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PROPERTY (gR) UNDER NILPOTENT COMMUTING PERTURBATION

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Abstract. The property (gR), introduced in [Aiena, P., Guillen, J. and Peña, P., *Property (gR) and perturbations*, to appear in Acta Sci. Math. (Szeged), 2012], is an extension to the context of B-Fredholm theory, of property (R), introduced in [Aiena, P., Guillen, J. and Peña, P., *Property (R) for bounded linear operators*, Mediterr. J. Math. **8** (4), 491-508, 2011]. In this paper we continue the study of property (gR) and we consider its preservation under perturbations by finite rank and nilpotent operators. We also prove that if T is left polaroid (resp. right polaroid) and N is a nilpotent operator which commutes with T then $T+N$ is also left polaroid (resp. right polaroid).

1. Introduction and preliminaries

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For $T \in L(X)$, we denote by $N(T)$ the null space of T and by $R(T) = T(X)$ the range of T. We denote by $\alpha(T) := \dim N(T)$ the nullity of T and by $\beta(T) := \text{codim } R(T) = \dim X/R(T)$ the defect of T. Other two classical quantities in operator theory are the *ascent* $p = p(T)$ of an operator T, defined as the smallest non-negative integer p such that $N(T^p) = N(T^{p+1})$ (if such an integer does not exist, we put $p(T) = \infty$), and the *descent* $q = q(T)$, defined as the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$ (if such an integer does not exist, we put $q(T) = \infty$). It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Furthermore, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ is a pole of the resolvent, see [14, Prop. 50.2]. An operator $T \in L(X)$ is said to be *Fredholm* (respectively, *upper semi-Fredholm*, *lower semi-Fredholm*), if $\alpha(T)$, $\beta(T)$ are both finite (respectively, $R(T)$ closed and $\alpha(T) < \infty$, $\beta(T) < \infty$). $T \in L(X)$ is said to be *semi-Fredholm* if T is either an upper semi-Fredholm or a lower semi-Fredholm operator. If T is semi-Fredholm, the *index* of T is defined by $\operatorname{ind} T := \alpha(T) - \beta(T)$. Other two important classes of operators in Fredholm theory are the classes of semi-Browder operators. These classes are defined as follows. $T \in L(X)$ is said to be *Browder*

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(resp. *upper semi-Browder*, *lower semi-Browder*) if T is a Fredholm (respectively, upper semi-Fredholm, lower semi-Fredholm) and both $p(T)$, $q(T)$ are finite (respectively, $p(T) < \infty$, $q(T) < \infty$). A bounded operator $T \in L(X)$ is said to be *upper semi-Weyl* (respectively, *lower semi-Weyl*) if T is upper Fredholm operator (respectively, lower semi-Fredholm) and index $\operatorname{ind} T \leq 0$ (respectively, $\operatorname{ind} T \geq 0$). $T \in L(X)$ is said to be *Weyl* if T is both upper and lower semi-Weyl, i.e. T is a Fredholm operator having index 0. The *Fredholm spectrum*, the *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{f}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\},$$
$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\},$$
$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}.$$

Since every Browder operator is Weyl then $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. Analogously, the *upper semi-Browder spectrum* and the *upper semi-Weyl spectrum* are defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$
$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\}.$$

A bounded operator $R \in L(X)$ is said to be Riesz if $\lambda I - T$ is a Fredholm operator for all $\lambda \neq 0$, i.e. $\sigma_f(T) \subseteq \{0\}$. The classical Riesz-Schauder theory of compact operators shows that every compact operator is Riesz. Also quasi-nilpotent operators (in particular nilpotent operators) are Riesz, since $\sigma_f(Q) \subseteq \sigma(Q) = \{0\}$ for any operator quasi-nilpotent $Q \in L(X)$. Browder spectra and Weyl spectra are invariant under commuting Riesz perturbations (see [15, 16]), i.e. if R is a Riesz operator such that $TR = RT$,

$$\sigma_{ub}(T) = \sigma_{ub}(T + R) \quad \text{and} \quad \sigma_{uw}(T) = \sigma_{uw}(T + R).$$

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum* defined by

$$\sigma_{\mathrm{ap}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

Note that if $\sigma_{\mathrm{s}}(T)$ denotes the *surjectivity spectrum*

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\}.$$

Obviously, $\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{s}}(T)$. Furthermore $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{s}}(T^*)$ and $\sigma_{\mathrm{s}}(T) = \sigma_{\mathrm{ap}}(T^*)$, where T^* is the dual of T.

Theorem 1.1. [1] *If $T \in L(X)$ and Q is a quasi-nilpotent operator commuting with T then*

(i) $\sigma(T) = \sigma(T + Q)$,

(ii) $\sigma_{ap}(T) = \sigma_{ap}(T + Q)$,

(iii) $\sigma_s(T) = \sigma_s(T + Q)$.

2. Semi B-Browder spectra under nilpotent perturbations

Given $n \in \mathbb{N}$, we denote by T_n the restriction of $T \in L(X)$ on the subspace $R(T^n) = T^n(X)$. According to [10, 11], T is said to be semi *B-Fredholm* (respectively, *B-Fredholm*, *upper semi B-Fredholm*, *lower semi B-Fredholm*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n, viewed as an operator from the space $R(T^n)$ into itself, is a semi-Fredholm operator (respectively, Fredholm, upper semi-Fredholm, lower semi-Fredholm). Analogously, $T \in L(X)$ is said to be *B-Browder* (respectively, *upper semi B-Browder*, *lower semi B-Browder*), if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n is a Browder operator (respectively, upper semi-Browder, lower semi-Browder). If T_n is a semi-Fredholm operator, it follows from [11, Proposition 2.1] that also T_m is semi-Fredholm for every $m \geq n$, and $\operatorname{ind} T_m = \operatorname{ind} T_n$. This enables us to define the *index* of a semi B-Fredholm operator T as the index of the semi-Fredholm operator T_n. Thus, a bounded operator $T \in L(X)$ is said to be a *B-Weyl operator* if T is a B-Fredholm operator having index 0. $T \in L(X)$ is said to be *upper semi B-Weyl* if T is upper semi B-Fredholm with index $\operatorname{ind} T \leq 0$, and T is said to be *lower semi B-Weyl* if T is lower semi B-Fredholm with $\operatorname{ind} T \geq 0$. Note that if T is B-Fredholm then also T^* is B-Fredholm with $\operatorname{ind} T^* = -\operatorname{ind} T$.

The classes of operators defined above motivate the definitions of several spectra. The *upper semi B-Browder spectrum* is defined by

$$\sigma_{\mathrm{ubb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Browder}\}.$$

The *lower semi B-Browder spectrum* is defined by

$$\sigma_{\mathrm{lbb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Browder}\},$$

while the *B-Browder spectrum* is defined by

$$\sigma_{\mathrm{bb}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Browder}\}.$$

Clearly, $\sigma_{\mathrm{bb}}(T) = \sigma_{\mathrm{ubb}}(T) \cup \sigma_{\mathrm{lbb}}(T)$. The *B-Weyl spectrum* is defined by

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\},$$

the *upper semi B-Weyl spectrum* and *lower semi B-Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{ubw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\},$$

and

$$\sigma_{\mathrm{lbw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not lower semi B-Weyl}\}.$$

Definition 2.1. $T \in L(X)$ is said to be left (resp. right) Drazin invertible if $p = p(T) < \infty$ (resp. $q = q(T) < \infty$) and $T^{p+1}(X)$ (resp. $T^q(X)$)is closed. $T \in L(X)$ is said to be Drazin invertible if $p(T) = q(T) < \infty$. If $\lambda I - T$ is left (resp. right) Drazin invertible and $\lambda \in \sigma_{ap}(T)$ (resp. $\lambda \in \sigma_s(T)$) then λ is said to be a left (resp. right) pole.

Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p = p(T) = q(T) < \infty$, then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$ [14, Prop. 50.2]. The left Drazin spectrum is then defined as

$$\sigma_{\rm ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\},$$

the right Drazin spectrum is defined as

$$\sigma_{\rm rd}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not right Drazin invertible}\}$$

and Drazin spectrum is defined as

$$\sigma_{\rm d}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\}.$$

Obviously, $\sigma_{\rm d}(T) = \sigma_{\rm ld}(T) \cup \sigma_{\rm rd}(T)$. Furthermore $\sigma_{\rm ld}(T) = \sigma_{\rm rd}(T^*)$ and $\sigma_{\rm rd}(T) = \sigma_{\rm ld}(T^*)$, where T^* is the dual of T, see Theorem 2.1 of [3].

Theorem 2.2. [13] *If $T \in L(X)$ then we have*

(i) *T is right Drazin invertible if and only if there exists a $k \in \mathbb{N}$ such that $T^k(X)$ is closed and T_k is onto. In this case $T^j(X)$ is closed and T_j is onto for all naturals $j \geq k$.*

(ii) *T is left Drazin invertible if and only if T is upper semi B-Browder.*

(iii) *T is right Drazin invertible if and only if T is lower semi B-Browder.*

(iv) *T is Drazin invertible if and only if T is B-Browder.*

Corollary 2.3. *If $T \in L(X)$ then we have*

$$\sigma_{ubb}(T) = \sigma_{ld}(T), \quad \sigma_{lbb}(T) = \sigma_{rd}(T) \quad \text{and} \quad \sigma_{bb}(T) = \sigma_d(T).$$

It has been observed in [9], that the B-Browder spectrum is invariant under commuting finite dimensional perturbation. In the next propositions we prove that all Drazin spectra are invariant under nilpotent commuting perturbations.

Theorem 2.4. *Let $T \in L(X)$ and N be a nilpotent operator which commutes with T. Then $\sigma_{rd}(T+N) = \sigma_{lbb}(T+N) = \sigma_{lbb}(T) = \sigma_{rd}(T)$.*

Proof. Suppose that $\lambda \notin \sigma_{lbb}(T)$. By part (iii) of Theorem 2.2, $\lambda I - T$ is right Drazin invertible and hence, $q = q(\lambda I - T) < \infty$ and $(\lambda I - T)^q(X)$ is closed. Let $n \in \mathbb{N}$ be such that $N^n = 0$ and set $m_1 = \max\{q, n\}$. We claim that

$$[(\lambda I - T) + N]^{2k}(X) \subseteq (\lambda I - T)^q(X) \quad \text{for all } k \geq m_1. \tag{1}$$

To show this, let $y \in [(\lambda I - T) + N]^{2k}(X)$ be arbitrary, so that there exists $x \in X$ for which $[(\lambda I - T) + N]^{2k}(x) = y$. Then

$$\begin{aligned} y &= \sum_{i=0}^{2k} \mu_{i,k} N^i((\lambda I - T)^{2k-i}(x)) \\ &= \sum_{i=0}^{k} \mu_{i,k} N^i((\lambda I - T)^{2k-i}(x)) + \sum_{i=k+1}^{2k} \mu_{i,k} N^i((\lambda I - T)^{2k-i}(x)) \\ &= \sum_{i=0}^{k} \mu_{i,k} N^i((\lambda I - T)^{2k-i}(x)) \\ &= (\lambda I - T)^k \Big[\sum_{i=0}^{k} \mu_{i,k} N^i((\lambda I - T)^{k-i}(x))\Big]. \end{aligned}$$

Therefore $y \in (\lambda I - T)^k(X)$. Hence, since $k \geq q$,

$$[(\lambda I - T) + N]^{2k}(X) \subseteq (\lambda I - T)^k(X) = (\lambda I - T)^q(X). \tag{2}$$

To prove the opposite inclusion, observe, by using (2), that it also follows that

$$\begin{aligned}(\lambda I - T)^q(X) &= (\lambda I - T)^{4k}(X) = [(\lambda I - T) + N - N]^{4k}(X)\\ &\subseteq [(\lambda I - T) + N]^{2k}(X),\end{aligned}$$

from which the equality (1) follows. Consequently, $[(\lambda I - T)]^{2k}(X)$ is closed for all k sufficiently large. Now, from part (i) of Theorem 2.2, we can choose k such that the restriction $(\lambda I - T)_{2k}$ of $(\lambda I - T)$ to $M = (\lambda I - T)^{2k}(X) = [(\lambda I - T) + N]^{2k}(X)$ is onto. If N_{2k} denotes the restriction of N to M, then $(\lambda I - T)_{2k} + N_{2k} = [(\lambda I - T) + N]_{2k}$ is onto, so, by Theorem 2.2, part (i), $(\lambda I - T) + N$ is right Drazin invertible, or equivalently, lower semi B-Browder. This shows that $\sigma_{lbb}(T) \subseteq \sigma_{lbb}(T + N)$ and by symmetry the opposite inclusion holds, so the equality $\sigma_{lbb}(T + N) = \sigma_{lbb}(T)$. ∎

By duality we have

Corollary 2.5. *Let $T \in L(X)$ and N be a nilpotent operator which commutes with T. Then $\sigma_{ld}(T + N) = \sigma_{ubb}(T + N) = \sigma_{ubb}(T) = \sigma_{ld}(T)$ and $\sigma_d(T + N) = \sigma_{bb}(T + N) = \sigma_{bb}(T) = \sigma_d(T)$.*

Remark 2.6. Theorem 2.4 and Corollary 2.5 answer positively to a question from [6], in particular it improves Theorem 4.3, where the invariance of the spectrum $\sigma_{lbb}(T)$, under commuting nilpotent perturbations, was proved assuming that T has SVEP, while the invariance of $\sigma_{ubb}(T)$ was proved assuming that T^* has SVEP.

3. Property (gR) under nilpotent perturbations

For an operator $T \in L(X)$ define

$$\begin{aligned}E(T) &= \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T)\},\\ E^a(T) &= \{\lambda \in \text{iso } \sigma_{ap}(T) : 0 < \alpha(\lambda I - T)\},\\ \Pi_{00}(T) &= \sigma(T) \setminus \sigma_{bb}(T),\\ \Pi^a_{00}(T) &= \sigma_{ap}(T) \setminus \sigma_{ubb}(T).\end{aligned}$$

Definition 3.1. A bounded $T \in L(X)$ is said to satisfy:

(i) property (gR) if $\sigma_{ap}(T) \setminus \sigma_{ubb}(T) = E(T)$;
(ii) property (gR^a) if $\sigma_{ap}(T) \setminus \sigma_{ubb}(T) = E^a(T)$;
(iii) property (gw) if $\sigma(T)_{ap} \setminus \sigma_{ubw}(T) = E(T)$;
(iv) generalized a-Weyl's theorem if $\sigma_{ap}(T) \setminus \sigma_{ubw}(T) = E^a(T)$.

Also a-Browder's theorem admits a generalized version, the generalized a-Browder's theorem, which means that T satisfies $\sigma_{ubw}(T) = \sigma_{ubb}(T)$. However, a-Browder's theorem and generalized a-Browder's theorem are equivalent, for a proof see [4].

THEOREM 3.2. [7] *If $T \in L(X)$, then we have*

(i) *T satisfies property (gw) if and only if a-Browder's theorem and property (gR) holds for T;*

(ii) *T satisfies generalized a-Weyl's theorem if and only if a-Browder's theorem and property (gR^a) holds for T.*

THEOREM 3.3. *Let $T \in L(X)$ and N be a nilpotent operator which commutes with T. Then $E(T) = E(T+N)$ and $E^a(T) = E^a(T+N)$.*

Proof. Suppose that $N^n = 0$. It is easily seen that

$$N(\lambda I - T) \subseteq N(\lambda I - T + N)^n. \tag{3}$$

Indeed, if $x \in N(\lambda I - T)$ then for some suitable binomial coefficients $\mu_{n,j}$, we have

$$(\lambda I - T + N)^n x = \sum_{j=1}^{n} \mu_{n,j} (\lambda I - T)^j N^{n-j} x = 0,$$

hence $x \in N(\lambda I - T + N)^n$.

Now, let $\lambda \in E(T)$. Then $\lambda \in \text{iso}\, \sigma(T) = \text{iso}\, \sigma(T+N)$ and $\alpha(\lambda I - T) > 0$. Suppose that $\alpha(\lambda I - T + N) = 0$. Then $\alpha(\lambda I - T + N)^k = 0$ for all $k \in \mathbb{N}$. From the inclusion (3), we have $\alpha(\lambda I - T) = 0$ and this is impossible. Therefore $\alpha(\lambda I - T + N) > 0$. Consequently, $E(T) \subseteq E(T+N)$ and, again by symmetry, the opposite inclusion holds. Therefore, $E(T) = E(T+N)$. Similarly we can prove that $E^a(T) = E^a(T+N)$. ∎

THEOREM 3.4. *Let $T \in L(X)$ and N be a nilpotent operator which commutes with T. Then T satisfies the property (gR) if only if $T + N$ satisfies the property (gR).*

Proof. By Theorem 3.3 and Theorem 2.4, it follows that

$$E(T+N) = E(T) = \sigma_{ap}(T) \setminus \sigma_{ubb}(T) = \sigma_{ap}(T+N) \setminus \sigma_{ubb}(T+N),$$

hence $T + N$ satisfies property (gR). By symmetry the reciprocal holds. ∎

THEOREM 3.5. *Let $T \in L(X)$ and N be a nilpotent operator which commutes with T. Then T satisfies the property (gR^a) if only if $T + N$ satisfies the property (gR^a).*

Proof. By Theorem 3.3 and Theorem 2.4, it follows that

$$E^a(T+N) = E^a(T) = \sigma_{ap}(T) \setminus \sigma_{ubb}(T) = \sigma_{ap}(T+N) \setminus \sigma_{ubb}(T+N),$$

hence $T + N$ satisfies property (gR^a). By symmetry the reciprocal holds. ∎

DEFINITION 3.6. $T \in L(X)$ is said to be left (resp. right) polaroid if $\sigma_{ap}(T)$ is empty or every isolated point of $\sigma_{ap}(T)$ is a left pole (resp. $\sigma_s(T)$ is empty or every isolated point of $\sigma_s(T)$ is a right pole).

THEOREM 3.7. *If $T \in L(X)$ is a left polaroid and N is a nilpotent operator commuting with T, then T is a left polaroid if only if $T + N$ is a left polaroid.*

Proof. Obviously, by Corollary 2.3, we have iso $\sigma_{ap}(T) = \sigma_{ap}(T) \setminus \sigma_{ubb}(T)$. Therefore,

$$\begin{aligned} \text{iso } \sigma_{ap}(T+N) &= \text{iso } \sigma_{ap}(T) \\ &= \sigma_{ap}(T) \setminus \sigma_{ubb}(T) \\ &= \sigma_{ap}(T+N) \setminus \sigma_{ubb}(T+N). \end{aligned}$$

Thus $T+N$ is left polaroid. By symmetry the reciprocal holds. ■

REMARK 3.8. The result of Theorem 3.9 improves Corollary 2.12 of [2], where it was proved that $T+N$ is a left polaroid assuming that T is a left polaroid and T^* has SVEP at the points $\lambda \notin \sigma_{uw}(T)$.

THEOREM 3.9. *If $T \in L(X)$ is a right polaroid and N is a nilpotent operator commuting with T, then T is a right polaroid if only if $T+N$ is a right polaroid.*

Proof. Obviously, by Corollary 2.3, we have iso $\sigma_s(T) = \sigma_s(T) \setminus \sigma_{lbb}(T)$. Therefore,

$$\begin{aligned} \text{iso } \sigma_s(T+N) &= \text{iso } \sigma_s(T) \\ &= \sigma_s(T) \setminus \sigma_{lbb}(T) \\ &= \sigma_s(T+N) \setminus \sigma_{lbb}(T+N). \end{aligned}$$

Thus $T+N$ is a right polaroid. By symmetry the reciprocal holds. ■

REMARK 3.10. The result of Theorem 3.9 improves Corollary 2.12 of [2], where it was proved that $T+N$ is a right polaroid assuming that T is a right polaroid and T has SVEP at the points $\lambda \notin \sigma_{uw}(T)$.

As in the above theorems, for the (gw) property introduced in [8], we have the following result.

THEOREM 3.11. *Let $T \in L(X)$ and N be a nilpotent operator which commutes with T. Then T satisfies the property (gw) if only if $T+N$ satisfies the property (gw).*

Proof. Suppose that T satisfies property (gw). Then T satisfies generalized a-Browder's theorem, or equivalently a-Browder's theorem, i.e. $\sigma_{ub}(T) = \sigma_{uw}(T)$. Since these spectra are invariant under N, we have that $T+N$ satisfies a-Browder's theorem. Then, from Theorems 3.4 and 3.2, it follows that $T+N$ satisfies property (gw). By symmetry the reciprocal holds. ■

As in the above theorems, for the generalized a-Weyl theorem introduced in [12], we have the following result.

THEOREM 3.12. *Let $T \in L(X)$ and N be a nilpotent operator which commutes with T. Then T satisfies the generalized a-Weyl Theorem if only if $T+N$ satisfies the generalized a-Weyl Theorem.*

Proof. Suppose that T satisfies generalized a-Weyl's theorem. Then since a-Browder's theorem and property (gR) are invariant under N, it follows from Theorem 3.2, that $T+N$ satisfies the generalized a-Weyl's theorem. By symmetry the reciprocal holds. ■

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