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TWO INFINITE FAMILIES OF EQUIVALENCES OF THE CONTINUUM HYPOTHESIS

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Abstract. In this brief note we present two infinite families of equivalences of the Continuum Hypothesis, as follows:

• For every fixed $n \geq 2$, the Continuum Hypothesis is equivalent to the following statement: "There is an n-dimensional real normed vector space E including a subset A of size $\aleph_1$ such that $E \setminus A$ is not path connected".

• For every fixed T_1 first-countable topological space X with at least two points, the Continuum Hypothesis is equivalent to the following statement: "There is a point of the Tychonoff product $X^{\mathbb{R}}$ with a fundamental system of open neighbourhoods B of size $\aleph_1$".

1. The main theorems

Throughout this paper, the cardinality of a set X is denoted by $|X|$.

The Continuum Hypothesis (**CH**) is the statement "$\mathfrak{c} = \aleph_1$", where $\mathfrak{c} = |\mathbb{R}| = 2^{\aleph_0}$ and $\aleph_1$ is the first uncountable cardinal. **CH** is probably the most famous mathematical statement known to be independent of **ZFC** (Zermelo-Fraenkel Set Theory, with the Axiom of Choice).

As we will see, there are elementary statements from Analysis and Topology which cannot be settled without dealing with such set-theoretical hypothesis.

For instance, it is well-known that the following statements, denoted by (*) and (**), both hold in **ZFC**:

(*) *Whenever A is a countable subset of $\mathbb{R}^2$, $\mathbb{R}^2 \setminus A$ is path connected.*

(**) *If $f \in \mathbb{R}^{\mathbb{R}}$ and B is a countable subfamily of $\mathcal{P}(\mathbb{R}^{\mathbb{R}})$, then B is not a fundamental system of open neighbourhoods of the point f in the Tychonoff topology.*

In this paper we show that the analogous statements obtained by considering $|A| = |B| = \aleph_1$ are independent of **ZFC**; they are undecidable statements because they are closely related to the Continuum Hypothesis. More precisely, we prove the following two theorems, each one of them presenting an infinite family of equivalences of **CH**:

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Theorem 1.1. *For every fixed $n \geq 2$, **CH** is equivalent to the following statement:*

"There is an n-dimensional real normed vector space E including a subset A of size $\aleph_1$ such that $E \setminus A$ is not path connected".

Theorem 1.2. *For every fixed T_1 first-countable topological space X with at least two points, the Continuum Hypothesis is equivalent to the following statement:*

"There is a point of the Tychonoff product $X^{\mathbb{R}}$ with a fundamental system of open neighbourhoods B of size $\aleph_1$".

A number of statements from Analysis and Topology are known to be equivalences of **CH**: here we are presenting another ones. The reader may find several equivalences of **CH** in the seminal work of Sierpiński back in the 1930's [2] or in the recent book of Komjáth and Totik [1]. All terminology referring to normed spaces and topological spaces may be found at [3].

2. Proof of the Main Theorems

For the following result (which generalizes the statement (*)), the crucial hypothesis is $\kappa < \mathfrak{c}$. In what follows, for any pair of distinct points $a, b \in \mathbb{R}^2$ let $[a, b]$ denote the segment $\{a + t(b - a) : 0 \leq t \leq 1\}$.

Proposition 2.1. *Let $A \subseteq \mathbb{R}^2$ be a set of size $\kappa < \mathfrak{c}$. Then $\mathbb{R}^2 \setminus A$ is path connected.*

Proof. Let x, y be distinct points of $\mathbb{R}^2 \setminus A$ and fix a line m such that $x, y \notin m$. For every $z \in m$, consider a path φ_z whose image is $[x, z] \cup [z, y]$. As $|m| = \mathfrak{c} > \kappa = |A|$, there are no injective functions from m into A and it follows that at least one of the paths φ_z does not intersect A (otherwise we would be able to use the Axiom of Choice in order to define an injective function from m into A). Therefore, there is a path joining x and y which is contained in $\mathbb{R}^2 \setminus A$. ∎

Of course, the same geometric argument may be done in any 2-dimensional subspace of any given Euclidean space, or, more generally, in 2-dimensional subspaces of any given real normed vector space. So, the following corollary holds:

Corollary 2.2. *Let $n \geq 2$ and let E be an n-dimensional real normed vector space and let $A \subseteq E$ be a set with $|A| < \mathfrak{c}$. Then $E \setminus A$ is path connected.*

Now our first main theorem is easily proved.

Proof of Theorem 1.1. Let $n \geq 2$ be fixed. Assuming **CH**, we may take $E = \mathbb{R}^n$ and take A to be any $(n-1)$-dimensional subspace of E. For the opposite implication, note that under $\neg$**CH** (i.e., under $\aleph_1 < \mathfrak{c}$) the preceding corollary ensures that for every n-dimensional normed space E and for every subset A of size $\aleph_1$ one has $E \setminus A$ path connected, and so we are done. ∎

Let us turn to the second main theorem. In the following proposition, $[\mathbb{R}]^{<\omega}$ denote the family of all finite subsets of $\mathbb{R}$.

Proposition 2.3. *If X is a first-countable topological space and $f \in X^{\mathbb{R}}$, then f has a fundamental system of open neighbourhoods of size not larger than $\mathfrak{c}$ in the Tychonoff product $X^{\mathbb{R}}$.*

Proof. For every $x \in X$ fix a countable local base $\mathcal{V}_x$ of x. For every non-empty $A \in [\mathbb{R}]^{<\omega}$, say $A = \{r_1, r_2, \dots, r_n\}$, let $\mathcal{U}_A$ be the family of basic open sets of $X^{\mathbb{R}}$ given by

$$\mathcal{U}_A = \{V \times {}^{\mathbb{R}\setminus A}X : V \in \prod_{1\leq i\leq n} \mathcal{V}_{f(r_i)}\}.$$

Each $\mathcal{U}_A$ is countable, and, as the family $[\mathbb{R}]^{<\omega}$ has size $\mathfrak{c}$, the family of open sets

$$\bigcup_{A\in[\mathbb{R}]^{<\omega}} \mathcal{U}_A$$

is (clearly) a local base of f of size not larger than $\mathfrak{c}$. ■

Notice that, in the preceding proposition, nothing restrains the existence of a point of $X^{\mathbb{R}}$ with a local base of size $\aleph_1$.

In T_1 spaces, the intersection of a local base at a point must reduce to a singleton, so (as T_1 is a productive property) the following proposition ensures that, if X is a T_1 space with at least two points, then every subfamily of $\mathcal{P}(X^{\mathbb{R}})$ of size less than $\mathfrak{c}$ cannot be a local base of any given point of the Tychonoff product. In particular, the following is a strengthening of (**).

Proposition 2.4. *Let X be a T_1 space with at least two points and let $f \in X^{\mathbb{R}}$. Suppose B is a non-empty family of basic open neighbourhoods of f such that $|B| < \mathfrak{c}$. Then $\bigcap B \neq \{f\}$. In particular, B is not a local base at the point f.*

Proof. For every $U \in B$, let C_U be the finite set of detached coordinates of U, meaning that if $U = \prod_{r\in\mathbb{R}} U_r$ then $C_U = \{r \in \mathbb{R} : U_r \neq X\}$. As $|B| < \mathfrak{c}$, the set $C = \bigcup_{U\in B} C_U$ has also size less than $\mathfrak{c}$, and therefore $\mathbb{R} \setminus C \neq \emptyset$. Define a function $g : \mathbb{R} \to X$ such that $g(x) = f(x)$ if $x \in C$ and $g(x) \neq f(x)$ otherwise; here we are using the hypothesis of X having more than one point. As $\mathbb{R} \setminus C \neq \emptyset$, one has $g \neq f$ and $g \in \bigcap B$, and this suffices for us. ■

Notice that, in the preceding proposition, nothing ensures that there is a local base at f of size $\mathfrak{c}$.

The two preceding propositions were stated for, respectively, first-countable spaces and T_1 spaces with at least two points. Considering both hypothesis simultaneously, we prove our second main theorem.

Proof of Theorem 1.2. Let X be a fixed T_1 first-countable topological space with at least two points. Assuming **CH**, by Proposition 2.3 we have—as X is first-countable—that every point has a local base of size not larger than $\mathfrak{c} = \aleph_1$; and, as X is T_1 with more than one point, there are no points of X with a countable local base (by Proposition 2.4—or even (*)). In this case, *every* point of the product $X^{\mathbb{R}}$ has a local base of size $\aleph_1$. On the other hand, assume $\neg$**CH**: by Proposition 2.4, there is no point of $X^{\mathbb{R}}$ with a local base of size $\aleph_1$, and this finishes the proof. ■

Remark. Our hypothesis "first-countable" was used, mainly, for showing that the cardinality of the continuum $\mathfrak{c}$ is an upper bound for the possible sizes of certain local bases at arbitrary points of $X^{\mathbb{R}}$. Adapting the arguments, one can easily prove the following: for every T_1 topological space $\langle X, \tau \rangle$ with at least two points and satisfying $|\tau| \leq \mathfrak{c}$ (or even $|\tau| \leq \aleph_1$), **CH** is equivalent to the statement: "There is a point of $X^{\mathbb{R}}$ with a local base of size $\aleph_1$". Notice that spaces with a countable net satisfy $|\tau| \leq \mathfrak{c}$. (A *net* for a topological space is a family of (not necessarily open) subsets such that every open set may be written as an union of a subfamily of the net.)

We also would like to remark that one could, of course, write down versions of our assertions (related to the second main theorem) stated in terms of suitable families of topological spaces $\{X_r : r \in \mathbb{R}\}$, and this procedure would provide another equivalences of **CH**.

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