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A NEW CLASS OF MEROMORPHIC FUNCTIONS RELATED TO CHO-KWON-SRIVASTAVA OPERATOR

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Abstract. In the present paper, we introduce a new class of meromorphic functions defined by means of the Hadamard product of Cho-Kwon-Srivastava operator and we define here a similar transformation by means of an operator introduced by Ghanim and Darus. We investigate a number of inclusion relationships of this class. We also derive some interesting properties of this class.

1. Introduction

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured unit disk $U = \{z : 0 < |z| < 1\}$. For $0 \leq \beta$, we denote by $S^*(\beta)$ and $k(\beta)$, the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in U.

For functions $f_j(z)(j = 1; 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \tag{1.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{1.3}$$

Let us define the function $\tilde{\phi}(\alpha, \beta; z)$ by

$$\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left| \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \right| z^n, \tag{1.4}$$

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for $\beta \neq 0, -1, -2, \ldots$, and $\alpha \in \mathbb{C}/\left\{0\right\}$, where $(\lambda)n = \lambda(\lambda+1)_{n+1}$ is the Pochhammer symbol. We note that

$$\tilde{\phi}\left(\alpha,\beta;z\right) = \frac{1}{z}{}_2F_1\left(1,\alpha,\beta;z\right)$$

where

$${}_2F_1\left(b,\alpha,\beta;z\right) = \sum_{n=0}^{\infty} \frac{\left(b\right)_n\left(\alpha\right)_n}{\left(\beta\right)_n}\frac{z^n}{n!}$$

is the well-known Gaussian hypergeometric function.

Let us put

$$q_{\lambda,\mu}\left(z\right) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} z^n, \qquad \left(\lambda > 0, \mu \geq 0\right).$$

Corresponding to the functions $\tilde{\phi}(\alpha,\beta;z)$ and $q_{\lambda,\mu}\left(z\right)$, and using the Hadamard product for $f(z) \in \Sigma$, we define a new linear operator $L(\alpha,\beta,\lambda,\mu)$ on Σ by

$$\begin{aligned} L\left(\alpha,\beta,\lambda,\mu\right) f\left(z\right) &= \left(f\left(z\right) * \tilde{\phi}\left(\alpha,\beta;z\right) * q_{\lambda,\mu}\left(z\right)\right) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty}\left|\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right|\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} a_n z^n. \end{aligned} \tag{1.5}$$

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [4,5], Liu [10], Liu and Srivastava [13–15], Cho and Kim [1].

For a function $f \in L\left(\alpha,\beta,\lambda,\mu\right) f\left(z\right)$ we define

$$I^{\mu,0}_{\alpha,\beta,\lambda} = L\left(\alpha,\beta,\lambda,\mu\right) f\left(z\right)$$

and for $k = 1, 2, 3, \ldots,$

$$\begin{aligned} I^{\mu,k}_{\alpha,\beta,\lambda} f\left(z\right) &= z\left(I^{k-1} L\left(\alpha,\beta,\lambda,\mu\right) f(z)\right)' + \frac{2}{z} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} n^k \left|\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right|\left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} a_n z^n. \end{aligned} \tag{1.6}$$

Note that if $n = \beta, k = 0$ the operator $I^{\mu,0}_{\alpha,n,\lambda}$ have been introduced by N.E. Cho, O.S. Kwon and H.M. Srivastava [2] for $\mu \in \mathbb{N}_0 = \mathbb{N} \cup 0$. It was known that the definition of the operator $I^{\mu,0}_{\alpha,n,\lambda}$ was motivated essentially by the Choi-Saigo-Srivastava operator [3] for analytic functions, which includes a simpler integral operator studied earlier by Noor [17] and others (cf. [11,12,18]). Note also the operator $I^{0,k}_{\alpha,\beta}$ have been recently introduced and studied by Ghanim and Darus [6–8]. To our best knowledge, the recent work regarding operator $I^{\mu,0}_{\alpha,n,\lambda}$ was charmingly studied by Piejko and Sokól [19]. Moreover, the operator $I^{\mu,k}_{\alpha,\beta,\lambda}$ was defined and studied by Ghanim and Darus [9]. In the same direction, we will study for the operator $I^{\mu,k}_{\alpha,\beta,\lambda}$ given in (1.6).

Now, it follows from (1.5) and (1.6) that

$$z\left(I^{\mu,k}_{\alpha,\beta,\lambda} f(z)\right)' = \alpha I^{\mu,k}_{\alpha+1,\beta,\lambda} f(z) - \left(\alpha+1\right) I^{\mu,k}_{\alpha,\beta,\lambda} f(z). \tag{1.7}$$

Let Ω be the class of analytic functions $h(z)$ with $h(0)=1$, which are convex and univalent in the open unit disk $U=U^{*}\cup\{0\}$. For functions f and g analytic in U, we say that f is subordinate to g and write $f\prec g$, if g is univalent in U, $f(0)=g(0)$ and $f(U)\subset g(U)$.

Definition 1.1. A function $f\in\Sigma$ is said to be in the class $\Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)$, if it satisfies the subordination condition

$$(1+\rho)\,z\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)+\rho z^{2}\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)'\prec h\,(z) \tag{1.8}$$

where ρ is a real or complex number and $h(z)\in\Omega$.

Let A be class of functions of the form

$$f\,(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n} \tag{1.9}$$

which are analytic in U. A function $h(z)\in A$ is said to be in the class $S^{*}(\mathfrak{a})$, if

$$\operatorname{Re}\left\{\frac{zf'\,(z)}{f\,(z)}\right\}>\mathfrak{a}\qquad(z\in U)\,.$$

For some $\mathfrak{a}(\mathfrak{a}<1)$. When $0<\mathfrak{a}<1, S^{*}(\mathfrak{a})$ is the class of starlike functions of order $\mathfrak{a}$ in U. A function $h(z)\in A$ is said to be prestarlike of order $\mathfrak{a}$ in U, if

$$\frac{z}{(1-z)^{2(1-\mathfrak{a})}}*f\,(z)\in S^{*}\,(\mathfrak{a})\qquad(\mathfrak{a}<1)$$

where the symbol $*$ means the familiar Hadamard product (or convolution) of two analytic functions in U. We denote this class by $R(\mathfrak{a})$ (see [20,24]). A function $f(z)\in A$ is in the class $R(0)$, if and only if $f(z)$ is convex univalent in U and

$$R\left(\frac{1}{2}\right)=S^{*}\left(\frac{1}{2}\right)$$

In this paper, we introduce and investigate various inclusion relationships and convolution properties of a certain class of meromorphic functions, which are defined in this paper by means of a linear operator.

2. Preliminary results

In order to prove our main results, we need the following lemmas.

Lemma 2.1. [16] *Let $g(z)$ be analytic in U, and $h(z)$ be analytic and convex univalent in U with $h(0)=g(0)$. If,*

$$g\,(z)+\frac{1}{\mathfrak{m}}zg'\,(z)\prec h(z) \tag{2.1}$$

where $\operatorname{Re}\mathfrak{m}\geq 0$ *and* $\mathfrak{m}\neq 0$*, then*

$$g\,(z)\prec\widetilde{h}(z)=\mathfrak{m}z^{-\mathfrak{m}}\int_{0}^{z}t^{\mathfrak{m}-1}h\,(t)\;dt\prec h\,(z)$$

and $\widetilde{h}(z)$ is the best dominant of (2.1).

Lemma 2.2. [20] *Let $a<1, f(z)\in S^{*}(a)$ and $g(z)\in R(\mathfrak{a})$. For any analytic function $F(z)$ in U, then*

$$\frac{g*(fF)}{g*f}(U)\subset\overline{co}\left(F\left(U\right)\right),$$

where $\overline{co}\left(F\left(U\right)\right)$ denotes the convex hull of $F(U)$.

3. Main results

Theorem 3.1. *For some real ρ, let $0\le\rho_1<\rho_2$. Then*

$$\Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho_2;h)\subset\Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho_1;h)$$

Proof. Let $0\le\rho_1<\rho_2$ and suppose that

$$g(z)=z\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right) \tag{3.1}$$

for $f(z)\in\Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho_2;h)$. Then the function $g(z)$ is analytic in U with $g(0)=1$. Differentiating both sides of (3.1) with respect to z and using (1.7), we have

$$(1+\rho_2)\,z\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)+\rho_2 z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)'=g(z)+\rho_2 zg'(z)\prec h(z). \tag{3.2}$$

Hence an application of Lemma 2.1 with $\mathfrak{m}=\frac{1}{\rho_2}>0$ yields

$$g(z)\prec h(z). \tag{3.3}$$

Noting that $0\le\frac{\rho_1}{\rho_2}<1$ and that $h(z)$ is convex univalent in U, it follows from (3.1)–(3.3) that

$$\begin{aligned}&(1+\rho_1)\,z\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)+\rho_1 z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)'\\&=\frac{\rho_1}{\rho_2}\left[(1+\rho_2)\,z\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)+\rho_2 z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k}f(z)\right)'\right]+\left(1-\frac{\rho_1}{\rho_2}\right)g(z)\prec h(z).\end{aligned}$$

Thus, $f(z)\in\Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho_1;h)$ and the proof of Theorem 3.1 is complete. ■

Theorem 3.2. *Let,*

$$\operatorname{Re}\left\{z\widetilde{\phi}(\alpha_1,\alpha_2;z)\right\}>\frac{1}{2}\qquad\left(z\in U;\alpha_2\notin\{0,-1,-2,\ldots\}\right), \tag{3.4}$$

where $\widetilde{\phi}(\alpha_1,\alpha_2;z)$ is defined as in (1.4). Then,

$$\Sigma_{\alpha_2,\beta}^{\mu,k,\lambda}(\rho;h)\subset\Sigma_{\alpha_1,\beta}^{\mu,k,\lambda}(\rho;h).$$

Proof. For $f(z)\in\Sigma$ it is easy to verify that

$$z\left(I_{\alpha_1,\beta,\lambda}^{\mu,k}f(z)\right)=\left(z\widetilde{\phi}(\alpha_1,\alpha_2;z)*\left(zI_{\alpha_2,\beta,\lambda}^{\mu,k}f(z)\right)\right) \tag{3.5}$$

and

$$z^2\left(I^{\mu,k}_{\alpha_1,\beta,\lambda}f(z)\right)' = \left(z\widetilde{\phi}\left(\alpha_1,\alpha_2;z\right) * z^2\left(I^{\mu,k}_{\alpha_2,\beta,\lambda}f(z)\right)'\right). \tag{3.6}$$

Let $f(z) \in \Sigma^{\mu,k,\lambda}_{\alpha_2,\beta}(\rho;h)$. Then from (3.5) and (3.6), we deduce that

$$(1+\rho)\,z\left(I^{\mu,k}_{\alpha_1,\beta,\lambda}f(z)\right) + \rho z^2\left(I^{\mu,k}_{\alpha_1,\beta,\lambda}f(z)\right)' = \left(z\widetilde{\phi}\left(\alpha_1,\alpha_2;z\right)\right) * \Psi(z) \tag{3.7}$$

and

$$\Psi(z) = (1+\rho)\,z\left(I^{\mu,k}_{\alpha_2,\beta,\lambda}f(z)\right) + \rho z^2\left(I^{\mu,k}_{\alpha_2,\beta,\lambda}f(z)\right)' \prec h(z) \tag{3.8}$$

In view of (3.4), the function $z\widetilde{\phi}(\alpha_1,\alpha_2;z)$ has the Herglotz representation

$$z\widetilde{\phi}\left(\alpha_1,\alpha_2;z\right) = \int_{|x|=1}\frac{d\mathfrak{m}(x)}{1-xz}\,(z\in U), \tag{3.9}$$

where $\mathfrak{m}(x)$ is a probability measure defined on the unit circle $|x|=1$ and

$$\int_{|x|=1} d\mathfrak{m}(x) = 1.$$

Since $h(z)$ is convex univalent in U, it follows from (3.7)–(3.9) that

$$(1+\rho)\,z\left(I^{\mu,k}_{\alpha_1,\beta,\lambda}f(z)\right) + \rho z^2\left(I^{\mu,k}_{\alpha_1,\beta,\lambda}f(z)\right)' = \int_{|x|=1}\Psi(xz)\;d\mathfrak{m}(x) \prec h(z).$$

This shows that $f(z) \in \Sigma^{\mu,k,\lambda}_{\alpha_1,\beta}(\rho;h)$ and the theorem is proved. ∎

THEOREM 3.3 *Let* $0<\alpha_1<\alpha_2$. *Then*

$$\Sigma^{\mu,k,\lambda}_{\alpha_2,\beta}(\rho;h) \subset \Sigma^{\mu,k,\lambda}_{\alpha_1,\beta}(\rho;h).$$

Proof. Define,

$$g(z) = z + \sum_{n=1}^{\infty}\left|\frac{(\alpha_1)_{n+1}}{(\alpha_2)_{n+1}}\right| z^{n+1}\,(z\in U; 0<\alpha_1<\alpha_2).$$

Then,

$$z^2\widetilde{\phi}\left(\alpha_1,\alpha_2;z\right) = g(z) \in A \tag{3.10}$$

where $\widetilde{\phi}(\alpha_1,\alpha_2;z)$ is defined as in (1.4), and

$$\frac{z}{(1-z)^{\alpha_2}} * g(z) = \frac{z}{(1-z)^{\alpha_1}}. \tag{3.11}$$

By (3.11), we see that

$$\frac{z}{(1-z)^{\alpha_2}} * g(z) \in S^*\left(1-\frac{\alpha_1}{2}\right) \subset S^*\left(1-\frac{\alpha_2}{2}\right)$$

for $0<\alpha_1<\alpha_2$, which implies that

$$g(z) \in R\left(1-\frac{\alpha_2}{2}\right) \tag{3.12}$$

Let $f(z) \in \Sigma_{\alpha_2,\beta}^{\mu,k,\lambda}(\rho;h)$. Then we deduce from (3.7), (3.8) and (3.10) that

$$(1+\rho)\,z\left(I_{\alpha_1,\beta,\lambda}^{\mu,k} f(z)\right) + \rho z^2 \left(I_{\alpha_1,\beta,\lambda}^{\mu,k} f(z)\right)' = \frac{g(z)}{z} * \Psi(z) = \frac{g(z) * (z\Psi(z))}{g(z) * z}, \tag{3.13}$$

where

$$\Psi(z) = (1+\rho)\,z\left(I_{\alpha_2,\beta,\lambda}^{\mu,k} f(z)\right) + \rho z^2 \left(I_{\alpha_2,\beta,\lambda}^{\mu,k} f(z)\right)' \prec h(z). \tag{3.14}$$

Since z belongs to $S^*\left(1-\frac{\alpha_2}{2}\right)$ and $h(z)$ is convex univalent in U, it follows from (3.12)–(3.14) and Lemma 2.2 that

$$(1+\rho)\,z\left(I_{\alpha_1,\beta,\lambda}^{\mu,k} f(z)\right) + \rho z^2 \left(I_{\alpha_1,\beta,\lambda}^{\mu,k} f(z)\right)' \prec h(z)$$

Thus, $f(z) \in \Sigma_{\alpha_1,\beta}^{\mu,k,\lambda}(\rho;h)$ and the proof is completed. ■

As a special case of Theorem 3.3, we have

$$\Sigma_{\alpha+1,\beta}^{\mu,k,\lambda}(\rho;h) \subset \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)\,(\alpha>0)$$

In Theorem 3.4 below we give a generalization of the above result.

THEOREM 3.4 *Let* $\operatorname{Re}\alpha \geq 0$ *and* $\alpha \neq 0$. *Then,*

$$\Sigma_{\alpha+1,\beta}^{\mu,k,\lambda}(\rho;h) \subset \Sigma_{\alpha,\beta}^{\mu,k,\lambda}\left(\rho;\widetilde{h}\right),$$

where

$$\widetilde{h}(z) = \alpha z^{-\alpha}\int_0^z t^{\alpha-1} h(t)\,dt \prec h(z).$$

Proof. Let us define

$$g(z) = (1+\rho)\,z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right) + \rho z^2 \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)' \tag{3.15}$$

for $f(z) \in \Sigma$. Then (1.7) and (3.15) lead to

$$\frac{g(z)}{z} = \alpha\rho\left(I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z)\right) + (1-\alpha\rho)\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right). \tag{3.16}$$

Differentiating both sides of (3.16) and using (1.7), we obtain the following

$$\begin{aligned} g'(z) - \frac{g(z)}{z} &= \alpha\rho z\left(I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z)\right)' \\ &\quad + (1-\alpha\rho)\left[\alpha\left(I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z)\right) - (1+\alpha)\left(I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z)\right)\right]. \end{aligned} \tag{3.17}$$

By (3.16) and (3.17), we get

$$g'(z) - \frac{\alpha g(z)}{z} = \alpha\rho z\left(I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z)\right)' + \alpha(1+\rho)\left(I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z)\right),$$

that is,

$$g(z) + \frac{z g'(z)}{\alpha} = (1+\rho)\,z\left(I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z)\right) + \rho z^2 \left(I_{\alpha+1,\beta,\lambda}^{\mu,k} f(z)\right)'. \tag{3.18}$$

If $f \in \Sigma_{\alpha+1,\beta}^{\mu,k,\lambda}(\rho;h)$, then it follows from (3.18) that

$$g(z) + \frac{zg'(z)}{\alpha} \prec h(z) \qquad (\operatorname{Re}\alpha \geq 0, \alpha \neq 0).$$

Hence an application of Lemma 2.1 yields

$$g(z) \prec \widetilde{h}(z) = \alpha z^{-\alpha} \int_0^z t^{\alpha-1} h(t)\, dt \prec h(z),$$

which shows that

$$f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}\left(\rho;\widetilde{h}\right) \subset \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h) \quad \blacksquare$$

THEOREM 3.5 *Let* $\rho > 0, \delta > 0$ *and* $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;\delta h + 1 - \delta)$. *If* $\delta \leq \delta_0$, *where*

$$\delta_0 = \frac{1}{2}\left(1 - \frac{1}{\rho}\int_0^1 \frac{u^{\frac{1}{\rho}-1}}{1+u}\, du\right)^{-1} \tag{3.19}$$

then $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)$.

Proof. Let us define

$$g(z) = z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right) \tag{3.20}$$

for $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;\delta h + 1 - \delta)$. with $\rho > 0$, and $\delta > 0$. Then we have

$$g(z) + \rho z g'(z) = (1+\rho) z \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right) + \rho z^2 \left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)' \prec \delta(h(z) - 1) + 1$$

Hence an application of Lemma 2.1 yields

$$g(z) \prec \frac{\delta}{\rho} z^{-\frac{1}{\rho}} \int_0^z t^{\frac{1}{\rho}-1} h(t)\, dt + 1 - \delta = (h * \Psi)(z), \tag{3.21}$$

where

$$\Psi(z) = \frac{\delta}{\rho} z^{-\frac{1}{\rho}} \int_0^z \frac{t^{\frac{1}{\rho}-1}}{1-t}\, dt + 1 - \delta \tag{3.22}$$

If $0 < \delta \leq \delta_0$, where $\delta_0 > 1$ is given by (3.19), then it follows from (3.22) that

$$\operatorname{Re}\Psi(z) = \frac{\delta}{\rho}\int_0^1 u^{\frac{1}{\rho}-1} \operatorname{Re}\left(\frac{1}{1-uz}\right) du + 1 - \delta > \frac{\delta}{\rho}\int_0^1 \frac{u^{\frac{1}{\rho}-1}}{1+u} du + 1 - \delta \geq \frac{1}{2}$$

$(z \in U)$. Now, by using the Herglotz representation for $\Psi(z)$, from (3.20) and (3.21) we get

$$z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right) \prec (h * \Psi)(z) \prec h(z)$$

because $h(z)$ is convex univalent in U. This shows that $f(z) \in \Sigma(\alpha,\beta,k,\rho;h)$. For $h(z) = \frac{1}{1-z}$ and $f(z) \in \Sigma$ defined by

$$z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right) = \frac{\delta}{\rho} z^{-\frac{1}{\rho}} \int_0^z \frac{t^{\frac{1}{\rho}-1}}{1-t}\, dt + 1 - \delta,$$

it is easy to verify that

$$(1+\rho)\, z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)+\rho z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)^{\prime}=\delta\,(h(z)-1)+1$$

Thus, $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;\delta h+1-\delta)$. Also, for $\delta>\delta_0$, we have

$$\operatorname{Re} z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right) \rightarrow \frac{\delta}{\rho} \int_0^1 \frac{u^{\frac{1}{\rho}-1}}{1+u}\, du+1-\delta<\frac{1}{2}\,(z \rightarrow-1),$$

which implies that $f(z) \notin \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)$. ■

4. Convolution properties

THEOREM 4.1. *Let* $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)$, $g(z) \in \Sigma$ *and* $\operatorname{Re}(zg(z))>\frac{1}{2}\,(z \in U)$. *Then,*

$$(f*g)(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)$$

Proof. For $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)$ and $g \in \Sigma$. we have

$$\begin{aligned}(1+\rho)\, z&\left(I_{\alpha,\beta,\lambda}^{\mu,k}(f*g)(z)\right)+\rho z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k}(f*g)(z)\right)^{\prime}\\ &=(1+\rho)\, zg(z)*z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)+\rho zg(z)*z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)^{\prime}=zg(z)*\Psi(z)\end{aligned} \tag{4.1}$$

where

$$\Psi(z)=(1+\rho)\, z\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)+\rho z^2\left(I_{\alpha,\beta,\lambda}^{\mu,k} f(z)\right)^{\prime} \prec h(z) \tag{4.2}$$

The remaining part of the proof of Theorem 4.1 is similar to that of Theorem 3.2 and hence we omit it. ■

COROLLARY 4.1. *Let* $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)$ *be given by* (1.1) *and let*

$$\omega_m(z)=\frac{1}{z}+\sum_{n=1}^{m-1} a_n z^{n-1}\,(m \in N \backslash\{1\}).$$

Then the function

$$\sigma_m(z)=\int_0^1 t \omega_m(tz)\, dt$$

is also in the class $\Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho;h)$.

Proof. We have

$$\sigma_m(z)=\frac{1}{z}+\sum_{n=1}^{m-1} \frac{a_n}{n+1} z^{n-1}=(f*g_m)(z) \qquad (m \in N \backslash\{1\}), \tag{4.3}$$

where

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^{n-1} \in \Sigma(\alpha,\beta,k,\rho;h)$$

and

$$g_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{z^{n-1}}{n+1} \in \Sigma.$$

Also, for $m \in N \setminus \{1\}$, it is known from [21] that

$$\operatorname{Re}\{z g_m(z)\} = \operatorname{Re}\left\{1 + \sum_{n=1}^{m-1} \frac{z^n}{n+1}\right\} > \frac{1}{2} \qquad (z \in U). \tag{4.4}$$

In view of (4.3) and (4.4), an application of Theorem 4.1 leads to $\sigma_m(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho; h)$. ■

THEOREM 4.2. *Let* $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho; h)$, $g(z) \in \Sigma$ *and* $z^2 g(z) \in R(\mathfrak{a})$ $(\mathfrak{a} < 1)$. *Then,*

$$(f * g)(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho; h).$$

Proof. For $f(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho; h)$ and$g(z) \in \Sigma$, from (4.1) (used in the proof of Theorem 4.1, we can write

$$(1+\rho) z \left(I_{\alpha,\beta,\lambda}^{\mu,k}(f * g)(z)\right) + \rho z^2 \left(I_{\alpha,\beta,\lambda}^{\mu,k}(f * g)(z)\right)' = \frac{z^2 g(z) * z\Psi(z)}{z^2 g(z) * z} \quad (z \in U), \tag{4.5}$$

where $\Psi(z)$ is defined as in (4.2).

Since $h(z)$ is convex univalent in U, $\Psi(z) \prec h(z)$, $z^2 g(z) \in R(\mathfrak{a})$ and $z \in S^*(\mathfrak{a})$ $(\mathfrak{a} < 1)$, the desired result follows from (4.5) and Lemma 2.2 ■

Taking $\mathfrak{a} = 0$ and $\mathfrak{a} = \frac{1}{2}$, Theorem 4.2 reduces to the following.

COROLLARY 4.2. *Let* $f(z) \in \Sigma(\alpha, \beta, k, \rho; h)$ *and let* $g(z) \in \Sigma$ *satisfy either of the following conditions*

(i) $z^2 g(z)$ *is convex univalent in* U *or*

(ii) $z^2 g(z) \in S^*\left(\frac{1}{2}\right)$.

Then, $(f * g)(z) \in \Sigma_{\alpha,\beta}^{\mu,k,\lambda}(\rho; h)$.

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