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ON $\mathcal{I}$ AND $\mathcal{I}^*$-EQUAL CONVERGENCE AND AN EGOROFF-TYPE THEOREM

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Abstract. In this paper we extend the notion of equal convergence of Császár and Laczkovich with the help of ideals of the set of positive integers and introduce the ideas of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence and prove certain properties. Throughout the investigation two classes of ideals, one satisfying "Chain Condition" and another called P-ideals play a very important role. We also introduce certain related notions of convergence and prove an Egoroff-type theorem for $\mathcal{I}^*$-equal convergence.

1. Introduction

We start by recalling the definition of asymptotic density as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of the set K_n. The asymptotic density of the subset K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$$

provided the limit exists.

Using this idea of asymptotic density, the idea of convergence of a real sequence had been extended to statistical convergence by Fast [22] (see also [33]) as follows: A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric space (X, ρ) is said to be statistically convergent to ℓ if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : d(x_k, \ell) \geq \varepsilon\}$ has asymptotic density zero. A lot of investigations have been done on this convergence and applications of these ideas in fields like Fourier Analysis, Measure Theory, Summability Theory, Functional Analysis etc. after the initial works by Fridy [23, 24] and Šalat [32] (for more reference see [2]).

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On the other hand, in [26] an interesting generalization of the notion of statistical convergence was proposed (though it was investigated before using filters in [25]). Namely it is easy to check that the family $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible (or free) ideal of $\mathbb{N}$ (recall that $\mathcal{I} \subset 2^N$ is called an ideal if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$. $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\emptyset\}$ and $\mathbb{N} \notin \mathcal{I}$. $\mathcal{I}$ is admissible (or free) if it contains all singletons. If $\mathcal{I}$ is a proper non-trivial ideal then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N}$: there exists $A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter in X. It is called the filter associated with the ideal). There have been several deep and impressive investigations on structures of ideals over the years (see for example [18, 20, 21, 34, 35, 36, 37]). Thus one may consider an arbitrary ideal $\mathcal{I}$ of $\mathbb{N}$ and define ideal ($\mathcal{I}$) convergence of a sequence by replacing the sets of density zero by the members of the ideal. For the last ten years a lot of work has been done on ideal convergence and applications of ideals in double sequences, nets, sequences of continuous functions etc. (see for example [2, 7, 8, 12, 13, 14, 15, 17, 27, 28] where many more references can be found).

In particular, in [2] certain types of statistical and ideal convergence notions were introduced for sequences of real measurable functions extending the well known ideas of pointwise and uniform convergence (see also [28]) and a statistical version of Egoroff theorem was presented which was subsequently extended to an ideal version very recently by Mrożek [29].

The interesting notion of equal convergence was introduced by Császár and Laczkovich in [10] for real functions (also known as quasi normal convergence [6]) which was shown to be between the notions of pointwise convergence and uniform convergence. A detailed investigation was carried out by Császár and Laczkovich in [10] and [11] on this convergence. The notion of equal convergence was later extended to uniform equal convergence by Papanastassiou in [30] (for more investigations in this line see also [19]) where he presented an Egoroff-type theorem for real valued measurable functions (for another version of Egoroff's theorem one can see [31]).

As a natural continuation of the above mentioned investigations, in this paper, we first unify the two approaches and extend the notion of equal convergence of Császár and Laczkovich with the help of ideals of the set of positive integers which actually produces two different ideas, namely, the ideas of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence and initiate certain investigations in line of [10]. We also concentrate on their inter-relationship and prove certain results where a class of ideals called P-ideals play a very important role. Finally we introduce the notion of $\mathcal{I}^*$-uniform equal convergence of sequences of real valued functions using ideals and prove an Egoroff type theorem for a sequence of real valued measurable functions following the line of [30]. Our result gives an $\mathcal{I}$-analogue of the main result of [30] which gives a more general version of that Egoroff-type theorem.

2. Preliminaries

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers and $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ [26]. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}^*$-convergent to $x \in \mathbb{R}$ if there is a set $M \in F(\mathcal{I}),\ M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $\lim_{k\to\infty} x_{m_k} = x$ [26].

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is called a P-ideal (or said to satisfy the condition (AP)) if for any sequence $\{A_1, A_2, \dots\}$ of mutually disjoint sets of $\mathcal{I}$ there is a sequence $\{B_1, B_2, \dots\}$ of sets such that $A_i \Delta B_i$ $(i = 1, 2, \dots)$ is finite and $B = \bigcup_{j\in\mathbb{N}} B_j \in \mathcal{I}$. Several examples of P-ideals can be seen from [26] and its importance in summability can be seen from [26], [13].

We also introduce the following definition which will be helpful in certain situations.

We say that $\mathcal{I}$ satisfies the "Chain Condition" (or CC in short) if there exists a sequence $\{C_k\}_{k\in\mathbb{N}} \subset \mathcal{I}$ with $C_1 \subset C_2 \subset C_3 \subset \cdots$, such that for any $A \in \mathcal{I}$ there exists $k \in \mathbb{N}$ such that $A \subset C_k$([20, 21]).

It should be noted that the "Chain Condition" is independent of the notion of P-ideal. The ideal of finite sets as well as the ideal defined in Theorem 3.5 are examples of ideals which have the "Chain Condition" but while the first ideal also is a P-ideal the second ideal is not a P-ideal. These ideals are also called countably generated ideals. Moreover, it is known (see e.g. Farah's book [21]) that there are only three pairwise nonisomorphic countably generated ideals.

We now recall the very important notion of equal convergence and another related notion of discrete convergence of sequences of real valued functions introduced in [10] which we intend to generalize using the notion of ideals. Let X be a nonempty set and let f_n, f be real valued functions defined on X. f is called the discrete limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if for every $x \in X$, there exists $n_0 = n_0(x)$ such that $f(x) = f_n(x)$ for $n \geq n_0$. The terminology is motivated by the fact that this condition means precisely the convergence of the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ to $f(x)$ with respect to the discrete topology of the real line. f is said to be the equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ ([10], it is called quasinormal limit in [6]) if there is a sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ tending to zero such that for every $x \in X$, there exists $n_0 = n_0(x)$ with $|f_n(x) - f(x)| < \varepsilon_n$ for $n \geq n_0$. $\{f_n\}_{n\in\mathbb{N}}$ is said to converge to f uniformly equally ([19], [30]) if there is a sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ tending to zero and a $M_1 \in \mathbb{N}$ such that for all $x \in X$, the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}$ can not exceed M_1.

We also recall that the ideas of pointwise convergence and uniform convergence of a sequence of real valued functions have already been extended through ideals in [2] which will be needed throughout the paper. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of real valued functions is said to be $\mathcal{I}$-pointwise convergent to f if for all $x \in X$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-convergent to $f(x)$ and in this case we write $f_n \xrightarrow{\mathcal{I}} f$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f if for any $\varepsilon > 0$ there exists $A \in \mathcal{I}$ such that for all $n \in A^c$ and for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$. In this case we write $f_n \xrightarrow{\mathcal{I}-u} f$.

3. $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence

We first present the notion of $\mathcal{I}$-equal convergence (which was introduced as $\mathcal{I}$-quasinormal convergence in [16]).

Definition 3.1. Let X be a non empty set and let f_n, f be real valued functions defined on X. We say that f is the $\mathcal{I}$-equal limit of the sequence $\{f_n\}_{n\in\mathbb{N}}$ if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\mathcal{I}\text{-}\lim_{n\to\infty}\varepsilon_n = 0$ such that for any $x \in X$, the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$. In this case we write $f_n \stackrel{\mathcal{I}-e}{\rightarrow} f$.

Below we observe that it is weaker than $\mathcal{I}$-uniform convergence which will also be needed in many results of this paper. Later we will give examples (Example 3.1) to show that the notion of $\mathcal{I}$-equal convergence is strictly stronger than the notion of $\mathcal{I}$-pointwise convergence and weaker than the notion of $\mathcal{I}$-uniform convergence.

Theorem 3.1. *$f_n \stackrel{\mathcal{I}-u}{\rightarrow} f$ implies $f_n \stackrel{\mathcal{I}-e}{\rightarrow} f$.*

Proof. We know that $f_n \stackrel{\mathcal{I}-u}{\rightarrow} f$ if and only if $\sup_{x\in X}|f_n(x) - f(x)| \stackrel{\mathcal{I}}{\rightarrow} 0$. Let $\varepsilon > 0$ be given. Then $A = \{n \in \mathbb{N} : \sup_{x\in X}|f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$.

Now define

$$\varepsilon_n = \begin{cases} \frac{1}{n} & \text{if } n \in A \\ \sup_{x\in X}|f_n(x) - f(x)| + \frac{1}{n} & \text{if } n \in A^c. \end{cases}$$

Then clearly $\varepsilon_n \stackrel{\mathcal{I}}{\rightarrow} 0$ and $|f_n(x) - f(x)| < \varepsilon_n$ for all $n \in A^c$ which implies $f_n \stackrel{\mathcal{I}-e}{\rightarrow} f$. ■

We will now recall some results from [16]. We give the results with complete proofs as we are using here a different name and for the easy reference for the readers the following equivalent characterization of $\mathcal{I}$-equal convergence (inspired by Theorem 5.1 [10] and Theorem 1.2 [6]) which will also be needed to establish Example 3.1.

Theorem 3.2. ([16]) *Let $\mathcal{I}$ be an ideal satisfying the Chain Condition. Let f, f_n, $n = 1, 2, \ldots$ be real valued functions defined on a set X. The following conditions are equivalent.*

(i) $f_n \stackrel{\mathcal{I}-e}{\rightarrow} f$ on X.

(ii) There are sets $X_k \subset X$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $f_n \stackrel{\mathcal{I}-u}{\rightarrow} f$ on X_k for every $k = 1, 2, \ldots$.

(iii) There are sets $X_k \subset X$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$, $X_1 \subset X_2 \subset \cdots$ and $f_n \stackrel{\mathcal{I}-u}{\rightarrow} f$ on X_k for every $k = 1, 2, \ldots$.

If X is a topological space and f_n, $n = 1, 2, \ldots$ are continuous, then (i), (ii) and (iii) are equivalent to

(iv) There are closed sets $X_k \subset X$, $k = 1, 2, \ldots$, $X = \bigcup_{k\in\mathbb{N}} X_k$, $X_1 \subset X_2 \subset \cdots$ and $f_n \stackrel{\mathcal{I}-u}{\rightarrow} f$ on X_k for every $k = 1, 2, \ldots$.

Proof. $(i) \Rightarrow (iii)$. Assume (i), i.e. $f_n \stackrel{\mathcal{I}-e}{\to} f$. Then there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\mathcal{I}\text{-}\lim_{n\to\infty}\varepsilon_n = 0$ and for every $x \in X$ there is a set $A_x \in \mathcal{I}$ such that $|f_n(x) - f(x)| < \varepsilon_n$ for all $n \in A_x^c$. Since $\mathcal{I}$ satisfies the Chain Condition, there exists a sequence $\{C_k\}_{k\in\mathbb{N}}$ in $\mathcal{I}$ with $C_1 \subset C_2 \subset \cdots$ such that for every $A \in \mathcal{I}$ there exists some $C_k \in \mathcal{I}$ with $A \subset C_k$. Now define $X_k = \{x \in X : |f_n(x) - f(x)| < \varepsilon_n \text{ for all } n \in C_k^c\}$, $k \in \mathbb{N}$. Then clearly $X_1 \subset X_2 \subset \cdots$. Further observe that for any $x \in X$, if $A_x \in \mathcal{I}$ is the set witnessing $\mathcal{I}$-equal convergence as defined above, then $A_x \subset C_k$ for some $k \in \mathbb{N}$. Consequently $x \in X_k$. Hence $X = \bigcup_{k\in\mathbb{N}} X_k$. It is now easy to observe that $f_n \stackrel{\mathcal{I}-u}{\to} f$ on X_k. This proves (iii).

$(ii) \Rightarrow (i)$. Now assume (ii), i.e. suppose that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $|f_n(x) - f(x)| \le \varepsilon_{in}$ for all $x \in X_i$ when $n \notin M(i) \in \mathcal{I}$, where $\mathcal{I}\text{-}\lim_{n\to\infty}\varepsilon_{in} = 0$ for a fixed i. We can select sets $M_k \in \mathcal{I}$ such that $M_1 \subset M_2 \subset \cdots \subset M_k \subset \cdots$ and $\varepsilon_{kn} < \frac{1}{k}$ whenever $n \notin M_k$, for $k = 1, 2, \ldots$. Define

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \in M_2 \\ \frac{1}{k} & \text{if } n \in M_{k+1} \setminus M_k \\ \frac{1}{n} & \text{if } n \notin \bigcup_{k\in\mathbb{N}} M_k. \end{cases}$$

Then $\mathcal{I}\text{-}\lim_{n\to\infty}\varepsilon_n = 0$ and furthermore $|f_n(x) - f(x)| \le \varepsilon_{in} < \varepsilon_n$ for $x \in X_i$ and if $n \notin M(i) \cup M_i \in \mathcal{I}$ which shows that $f_n \stackrel{\mathcal{I}-e}{\to} f$. So (i) follows. Since $(iii) \Rightarrow (ii)$, so it now follows that (i), (ii), (iii) are equivalent.

Now let X be a topological space and f_n, $n = 1, 2, \ldots$ be continuous. Evidently (iv) implies (iii). Assume (i). Let us define $X_k = \{x \in X : |f_n(x) - f_m(x)| \le \varepsilon_n + \varepsilon_m \text{ for all } m, n \in C_k^c\}$, $k \in \mathbb{N}$. Suppose as before $\mathcal{I}$ satisfies the Chain Condition with the sequence $\{C_k\}_{k\in\mathbb{N}}$ in $\mathcal{I}$. Clearly X_k is closed for $k = 1, 2, \ldots$ as f_n's are continuous functions and $X_1 \subset X_2 \subset X_3 \subset \cdots$. If $x \in X$ then from the proof of $(i) \Rightarrow (iii)$, it readily follows that $x \in X_k$ for some $k \in \mathbb{N}$ and $f_n \stackrel{\mathcal{I}-u}{\to} f$ on each X_k. So (iv) is proved. Hence (i), (ii), and (iii) are equivalent to (iv). ∎

Remark 3.1. It should be noted that for the implication $(ii) \Rightarrow (i)$, it is not necessary to assume that the ideal $\mathcal{I}$ satisfies the Chain Condition. It is not clear whether the implication $(i) \Rightarrow (ii)$ holds for every ideal and we leave it as an open problem.

We are now in a position to give an example which establishes the fact that $\mathcal{I}$-equal convergence is stronger than $\mathcal{I}$-pointwise convergence.

Example 3.1. ([16]) This example shows that there exist functions f and f_n, $n = 1, 2, \ldots$ such that $f_n \stackrel{\mathcal{I}}{\to} f$ but $f_n \stackrel{\mathcal{I}-e}{\nrightarrow} f$. Let $\mathcal{I}$ ($\mathcal{I} \ne \mathcal{I}_{fin}$) be an admissible ideal satisfying the Chain Condition. Let C be an infinite member of $\mathcal{I}$. Let $\mathbb{Q} = \{r_k : k \in \mathbb{N} \cup \{0\}\}$ be a one to one enumeration of rational numbers. Let

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 2^{-k} & \text{if } x = r_k,\ k = 0, 1, 2, \ldots. \end{cases}$$

Clearly, f is not continuous on any interval. For every $n \in C^c$ (where c stands for the complement) choose a positive real $\delta_n \le 2^{-n}$ such that $\delta_n \le \frac{1}{2}|r_i - r_j|$,

$i = 0, 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, n$, $i \neq j$. Let

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \bigcup_{i=0}^{n} (r_i - \delta_i, r_i + \delta_i) \\ 2^{-i} & \text{for } x = r_i \ , i = 0, 1, 2, \ldots, n \\ 2^{-i}\left(1 - \frac{|x - r_i|}{\delta_i}\right) & \text{for } x \in (r_i - \delta_i, r_i + \delta_i) \ , i = 0, 1, 2, \ldots, n \end{cases}$$

for $n \in C^c$ and $f_n = n$ for each $n \in C$.

Clearly $f_n \xrightarrow{\mathcal{I}} f$ (though f_n does not converge to f pointwise) on $\mathbb{R}$. But $f_n \overset{\mathcal{I}-e}{\nrightarrow} f$ on $\mathbb{R}$, for otherwise if $f_n \xrightarrow{\mathcal{I}-e} f$ on $\mathbb{R}$ then by Theorem 3.2, $\mathbb{R} = \bigcup_{k=0}^{\infty} E_k$ where E_k's are closed and $f_n \xrightarrow{\mathcal{I}-u} f$ on every E_k for $k = 0, 1, 2 \ldots$. By the Baire category theorem, there is k such that $Int\ E_k \neq \emptyset$, i.e. there are $a < b$ such that $[a, b] \subseteq E_k$. Since each f_n is continuous and $f_n \xrightarrow{\mathcal{I}-u} f$ on $[a, b]$, it follows that f being the $\mathcal{I}$-uniform limit of continuous functions on $[a, b]$ is continuous on $[a, b]$ (see [2]), which is a contradiction. We do not know whether an example can be constructed corresponding to any arbitrary ideal $\mathcal{I}$ and leave it as an open problem.

When Kostyrko et al. [26] extended the notion of usual and statistical convergence using ideals, they observed that it can be done in two ways and so introduced the notions of $\mathcal{I}$ and $\mathcal{I}^*$-convergence of sequences. One of the most interesting problem was to investigate the relation between these two concepts. Subsequently such investigations have been carried out in many contexts (see for example [12], [13]). In this section we intend to proceed with similar investigations in respect of $\mathcal{I}$ and $\mathcal{I}^*$-equal convergence.

Definition 3.2. f is said to be the $\mathcal{I}^*$-pointwise limit of $\{f_n\}_{n\in\mathbb{N}}$ if for each $x \in X$, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in F(\mathcal{I})$ such that $f(x)$ is the pointwise limit of the subsequence $\{f_{m_k}(x)\}_{k\in\mathbb{N}}$. In this case we write $f_n \xrightarrow{\mathcal{I}^*} f$.

Definition 3.3. f is said to be the $\mathcal{I}^*$-uniform limit of $\{f_n\}_{n\in\mathbb{N}}$ if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in F(\mathcal{I})$ such that f is the uniform limit of the subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-u} f$.

Definition 3.4. f is said to be the $\mathcal{I}^*$-equal limit of $\{f_n\}_{n\in\mathbb{N}}$ if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in F(\mathcal{I})$ such that f is the equal limit of the subsequence $\{f_{m_k}\}_{k\in\mathbb{N}}$. In this case we write $f_n \xrightarrow{\mathcal{I}^*-e} f$.

Theorem 3.3. *Let $\mathcal{I}$ be a P-ideal. Let f, f_n, $n = 1, 2, \ldots$ be real valued functions defined on a set X. The following conditions are equivalent.*

(i) $f_n \xrightarrow{\mathcal{I}^*-e} f$ on X.

(ii) There are sets $X_k \subset X$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$ and $f_n \xrightarrow{\mathcal{I}^*-u} f$ on X_k for every $k = 1, 2, \ldots$.

(iii) There are sets $X_k \subset X$ such that $X = \bigcup_{k\in\mathbb{N}} X_k$, $X_1 \subset X_2 \subset \cdots$ and $f_n \xrightarrow{\mathcal{I}^*-u} f$ on X_k for every $k = 1, 2, \ldots$.

If X is a topological space and f_n, $n = 1, 2, \dots$ are continuous, then (i), (ii), (iii) *are equivalent to*

(iv) *There are closed sets $X_k \subset X$, $k = 1, 2, \dots$, $X = \bigcup_{k\in\mathbb{N}} X_k$, $X_1 \subset X_2 \subset \cdots$ and $f_n \overset{\mathcal{I}^*-u}{\to} f$ on X_k for every $k = 1, 2, \dots$.*

Proof. $(i) \Rightarrow (iii)$. Assume (i), i.e. $f_n \overset{\mathcal{I}^*-e}{\to} f$. Then there is a set $M = \{p_1 < p_2 < p_3 < \cdots\} \in F(\mathcal{I})$ such that for all $x \in X$, $f(x)$ is the equal limit of the sequence $\{f_{p_n}\}_{n\in\mathbb{N}}$. Hence there exists a sequence $\{\varepsilon_{p_n}\}_{n\in\mathbb{N}}$ of positive real numbers with $\lim_{n\to\infty}\varepsilon_{p_n} = 0$ and for every $x \in X$ there is a number $k > 0$ such that $|f_{p_n}(x) - f(x)| < \varepsilon_{p_n}$ for all $n \geq k$. Now define $X_k = \{x \in X : |f_{p_n}(x) - f(x)| < \varepsilon_{p_n}$ for all $n \geq k\}$, $k \in \mathbb{N}$. Then clearly $X_1 \subset X_2 \subset X_3 \subset \cdots$. Further observe that for any $x \in X$, $x \in X_k$ for some $k \in \mathbb{N}$. Hence $X = \bigcup_{k\in\mathbb{N}} X_k$. It is now easy to observe that $f_n \overset{\mathcal{I}^*-u}{\to} f$ on X_k for every k. This proves (iii).

Evidently, $(iii) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$. Let $X_k \subset X$, $X = \bigcup_{k\in\mathbb{N}} X_k$ and $f_n \overset{\mathcal{I}^*-u}{\to} f$ on X_k for every $k = 1, 2, \dots$ Then $|f_{p^i_n}(x) - f(x)| \leq \varepsilon_{p^i_n}$ for all $x \in X_i$ when $n \geq k(i)$ with $\lim_{n\to\infty}\varepsilon_{p^i_n} = 0$ for a fixed i and $\{p^i_n\}_{n\in\mathbb{N}} = M_i \in F(\mathcal{I})$. Now since $\mathcal{I}$ is a P-ideal then there exists a set $M_0 \in F(\mathcal{I})$ such that $M_0 \setminus M_i$ is finite for all i and a sequence $\{\varepsilon_{p_n}\}_{n\in\mathbb{N}}$ of positive real numbers with $\lim_{n\to\infty}\varepsilon_{p_n} = 0$ such that for every $x \in X$, $|f_{p_n}(x) - f(x)| < \varepsilon_{p_n}$ for all $p_n \in M_0 = \{p_1 < p_2 < p_3 < \cdots\}$ except for finite indices. Hence $f_n \overset{\mathcal{I}^*-e}{\to} f$. So (i) follows. So it now follows that (i), (ii), (iii) are equivalent.

Evidently $(iv) \Rightarrow (iii)$.

$(i) \Rightarrow (iv)$. Now let X be a topological space and f_n, $n = 1, 2, \dots$ be continuous. Assume (i). Let us define $X_k = \{x \in X : |f_{p_n}(x) - f_{p_m}(x)| \leq \varepsilon_{p_n} + \varepsilon_{p_m}$ for all $m, n \geq k\}$, $k \in \mathbb{N}$. Clearly X_k is closed for $k = 1, 2, \dots$ as f_n's are continuous functions and $X_1 \subset X_2 \subset X_3 \subset \cdots$. If $x \in X$ then from the proof of $(i) \Rightarrow (iii)$, it readily follows that $x \in X_k$ for some $k \in \mathbb{N}$ and $f_n \overset{\mathcal{I}^*-u}{\to} f$ on each X_k. So (iv) is proved. Hence (i), (ii) and (iii) are equivalent to (iv). ∎

Remark 3.2. Note that the implications $(i) \Rightarrow (iii)$, $(iii) \Rightarrow (ii)$, $(iv) \Rightarrow (iii)$, $(i) \Rightarrow (iv)$ are true for any arbitrary ideal. Comparing Theorem 3.2 and Theorem 3.3 it is interesting to observe that in order to prove similar results for $\mathcal{I}$-equal convergence and $\mathcal{I}^*$-equal convergence two different assumptions on the ideal are required. We do not know whether they can be proved under same assumption or can actually be proved without any assumption which we leave as an open problem.

Example 3.2. Let $\mathcal{I}$ $(\mathcal{I} \neq \mathcal{I}_{fin})$ be an admissible ideal and $\mathbb{Q} = \{r_k : k \in \mathbb{N} \cup \{0\}\}$ be a one to one enumeration of rational numbers. Taking the function $f(x)$, C the same as in Example 3.1 and defining the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ as follows:

$f_n = 0$ for each $n \in C$ and for $n \in C^c$

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \bigcup_{i=0}^{n}(r_i - \delta_i, r_i + \delta_i) \\ 2^{-i} & \text{for } x = r_i,\ i = 0,1,2,\dots,n \\ 2^{-i}\left(1 - \frac{|x-r_i|}{\delta_i}\right) & \text{for } x \in (r_i - \delta_i, r_i + \delta_i),\ i = 0,1,2,\dots,n, \end{cases}$$

it can be shown that $f_n \xrightarrow{\mathcal{I}^*} f$ but $f_n \overset{\mathcal{I}^*-e}{\nrightarrow} f$.

Theorem 3.4. *Let $\mathcal{I}$ be an admissible ideal. If $f_n \xrightarrow{\mathcal{I}^*-e} f$ then $f_n \xrightarrow{\mathcal{I}-e} f$.*

Proof. The proof is straightforward and so is omitted. ■

However the converse is not generally true as shown by the following Theorem.

Theorem 3.5. *There exist an admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ and a sequence $\{g_n\}_{n\in\mathbb{N}}$ such that $g_n \xrightarrow{\mathcal{I}-e} f$ but $g_n \overset{\mathcal{I}^*-e}{\nrightarrow} f$.*

Proof. Consider a function f and a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined on X such that $f_n \xrightarrow{u} f$ and $f_n \neq f$ for any $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then there exists an $m \in \mathbb{N}$ such that for all $x \in X$, $|f_n(x) - f(x)| < \varepsilon$ for all $n > m$. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} \Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. Denote by $\mathcal{I}$ the class of all $A \subset \mathbb{N}$ that intersect only a finite number of Δ_j's. Then $\mathcal{I}$ is a non-trivial admissible ideal. Define a sequence $\{g_n\}_{n\in\mathbb{N}}$ by

$$g_n = f_j \quad \text{if } n \in \Delta_j.$$

Then for all $x \in X$, the set $\{n \in \mathbb{N} : |g_n(x) - f(x)| \geq \varepsilon\} \subset \Delta_1 \cup \Delta_2 \cup \dots \cup \Delta_m \in \mathcal{I}$ which shows that $g_n \xrightarrow{\mathcal{I}-u} f$. Hence $g_n \xrightarrow{\mathcal{I}-e} f$ (by Theorem 3.1).

Suppose now that $g_n \xrightarrow{\mathcal{I}^*-e} f$. Proceeding as in Theorem 3.1 (ii) [26] we can arrive at a contradiction. ■

Theorem 3.6. *Let X be a countable set and $\mathcal{I}$ be a P-ideal (i.e. it satisfies the condition (AP)). Then $f_n \xrightarrow{\mathcal{I}-e} f$ implies $f_n \xrightarrow{\mathcal{I}^*-e} f$.*

Proof. Since $f_n \xrightarrow{\mathcal{I}-e} f$, there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ such that for each $x \in X$ there exists $M(x) \in F(\mathcal{I})$ and for all $n \in M(x)$, $|f_n(x) - f(x)| < \varepsilon_n$. Since $\mathcal{I}$ is a P-ideal, $\varepsilon_n \xrightarrow{\mathcal{I}} 0$ implies $\varepsilon_n \xrightarrow{\mathcal{I}^*} 0$. Hence there exists a set $A \in F(\mathcal{I})$ such that $\{\varepsilon_n\}_{n\in A} \to 0$. Since X is countable we can write $X = \{x_1, x_2, x_3, \dots\}$. Now from hypothesis, for every x_i there exists a set $M_i = M(x_i) \in F(\mathcal{I})$ such that $|f_n(x_i) - f(x_i)| < \varepsilon_n$ for all $n \in M_i$. Since $\mathcal{I}$ is a P-ideal, there exists a set $M_0 \in F(\mathcal{I})$ such that $M_0 \setminus M_i$ is finite for all i. Therefore $|f_n(x) - f(x)| < \varepsilon_n$ for all $n \in M_0 \cap A$ except for finite indices. Hence $f_n \xrightarrow{\mathcal{I}^*-e} f$. ■

Open Problem. It is not clear whether the result remains true when X is uncountable and we leave it as an open problem.

Theorem 3.7. *If $\mathcal{I}$-equal and $\mathcal{I}^*$-equal convergence coincide then $\mathcal{I}$ is a P-ideal.*

Proof. Consider a function f and a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined on X such that $f_n \xrightarrow{u} f$ and $f_n \neq f$ for any $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Then there exists a $m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > m$ and $\forall x \in X$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a disjoint family of non-empty sets from $\mathcal{I}$. Define a sequence $\{g_n\}_{n\in\mathbb{N}}$ by

$$g_n = \begin{cases} f_j & \text{if } n \in A_j \\ f & \text{if } n \notin A_j \text{ for any } j \in \mathbb{N}. \end{cases}$$

Then for all $x \in X$, observe that for $\varepsilon > 0$ given, $A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_m \in \mathcal{I}$ and for all $n \in A^c$ we have $|g_n(x) - f(x)| < \varepsilon$. Hence $g_n \xrightarrow{\mathcal{I}-u} f$ which implies $g_n \xrightarrow{\mathcal{I}-e} f$ (by Theorem 3.1). Consequently by our assumption $g_n \xrightarrow{\mathcal{I}^*-e} f$. So there exists a set $B \in \mathcal{I}$ such that $M = \mathbb{N} \setminus B = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ and $g_{m_k} \xrightarrow{e} f$. Now proceeding as in Theorem 3.2 (ii) [26] we can derive the result. ∎

Let Φ be an arbitrary class of real valued functions defined on X. We denote by $\Phi^{\mathcal{I}-e}$ the class of all functions defined on X which are $\mathcal{I}$-equal limits of sequences of functions belonging to Φ. Also for any function class Φ on X we recall the following definitions from [11].

Definition 3.5. (a) Φ is called a lattice if Φ contains all constants and $f, g \in \Phi$ implies $\max(f, g) \in \Phi$ and $\min(f, g) \in \Phi$.

(b) Φ is called a translation lattice if it is a lattice and $f \in \Phi, c \in \mathbb{R}$ implies $f + c \in \Phi$.

(c) Φ is called a congruence lattice if it is a translation lattice and $f \in \Phi$ implies $-f \in \Phi$.

(d) Φ is called a subtractive lattice if it is a lattice and $f, g \in \Phi$ implies $f - g \in \Phi$.

(e) Φ is called an ordinary class if it is a subtractive lattice, $f, g \in \Phi$ implies $f \cdot g \in \Phi$ and $f \in \Phi$, $f(x) \neq 0$, for all $x \in X$ implies $1/f \in \Phi$.

Theorem 3.8. *If Φ is an ordinary class then $\Phi^{\mathcal{I}-e}$ is also so.*

Proof. The proof is patterned after Proposition 3 [10]. It is obvious that $\Phi^{\mathcal{I}-e}$ is a subtractive lattice. Suppose $f, g \in \Phi^{\mathcal{I}-e}$. Then there exist two sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ in Φ and $A_x \in \mathcal{I}$ such that $|f_n(x) - f(x)| < \frac{1}{n^2}$ and $|g_n(x) - g(x)| < \frac{1}{n^2}$ for every $x \in X$ and for all $n \in A_x^c$. Now $|f_n(x)g_n(x) - f(x)g(x)| \leq |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \leq \frac{1}{n^2}(|g(x)| + 1) + \frac{1}{n^2}(|f(x)|) < \frac{1}{n}$ for $n \notin A_x \cup \{1, 2, 3, \ldots, n_0\}$ where $n_0 = \max\{2[|g(x)| + 1], 2[|f(x)|]\}$. Therefore $f \cdot g$ is the $\mathcal{I}$-equal limit of $\{f_n \cdot g_n\}_{n\in\mathbb{N}}$. Hence $f \cdot g \in \Phi^{\mathcal{I}-e}$. Let $f \in \Phi^{\mathcal{I}-e}$, $f(x) \neq 0$ for all $x \in X$. Then $f^2 \in \Phi^{\mathcal{I}-e}$. Therefore there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \Phi$ and $A_x \in \mathcal{I}$ such that for all $n \in A_x^c$, $|f_n(x) - f^2(x)| < \frac{1}{n^3}$. If $g_n = \max\{f_n, \frac{1}{n}\}$, then $g_n \in \Phi$ and $g_n \geq \frac{1}{n}$, also $|g_n(x) - f^2(x)| < \frac{1}{n^3}$ when $n \notin A_x \cup \{1, 2, 3, \ldots, n_0\}$ where $n_0 = 2[f(x)^{-2}] + 1$. Thus for $h_n = {g_n}^{-1}$ we have $h_n \in \Phi$ and $|h_n(x) - f(x)^{-2}| \leq |g_n(x) - f(x)^2||g_n(x)^{-1}||f(x)^{-2}| < \frac{1}{n^3} \cdot n \cdot n = \frac{1}{n}$ for all $n \notin A_x \cup \{1, 2, 3, \ldots, n_0\}$. Therefore $f^{-2} \in \Phi^{\mathcal{I}-e}$. Hence $f^{-1} = f \cdot f^{-2} \in \Phi^{\mathcal{I}-e}$. Hence $\Phi^{\mathcal{I}-e}$ is an ordinary class. ∎

4. On an Egoroff-type theorem for $\mathcal{I}^*$-equal convergence

We first introduce the following definition.

DEFINITION 4.1. $\{f_n\}_{n\in\mathbb{N}}$ is said to converge to f $\mathcal{I}^*$-uniformly equally if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$, $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ and $k(\{\varepsilon_n\}) \in \mathbb{N}$ such that $|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}|$ is at most $k = k(\{\varepsilon_n\})$ for all $x \in X$.

Clearly, $\mathcal{I}^*$-equal convergence is weaker than $\mathcal{I}^*$-uniform equal convergence which is again weaker than $\mathcal{I}^*$-uniform convergence.

EXAMPLE 4.1 Let $A \in \mathcal{I}$. Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of functions on $\mathbb{R}$ defined by: f_n is the characteristic function of $[n, n+\frac{1}{n}]$ for all $n \in \mathbb{N} \setminus A$, f_n is the constant function 1 on $\mathbb{R}$ for all $n \in A$. Now clearly $\sup_{x\in\mathbb{R}}|f_n(x)| = 1$ for all n and so $\{f_n\}_{n\in\mathbb{N}}$ cannot converge to the constant function $f \equiv 0$, $\mathcal{I}^*$-uniformly. But since for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\lim_n \varepsilon_n = 0$, $\{n \in \mathbb{N} \setminus A : f_n(x) \geq \varepsilon_n\}$ has cardinality at most 1 for all $x \in \mathbb{R}$, so $\{f_n\}_{n\in\mathbb{N}}$ obviously converges to $f \equiv 0$ $\mathcal{I}^*$-uniformly equally. Note that if A is infinite (i.e. for all ideals containing $\mathcal{I}_{fin}$ properly) then $\{f_n\}_{n\in\mathbb{N}}$ does not converge to $f \equiv 0$ uniformly equally and so equally.

From now on (X, S, μ) will stand for a measure space and $\{f_n\}_{n\in\mathbb{N}}$, f are always real valued measurable functions on X.

DEFINITION 4.2. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable functions on X is said to converge to a measurable function f $\mathcal{I}^*$-almost uniformly equally if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$, $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ such that for every $\delta > 0$, there exists a $A_\delta \in S$ with $\mu(A_\delta) < \delta$ and a $k = k(\{\varepsilon_n\}, \delta) \in \mathbb{N}$ such that $|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}|$ is at most k for all $x \in X \setminus A_\delta$.

For measurable functions $\{f_n\}_{n\in\mathbb{N}}$, f on X and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers, modifying similar notions from [30], we define

$$A_x^f(\{\varepsilon_n\})_M = \{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\},\ M \subset \mathbb{N},\ x \in X,$$
$$B_k^f(\{\varepsilon_n\})_M = \{x \in X : |A_x^f(\{\varepsilon_n\})_M| > k\},\ k = 1, 2, \dots,\ M \subset \mathbb{N}.$$

Clearly $\{B_k^f(\{\varepsilon_n\})_M\}_{k\in\mathbb{N}}$ is a decreasing sequence of measurable sets for a fixed $M \subset \mathbb{N}$. Also if $M_1 \subset M_2$ then

$$A_x{}^f(\{\varepsilon_n\})_{M_1} \subset A_x{}^f(\{\varepsilon_n\})_{M_2},\ B_k{}^f(\{\varepsilon_n\})_{M_1} \subset B_k{}^f(\{\varepsilon_n\})_{M_2}$$

for all $x \in X$, $k = 1, 2, \dots$.

DEFINITION 4.3. $\{f_n\}_{n\in\mathbb{N}}$ is said to satisfy the $\mathcal{I}^*$-quasi vanishing restriction with respect to f if there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$ and $M \in F(\mathcal{I})$ such that $\lim_k \mu(B_k^f(\{\varepsilon_n\})_M) = 0$.

We are now in a position to prove the following Egoroff-type theorem.

THEOREM 4.1. (cf. Theorem 2.7 [30]) *Let $\{f_n\}_{n\in\mathbb{N}}$, f be all measurable. The following are equivalent:*

(i) $\{f_n\}_{n\in\mathbb{N}}$ converges to f $\mathcal{I}^*$-almost uniformly equally.

(ii) $\{f_n\}_{n\in\mathbb{N}}$ converges to f $\mathcal{I}^*$-equally almost everywhere and there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \gamma_n = 0$ and $M \in F(\mathcal{I})$, $m_0 \in \mathbb{N}$ such that $\mu(B^f_{m_0}(\{\gamma_n\})_M) < \infty$.

(iii) $\{f_n\}_{n\in\mathbb{N}}$ satisfies $\mathcal{I}^*$-quasi vanishing restriction with respect to f.

Proof. $(i) \Rightarrow (ii)$. First suppose that $\{f_n\}_{n\in\mathbb{N}}$ converges $\mathcal{I}^*$-almost uniformly equally to f. Then there exists a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive reals with $\lim_n \varepsilon_n = 0$ and $M = M(\{\varepsilon_n\}) \in F(\mathcal{I})$ such that for every $\delta > 0$, there exists a $A_\delta \in S$ with $\mu(A_\delta) < \delta$ and a $k = k(\{\varepsilon_n\}, \delta) \in \mathbb{N}$ such that

$$|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq k(\{\varepsilon_n\}, \delta) \text{ for all } x \in X \setminus A_\delta.$$

Hence $B^f_{k(\{\varepsilon_n\},\delta)}(\{\varepsilon_n\})_M \subset A_\delta$ and so $\mu(B^f_{k(\{\varepsilon_n\},\delta)}(\{\varepsilon_n\})_M) < \delta$ which proves the second part of (ii). Now proceeding as in usual measure theory one can show that $\{f_n\}_{n\in\mathbb{N}}$ converges to f $\mathcal{I}^*$-equally almost everywhere.

$(ii) \Rightarrow (iii)$. Suppose that the conditions in (ii) hold. Since $\{f_n\}_{n\in\mathbb{N}}$ converges to f $\mathcal{I}^*$-equally almost everywhere so there is a set $G \in S$ with $\mu(G) = 0$ such that we can find a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\lim_n \varepsilon_n = 0$ and a set $M_1(\{\varepsilon_n\}) \in F(\mathcal{I})$ and for each $x \in X \setminus G$, there exists a finite set $P_x \subset M_1$ such that $|f_n(x) - f(x)| < \varepsilon_n$ for all $n \in M_1 \setminus P_x$.

Set $\lambda_n = \max\{\varepsilon_n,\ \gamma_n\}$. Then $0 \leq \varepsilon_n,\ \gamma_n \leq \lambda_n$ and $\lim_n \lambda_n = 0$. Now for $x \in X \setminus G$,

$$|f_n(x) - f(x)| < \lambda_n \text{ for all } n \in M_1 \setminus P_x. \tag{1}$$

Since $\lambda_n \geq \gamma_n \geq 0$ for every $n \in \mathbb{N}$, so we have

$$\{n \in M : |f_n(x) - f(x)| \geq \lambda_n\} \subset \{n \in M : |f_n(x) - f(x)| \geq \gamma_n\}$$

which implies that $B^f_k(\{\lambda_n\})_M \subset B^f_k(\{\gamma_n\})_M$ for all $k = 1, 2, \ldots$, where $M \in F(\mathcal{I})$ is the set coming from our assumption. Further by our assumption there is a $m_0 \in \mathbb{N}$ for which $\mu(B^f_{m_0}(\{\gamma_n\})_M) < \infty$ and so $\mu(B^f_{m_0}(\{\lambda_n\})_M) < \infty$.

Let $M_0 = M \cap M_1$. Then $M_0 \in F(\mathcal{I})$. Also since $B^f_{m_0}(\{\lambda_n\})_{M_0} \subset B^f_{m_0}(\{\lambda_n\})_M$ so we have $\mu(B^f_{m_0}(\{\lambda_n\})_{M_0}) < \infty$. Clearly (1) implies that $|f_n(x) - f(x)| < \lambda_n$ for all $n \in M_0 \setminus P_x$ and so if $|P_x| = l(x)$ then it follows that $x \notin B^f_{l(x)}(\{\lambda_n\})_{M_0}$ and so $x \notin \bigcap_{k=1}^{\infty} B^f_k(\{\lambda_n\})_{M_0}$. Since this is true for each $x \in X \setminus G$, so $\bigcap_{k=1}^{\infty} B^f_k(\{\lambda_n\})_{M_0} \subset G$ and therefore $\mu(\bigcap_{k=1}^{\infty} B^f_k(\{\lambda_n\})_{M_0}) = 0$.

But as $\{B^f_k(\{\lambda_n\})_{M_0}\}_{k\in\mathbb{N}}$ is a decreasing sequence of measurable sets and we have $\mu(B^f_{m_0}(\{\lambda_n\})_{M_0}) < \infty$ so $\lim_k\ (\mu(B^f_k(\{\lambda_n\})_{M_0}) = 0$.

This shows that $\{f_n\}_{n\in\mathbb{N}}$ satisfies $\mathcal{I}^*$-quasi vanishing restriction with respect to f.

$(iii) \Rightarrow (i)$. Finally, suppose that $\{f_n\}_{n\in\mathbb{N}}$ satisfies the $\mathcal{I}^*$-quasi vanishing restriction with respect to f. Then there are a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers with $\lim_n \varepsilon_n = 0$ and $M \in F(\mathcal{I})$ such that $\lim_k \mu(B_k^f(\{\varepsilon_n\})_M) = 0$. So given $\delta > 0$, we can find a $k \in \mathbb{N}$ such that $\mu(B_k^f(\{\varepsilon_n\})_M) < \delta$. Take $A_\delta = B_k^f(\{\varepsilon_n\})_M$. Then $\mu(A_\delta) < \delta$. Also if $x \in X \setminus A_\delta$ then $x \notin B_k^f(\{\varepsilon_n\})_M$ and so

$$|\{n \in M : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq k.$$

This shows that $\{f_n\}_{n\in\mathbb{N}}$ converges to f $\mathcal{I}^*$-almost uniformly equally. ∎

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