\kappa+2)|c|} + M\right) \leq 1,$$

then the operator $T_\nu(u,g)(z)$ defined by (11) is in the class of univalent function $\mathcal{S}$.

Proof. From the definition of $T_\nu(u,g)(z)$ we obtain that

$$\frac{zT_\nu''(u,g)(z)}{T_\nu'(u,g)(z)} = \alpha \left(\frac{zu_{\nu,b,c}''(z)}{u_{\nu,b,c}(z)} + zg'(z)\right)$$

and

$$\left|\frac{zT_\nu''(u,g)(z)}{T_\nu'(u,g)(z)}\right| \leq |\alpha| \left(\left|\frac{zu_{\nu,b,c}''(z)}{u_{\nu,b,c}(z)}\right| + |zg'(z)|\right). \tag{18}$$

For the function $u_{\nu,b,c}(z)$ we will use the relation (6) from Lemma 1.1 and we obtain that

$$\left|\frac{zu_{\nu,b,c}''(z)}{u_{\nu,b,c}(z)}\right| \leq \frac{4(\kappa+1)|c| + |c|^2}{8\kappa(\kappa+1) - 2(3\kappa+2)|c|}.$$

Now using the hypothesis that $|zg(z)| \leq M$ and the above relation from (18) we obtain

$$\left|\frac{zT_\nu''(u,g)(z)}{T_\nu'(u,g)(z)}\right| \leq |\alpha| \left(\frac{4(\kappa+1)|c| + |c|^2}{8\kappa(\kappa+1) - 2(3\kappa+2)|c|} + M\right).$$

To prove the univalence we will use Theorem 1.1. So from Theorem 1.1 and from the theorem hypothesis it follows that

$$\left|d|z|^2 + (1-|z|)^2 \frac{zT_\nu''(u,g)(z)}{T_\nu'(u,g)(z)}\right| \leq |d| + |\alpha| \left(\frac{4(\kappa+1)|c| + |c|^2}{8\kappa(\kappa+1) - 2(3\kappa+2)|c|} + M\right) \leq 1,$$

which implies that the operator is in the univalent function class $\mathcal{S}$. ■

THEOREM 2.6. *Let $\nu, b, c \in \mathbb{R}$, $\kappa > \frac{|c|}{8} - 1$ for $\kappa = \nu + \frac{b+1}{2}$ and the function $u_{\nu,b,c}(z)$ be defined by (3). If $\gamma \in \mathbb{C} - 0$, $d \in \mathbb{C}$, $d \neq -1$, $g(z) \in \mathcal{A}$ with $|zg'(z)| \leq M$ for $M \geq 1$ and if we have the inequality*

$$|d| + |\alpha| \times$$

$$\times \left(\frac{|c|}{2} \cdot \frac{64(\kappa+1)^2[8(\kappa+2) - |c|] + 128(\kappa+1)(\kappa+2) - [8(\kappa+2) + |c|]|c|^2}{2(\kappa+1)[8(\kappa+1) - |c|]16(\kappa+1)(2\kappa - |c|) - |c|(4\kappa + |c|)} + M\right)$$

$$\leq 1,$$

then the operator $T_\nu(u,g)(z)$ *defined by* (11) *is in the univalent function class* $\mathcal{S}$.

Proof. The proof is similar with the proof of Theorem 2.5, or the results from Lemma 1.2 can be used. ■

Acknowledgements. This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People. The author would like to thank the referees for the useful comments and suggestions.

REFERENCES

[**1**] L.V. Ahlfors, *Sufficient conditions for quasiconformal extension*, Proc. 1973 Conf. Univ. of Maryland, Ann. of Math. Studies, **79**, 23–29.

[**2**] J. Becker, *Löwnersche Differentialgleichung und Schlichtheits-Kriterion*, Math. Ann. **202** (1973), 321–335.

[**3**] Á. Baricz, *Generalized Bessel functions of the first kind*, Lecture Notes in Mathematics, Vol. 1994, Springer-Verlag, Berlin, 2010.

[**4**] Á. Baricz, *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen **73** (2008), 155–178.

[**5**] Á. Baricz, *Some inequalities involving generalized Bessel functions*, Math. Inequal. Appl. **10** (2007), 827–842.

[**6**] Á. Baricz, B.A. Frasin, *Univalence of integral operators involving Bessel functions*, Appl. Math. Lett. **23** (2010), 371-376.

[**7**] Á. Baricz, S. Ponnusamy, *Starlikeness and convexity of generalized Bessel functions*, Integral Transforms Spec. Funct. **21** (2010), 641–653.

[**8**] E. Deniz, H. Orhan, H.M. Srivastava, *Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions*, Taiwanese J. Math. **15** (2011), 883–917.

[**9**] N. Ularu, D. Breaz, *Univalence conditions and some properties for some new integral operators*, submitted.

[**10**] N. Ularu, D. Breaz, *Univalence criterion and convexity for an integral operator*, Appl. Math. Letters **25** (2012), 658–661.

(received 20.01.2012; in revised form 19.05.2012; available online 01.10.2012)

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