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SET-VALUED PREŠIĆ-ĆIRIĆ TYPE CONTRACTION IN 0-COMPLETE PARTIAL METRIC SPACES

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Abstract. The purpose of this paper is to introduce the set-valued Prešić-Ćirić type contraction in 0-complete partial metric spaces and to prove some coincidence and common fixed point theorems for such mappings in product spaces, in partial metric case. Results of this paper extend, generalize and unify several known results in metric and partial metric spaces. An example shows how the results of this paper can be used while the existing one cannot.

1. Introduction and preliminaries

There are a number of generalizations of Banach contraction principle. One such generalization is given by S.B. Prešić [28,29] in 1965. Prešić proved following theorem.

THEOREM 1. *Let (X,d) be a complete metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition:*

$$d(T(x_1,x_2,\dots,x_k),T(x_2,x_3,\dots,x_{k+1})) \le \sum_{i=1}^{k} q_i d(x_i,x_{i+1}) \tag{1}$$

for every $x_1,x_2,\dots,x_{k+1} \in X$, where $q_1,q_2,\dots,q_k$ are nonnegative constants such that $q_1+q_2+\dots+q_k < 1$. Then there exists a unique point $x \in X$ such that $T(x,x,\dots,x) = x$. Moreover, if $x_1,x_2,\dots,x_k$ are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = T(x_n,x_{n+1},\dots,x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n,\lim x_n,\dots,\lim x_n)$.

Note that condition (1) in the case $k = 1$ reduces to the well-known Banach contraction mapping principle. So, Theorem 1 is a generalization of the Banach fixed point theorem. Some generalizations and applications of Theorem 1 can be seen in [11,13,16,19,20,25–27,33,35,36,38–40].

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Inspired by the results in Theorem 1, Ćirić and Prešić [13] proved following theorem.

THEOREM 2. *Let* (X, d) *be a complete metric space,* k *a positive integer and* $T : X^k \to X$ *a mapping satisfying the following contractive type condition;*

$$d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \le \lambda \max\{d(x_i, x_{i+1}) : 1 \le i \le k\},$$

where $\lambda \in [0, 1)$ *is a constant and* $x_1, x_2, \dots, x_{k+1}$ *are arbitrary points in* X*. Then there exists a point* x *in* X *such that* $T(x, x, \dots, x) = x$*. Moreover, if* $x_1, x_2, \dots, x_k$ *are arbitrary points in* X *and for* $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$*, then the sequence* $\{x_n\}$ *is convergent and* $\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$*. If in addition we suppose that on diagonal* $\triangle \subset X^k$, $d(T(u, u, \dots, u), T(v, v, \dots, v)) < d(u, v)$ *holds for* $u, v \in X$*, with* $u \ne v$*, then* x *is a unique fixed point satisfying* $x = T(x, x, \dots, x)$*.*

Nadler [24] generalized the Banach contraction mapping principle to set-valued functions and proved the following fixed point theorem.

THEOREM 3. *Let* (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $CB(X)$ *(here* $CB(X)$ *denotes the set of all nonempty closed bounded subset of* X*) such that for all* $x, y \in X$*,*

$$H(Tx, Ty) \le \lambda d(x, y)$$

where, $0 \le \lambda < 1$*. Then* T *has a fixed point.*

After the work of Nadler, several authors proved fixed point results for set-valued mappings (see, e.g., [5,6,8,10,12,15,23,39–41]).

Recently, in [39], the author introduced the notion of weak compatibility of set-valued Prešić type mappings with a single-valued mapping and proved some coincidence and common fixed point theorems for such mappings in product spaces. The following theorem was one of the main results of [39].

THEOREM 4. *Let* (X, d) *be any complete metric space,* k *a positive integer. Let* $f : X^k \to CB(X)$ *and* $g : X \to X$ *be two mappings such that* $g(X)$ *is a closed subspace of* X *and* $f(x_1, x_2, \dots, x_k) \subset g(X)$ *for all* $x_1, x_2, \dots, x_k \in X$*. Suppose that the following condition holds:*

$$H(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \le \sum_{i=1}^{k} \alpha_i d(gx_i, gx_{i+1}),$$

for all $x_1, x_2, \dots, x_{k+1} \in X$*, where* α_i *are nonnegative constants such that* $\sum_{i=1}^{k} \alpha_i < 1$*. Then* f *and* g *have a point of coincidence* $v \in X$*.*

The above theorem generalizes the results of Prešić and Nadler in product spaces in metric case. A generalization of the above theorem can be seen in [40].

On the other hand, Matthews [22] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, with the interesting property of "non-zero self distance" in the space. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. Subsequently, several authors (see, e.g., [1–4,6,7,9,14,17,18,30–32,34,37]) derived fixed point theorems in partial metric spaces. Romaguera [30] introduced the notion of 0-Cauchy sequence, 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

Recently, Aydi et al. [6] introduced the notion of partial Hausdorff metric and extended the Nadler's theorem to partial metric spaces.

In the present paper, we prove some coincidence and common fixed point theorems for the mappings satisfying Prešić-Ćirić type contractive conditions (see [13]) in 0-complete partial metric spaces. Our results extend, generalize and unify the results of Matthews [22], Prešić [28], Ćirić and Prešić [13], Nadler [24] and recent results of Shukla et al. [39] and Aydi et al. [6] to 0-complete partial metric spaces.

Consistent with [4,6,18,22,30,32], the following definitions and results will be needed in the sequel.

Definition 1. A partial metric on a nonempty set X is a function $p : X\times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(P2) $p(x, x) \le p(x, y)$,

(P3) $p(x, y) = p(y, x)$,

(P4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Also every metric space is a partial metric space, with zero self distance.

Example 1. If $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x, y) = \max\{x, y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.

Some more examples of partial metric space can be seen in [6,18,22].

Each partial metric on X generates a T_0 topology τ_p on X which has a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X,\ \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Theorem 5. [22]*For each partial metric $p : X \times X \to \mathbb{R}^+$ the pair (X, d) where, $d(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, is a metric space.*

Here (X, d) is called the induced metric space and d is the induced metric. In further discussion until unless specified (X, d) will represent the induced metric space.

Let (X,p) be a partial metric space.

(1) A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$.

(2) A sequence $\{x_n\}$ in (X,p) is called Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$.

(3) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(4) A sequence $\{x_n\}$ in (X,p) is called 0-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. The space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = 0$.

Lemma 1. [22,30,32] *Let (X,p) be a partial metric space and $\{x_n\}$ be any sequence in X.*

(i) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in metric space (X,d).*

(ii) *(X,p) is complete if and only if the metric space (X,d) is complete. Furthermore, $\lim_{n\to\infty} d(x_n,x) = 0$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.*

(iii) *Every 0-Cauchy sequence in (X,p) is Cauchy in (X,d).*

(iv) *If (X,p) is complete then it is 0-complete.*

The converse assertions of (iii) and (iv) do not hold. Indeed the partial metric space $(\mathbb{Q}\cap[0,\infty),p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x,y) = \max\{x,y\}$, provides an easy example of a 0-complete partial metric space which is not complete. It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

Let (X,p) be a partial metric space. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X,p), induced by the partial metric p. Note that closedness is taken in the sense of (X,τ_p) (τ_p is the topology induced by p) and boundedness is given as follows: A is a bounded subset in (X,p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(a,a) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define

$$p(x,A) = \inf\{p(x,a) : a \in A\}, \qquad \delta_p(A,B) = \sup\{p(a,B) : a \in A\}.$$

Lemma 2. [4] *Let (X,p) be a partial metric space, $A \subset X$. Then $a \in \overline{A}$ if and only if $p(a,A) = p(a,a)$.*

Proposition 1. [6] *Let (X,p) be a partial metric space. For any $A,B,C \in CB^p(X)$, we have the following:*

(i) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$;

(ii) $\delta_p(A,A) \leq \delta_p(A,B)$;

(iii) $\delta_p(A,B) = 0$ *implies that* $A \subseteq B$;

(iv) $\delta_p(A,B) \leq \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Let (X,p) be a partial metric spaces. For $A,B \in CB^p(X)$, define

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}.$$

Proposition 2. [6] *Let (X,p) be a partial metric space. For $A,B,C \in CB^p(X)$, we have*

(h1) $H_p(A,A) \le H_p(A,B)$;

(h2) $H_p(A,B) = H_p(B,A)$;

(h3) $H_p(A,B) \le H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(c,c)$.

Corollary 1. [6] *Let (X,p) be a partial metric space. For $A,B \in CB^p(X)$ the following holds*

$$H_p(A,B) = 0 \quad \text{implies that} \quad A = B.$$

In view of Proposition 2 and Corollary 1, we call the mapping $H_p : CB^p(X) \times CB^p(X) \to [0,\infty)$, a partial Hausdorff metric induced by p.

Lemma 3. [6] *Let (X,p) be a partial metric space and $A,B \in CB^p(X)$ and $h > 1$. For any $a \in A$ there exists $b = b(a) \in B$ such that $p(a,b) \le hH_p(A,B)$.*

Definition 2. [39] Let X be a nonempty set, k a positive integer, $f : X^k \to 2^X$ and $g : X \to X$ be mappings.

(a) If $x \in f(x,\dots,x)$, then $x \in X$ is called a fixed point of f.

(b) An element $x \in X$ said to be a coincidence point of f and g if $gx \in f(x,\dots,x)$.

(c) If $w = gx \in f(x,\dots,x)$, then w is called a point of coincidence of f and g.

(d) If $x = gx \in f(x,\dots,x)$, then x is called a common fixed point of f and g.

(e) Mappings f and g are said to be commuting if $g(f(x,\dots,x)) = f(gx,\dots,gx)$ for all $x \in X$.

(f) Mappings f and g are said to be weakly compatible if $gx \in f(x,\dots,x)$ implies $g(f(x,\dots,x)) \subseteq f(gx,\dots,gx)$.

2. Main results

Theorem 6. *Let (X,p) be a 0-complete partial metric space, k a positive integer. Let $f : X^k \to CB^p(X)$ and $g : X \to X$ be two mappings such that $g(X)$ is a closed subspace of X and $f(x_1,x_2,\dots,x_k) \subset g(X)$ for all $x_1,x_2,\dots,x_k \in X$. Suppose following condition holds:*

$$H_p(f(x_1,x_2,\dots,x_k), f(x_2,x_3,\dots,x_{k+1})) \le \lambda \max\{p(gx_i,gx_{i+1}), 1 \le i \le k\} \quad (2)$$

for all $x_1,x_2,\dots,x_{k+1} \in X$, where $\lambda \in [0,1)$. Then f and g have a point of coincidence $v \in X$.

Proof. We define a sequence $\{y_n\} = \{gx_n\}$ in X as follows: let $x_1,x_2,\dots,x_k \in X$ be arbitrary and $y_n = gx_n$ for $n = 1,2,\dots,k$. As $f(x_1,\dots,x_k) \in CB^p(X)$ and $f(x_1,\dots,x_k) \subset g(X)$, we can assume $y_{k+1} = gx_{k+1} \in f(x_1,\dots,x_k)$, for some $x_{k+1} \in X$ also, $\lambda < 1$ so using Lemma 3 with $h = 1/\sqrt{\lambda}$, there exists $y_{k+2} = gx_{k+2} \in f(x_2, \dots, x_{k+1})$ such that

$$\begin{aligned} p(y_{k+1}, y_{k+2}) &= p(gx_{k+1}, gx_{k+2}) \\ &\le \frac{1}{\sqrt{\lambda}} H_p(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \\ &\le \sqrt{\lambda} \max\{p(gx_i, gx_{i+1}), 1 \le i \le k\} \\ &= \sqrt{\lambda} \max\{p(y_i, y_{i+1}), 1 \le i \le k\}. \end{aligned}$$

Similarly, there exists $y_{k+3} = gx_{k+3} \in f(x_3, \dots, x_{k+2})$ such that

$$\begin{aligned} p(y_{k+2}, y_{k+3}) &= p(gx_{k+2}, gx_{k+3}) \\ &\le \frac{1}{\sqrt{\lambda}} H_p(f(x_2, \dots, x_{k+1}), f(x_3, \dots, x_{k+2})) \\ &\le \sqrt{\lambda} \max\{p(y_i, y_{i+1}), 2 \le i \le k+1\}. \end{aligned}$$

Continuing this procedure we obtain a sequence $\{y_n\}$ such that $y_n = gx_n$ for $n = 1, 2, \dots, k$ and $y_{n+k} = gx_{n+k} \in f(x_n, \dots, x_{n+k-1})$ for $n = 1, 2, \dots$ with

$$p(y_{k+n}, y_{k+n+1}) \le \sqrt{\lambda} \max\{p(y_i, y_{i+1}), n \le i \le n+k-1\}. \tag{3}$$

for all $n \in \mathbb{N}$.

Set $p_n = p(gx_n, gx_{n+1}) = p(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} \mu &= \max\{\frac{p(gx_1, gx_2)}{\delta}, \frac{p(gx_2, gx_3)}{\delta^2}, \dots, \frac{p(gx_k, gx_{k+1})}{\delta^k}\} \\ &= \max\{\frac{p_1}{\delta}, \frac{p_2}{\delta^2}, \dots, \frac{p_k}{\delta^k}\} \end{aligned}$$

where $\delta = \lambda^{1/2k}$. By the method of mathematical induction we shall prove that

$$p_n \le \mu\delta^n \text{ for all } n \in \mathbb{N}. \tag{4}$$

By the definition of μ it is clear that (4) is true for $n = 1, 2, \dots, k$. Let the k inequalities $p_n \le \mu\delta^n, p_{n+1} \le \mu\delta^{n+1} \dots, p_{n+k-1} \le \mu\delta^{n+k-1}$ be the induction hypothesis. Using (3) we obtain

$$\begin{aligned} p_{n+k} &= p(y_{n+k}, y_{n+k+1}) \\ &\le \sqrt{\lambda} \max\{p(y_i, y_{i+1}), n \le i \le n+k-1\} \\ &= \sqrt{\lambda} \max\{p_i, n \le i \le n+k-1\} \\ &= \sqrt{\lambda} \max\{p_n, p_{n+1}, \dots, p_{n+k-1}\} \\ &\le \sqrt{\lambda} \max\{\mu\delta^n, \mu\delta^{n+1}, \dots, \mu\delta^{n+k-1}\} \\ &= \sqrt{\lambda}\mu\delta^n \qquad (\text{as } \ \delta = \lambda^{1/2k} < 1) \\ &= \mu\delta^{n+k}. \end{aligned}$$

Thus, inductive proof of (4) is complete. Now we shall show that the sequence $\{y_n\} = \{gx_n\}$ is a Cauchy sequence in $g(X)$. Let $m, n \in \mathbb{N}$ with $m > n$, then using (4) we obtain

$$p(y_n, y_m) \le p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \dots + p(y_{m-1}, y_m)$$

$$
\begin{aligned}
&\quad - [p(y_{n+1}, y_{n+1}) + p(y_{n+2}, y_{n+2}) + \cdots + p(y_{m-1}, y_{m-1})] \\
&\leq p_n + p_{n+1} + \cdots + p_{m-1} \\
&\leq \mu\delta^n + \mu\delta^{n+1} + \cdots + \mu\delta^{m-1} \\
&\leq \mu\delta^n[1 + \delta + \delta^2 + \cdots] = \frac{\mu\delta^n}{1-\delta}.
\end{aligned}
$$

As $\delta = \lambda^{1/2k} < 1$, therefore $\frac{\mu\delta^n}{1-\delta} \to 0$ as $n \to \infty$. So, it follows from above inequality that

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0.$$

Therefore $\{y_n\} = \{gx_n\}$ is a 0-Cauchy sequence in $g(X)$. As $g(X)$ is closed, there exists $u, v \in X$ such that $v = gu$ and

$$\lim_{n\to\infty} p(y_n, v) = \lim_{n,m\to\infty} p(y_n, y_m) = p(gu, gu) = p(v, v) = 0. \tag{5}$$

We shall show that u is a coincidence point of f and g.

Note that, $gx_{n+k} = y_{n+k} \in f(x_n, x_{n+1}, \ldots, x_{n+k-1})$ so, for any $n \in \mathbb{N}$ we have

$$
\begin{aligned}
p(v, f(u, \ldots, u)) &\leq p(v, y_{n+k}) + p(y_{n+k}, f(u, \ldots, u)) \\
&\leq p(v, y_{n+k}) + H_p(f(x_n, \ldots, x_{n+k-1}), f(u, \ldots, u)) \\
&\leq p(v, y_{n+k}) + H_p(f(x_n, \ldots, x_{n+k-1}), f(x_{n+1}, \ldots, x_{n+k-1}, u)) \\
&\quad + H_p(f(x_{n+1}, \ldots, x_{n+k-1}, u), f(x_{n+2}, \ldots, x_{n+k-1}, u, u)) \\
&\quad + \cdots + H_p(f(x_{n+k-1}, u, \ldots, u), f(u, \ldots, u)),
\end{aligned}
$$

and using (2) in the above inequality we obtain

$$
\begin{aligned}
p(v, f(u, \ldots, u)) &\leq p(v, y_{n+k}) + \lambda \max\{p_n, \ldots, p_{n+k-2}, p(gx_{n+k-1}, gu)\} \\
&\quad + \lambda \max\{p_{n+1}, \ldots, p_{n+k-2}, p(gx_{n+k-1}, gu), p(gu, gu)\} \\
&\quad + \cdots + \lambda \max\{p(gx_{n+k-1}, gu), p(gu, gu), \ldots, p(gu, gu)\} \\
&= p(v, y_{n+k}) + \lambda \max\{p_n, \ldots, p_{n+k-2}, p(y_{n+k-1}, v)\} \\
&\quad + \lambda \max\{p_{n+1}, \ldots, p_{n+k-2}, p(y_{n+k-1}, v), p(v, v)\} \\
&\quad + \cdots + \lambda \max\{p(y_{n+k-1}, v), p(v, v), \ldots, p(v, v)\}.
\end{aligned}
$$

In view of (5), it follows from the above inequality that $p(v, f(u, \ldots, u)) = 0 = p(v, v)$. As $f(u, \ldots, u) \in CB^p(X)$, by Lemma 2 we have $v = gu \in f(u, \ldots, u)$ i.e. u is a coincidence point and v is a point of coincidence of f and g. ■

Remark 1. If we take $p = d$, i.e., if we replace partial metric by metric in the above theorem, we obtain a generalization of the result of [35] in metric spaces.

Taking $g = I_X$ in Theorem 6, we obtain the following fixed point result for set-valued Prešić-Ćirić type contraction.

Corollary 2. *Let (X, p) be a 0-complete metric space, k a positive integer. Let $f : X^k \to CB^p(X)$ be a set-valued Prešić-Ćirić type contraction, i.e., let it satisfy the following contractive type condition*

$$H_p(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq \lambda \max\{p(x_i, x_{i+1}), 1 \leq i \leq k\} \tag{6}$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$, where $\lambda \in [0, 1)$. Then f has a fixed point $v \in X$.

REMARK 2. The above corollary is a set-valued version and generalization of the result of Prešić and Ćirić [13] for set-valued mappings in 0-complete partial metric spaces. Note that for $k = 1$ the above corollary reduces to the of result of Aydi et al. (see Theorem 3.2 of [6]), therefore it is a generalization of the result of Aydi et al. Also, it generalizes the result of Prešić (Theorem 1) for set-valued mappings.

The following theorem provides some sufficient conditions for the uniqueness of point of coincidence of mappings f and g.

THEOREM 7. *Let (X,p) be a 0-complete partial metric space, k a positive integer. Let $f : X^k \to CB^p(X)$ and $g : X \to X$ be two mappings such that, all the conditions of Theorem 6 are satisfied and for any coincidence point u of f and g we have $f(u,\dots,u) = \{gu\}$. If*

(i) on the diagonal $\triangle \subset X^k$,

$$H_p(f(x,\dots,x), f(y,\dots,y)) < p(gx, gy)$$

holds for all $x, y \in X$ with $x \neq y$, or

(ii) in condition (2) the constant $\lambda \in (0, \frac{1}{k})$.

Then, there exists a unique point of coincidence of f and g. Suppose in addition that f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. The existence of coincidence point u and point of coincidence $v = gu$ follows from Theorem 6.

First, suppose that (i) is satisfied. We shall show that the point of coincidence v is unique. If v' is another point of coincidence with coincidence point u' of f and g, then $f(u',\dots,u') = \{gu'\} = \{v'\}$ and we have

$$\begin{aligned} p(v,v') &= H_p(\{v\},\{v'\}) \\ &= H_p(f(u,\dots,u), f(u',\dots,u')) \\ &< p(gu,gu') = p(v,v'), \end{aligned}$$

a contradiction. So, the point of coincidence of f and g is unique.

Suppose (ii) is satisfied, then using (2) we obtain

$$\begin{aligned} p(v,v') &= H_p(\{v\},\{v'\}) \\ &= H_p(f(u,\dots,u), f(u',\dots,u')) \\ &\le H_p(f(u,\dots,u), f(u,\dots,u,u')) + H_p(f(u,\dots,u,u'), f(u,\dots,u,u',u')) \\ &\quad + \cdots + H_p(f(u,u',\dots,u'), f(u',\dots,u')) \\ &\le \lambda \max\{p(gu,gu),\dots,p(gu,gu),p(gu,gu')\} \\ &\quad + \lambda \max\{p(gu,gu),\dots,p(gu,gu),p(gu,gu'),p(gu',gu')\} \\ &\quad + \cdots + \lambda \max\{p(gu,gu'),p(gu',gu')\dots,p(gu',gu')\} \\ &= k\lambda p(gu,gu') = k\lambda p(v,v') < p(v,v'), \end{aligned}$$

again a contradiction. So, the point of coincidence of f and g is unique.

Suppose that f and g are weakly compatible. Then we have

$$g(f(u,\dots,u)) \subseteq f(gu,\dots,gu) = f(v,\dots,v) \text{ i.e. } \{gv\} \subseteq f(v,\dots,v).$$

Therefore $gv \in f(v,\dots,v)$, which shows that gv is another point of coincidence of f and g and by uniqueness we have $v = gv \in f(v,\dots,v)$. Thus v is a unique common fixed point of f and g. ■

The following is a simple example of set-valued Prešić-Ćirić contraction which illustrate the case when the results of this paper can be used while the existing one cannot.

EXAMPLE 2. Let $X = \mathbb{Q}\cap[0,1]$ be endowed with the partial metric $p : X\times X \to \mathbb{R}^+$ defined by

$$p(x,y) = |x-y| + \max\{x,y\} \quad \text{for all} \quad x,y \in X.$$

First, we shall show that the space (X,p) is 0-complete. If $\{x_n\}$ is any 0-Cauchy sequence in X, then $\lim_{n,m\to\infty} p(x_n,x_m) = 0$, i.e.,

$$\lim_{n,m\to\infty}[|x_n - x_m| + \max\{x_n,x_m\}] = 0. \tag{7}$$

Note that the partial metric space (X,p_1) is 0-complete, where $p_1(x,y) = \max\{x,y\}$ for all $x,y \in X$ (see [30]). Therefore it follows from (7) that $\lim_{n\to\infty} p_1(x_n,0) = 0 = p_1(0,0)$ and $\lim_{n\to\infty}|x_n - 0| = 0$. So we have $\lim_{n\to\infty} p(x_n,0) = 0 = p(0,0)$. As $0 \in X$, the space (X,p) is 0-complete.

Note that, the metric d induced by p is given by

$$d(x,y) = 2|x-y| + 2\max\{x,y\} - x - y = 3|x-y| \quad \text{for all} \quad x,y \in X,$$

and the metric space (X,d) is not complete, and so the partial metric space (X,p) is not complete. Note that, if $x \in X$ then the singleton subset $\{x\}$ of X is a closed subset with respect to p. Indeed, for any $y \in X$, we have

$$\begin{aligned} y \in \overline{\{x\}} &\Leftrightarrow p(y,y) = p(y,\{x\}) \\ &\Leftrightarrow p(y,y) = p(y,x) \\ &\Leftrightarrow y = |y-x| + \max\{y,x\} \\ &\Leftrightarrow y = x. \end{aligned}$$

Thus $\{x\}$ is closed. Now, for $k = 2$, define a mapping $T : X^2 \to X$ by

$$T(x,y) = \begin{cases} 0, & \text{if } x = y = 1; \\ \frac{x+y}{10}, & \text{otherwise,} \end{cases}$$

and a mapping $f : X^2 \to CB^p(X)$ by

$$f(x,y) = \{T(x,y)\} \cup \{0\} \text{ for all } x,y \in [0,1].$$

We shall show that f satisfies condition (6) of Corollary 2 with $\lambda \in [\frac{2}{5}, 1)$.

If $x_1,x_2,x_3 \in [0,1)$ with $x_1 \le x_2 \le x_3$ then

$$H_p(f(x_1,x_2), f(x_2,x_3)) = H_p(\{\frac{x_1+x_2}{10}, 0\}, \{\frac{x_2+x_3}{10}, 0\})$$

$$= \max\{\inf\{\frac{2x_3+x_2-x_1}{10}, \frac{2x_1+2x_2}{10}\},$$
$$\inf\{\frac{2x_3+x_2-x_1}{10}, \frac{2x_2+2x_3}{10}\}\}$$
$$= \frac{2x_3+x_2-x_1}{10} = \frac{1}{10}(2x_3-x_2+2x_2-x_1)$$
$$\leq \frac{1}{5}\max\{2x_2-x_1, 2x_3-x_2\}$$
$$= \frac{1}{5}\max\{p(x_1,x_2), p(x_2,x_3)\}.$$

Therefore (6) is satisfied with $\lambda \in [\frac{2}{5}, 1)$.

If $x_1, x_2, x_3 \in [0,1)$ with $x_3 \leq x_1 \leq x_2$ then

$$\begin{aligned} H_p(f(x_1,x_2), f(x_2,x_3)) &= H_p(\{\frac{x_1+x_2}{10}, 0\}, \{\frac{x_2+x_3}{10}, 0\}) \\ &= \max\{\inf\{\frac{2x_1+x_2-x_3}{10}, \frac{2x_1+2x_2}{10}\}, \\ &\quad \inf\{\frac{2x_1+x_2-x_3}{10}, \frac{2x_2+2x_3}{10}\}\} \\ &= \frac{2x_1+x_2-x_3}{10} \end{aligned}$$

and $\max\{p(x_1,x_2), p(x_2,x_3)\} = \max\{2x_2-x_1, 2x_2-x_3\} = 2x_2-x_3$. Therefore, (6) is satisfied with $\lambda \in [\frac{2}{5}, 1)$.

Similarly, if $x_1, x_2, x_3 \in [0,1)$ with $x_2 \leq x_3 \leq x_1$ or any one of x_1, x_2, x_3 is equal to 1, then with a similar process one can verify (6).

If any two of x_1, x_2, x_3 is equal to 1, e.g., let $x_1 = x_2 = 1$ and $x_3 \in [0,1)$, then

$$\begin{aligned} H_p(f(x_1,x_2), f(x_2,x_3)) &= H_p(\{0\}, \{\frac{1+x_3}{10}, 0\}) \\ &= \max\{0, \frac{2+2x_3}{10}\} = \frac{2+2x_3}{10} \end{aligned}$$

and $\max\{p(x_1,x_2), p(x_2,x_3)\} = \max\{1, 2-x_3\} = 2-x_3$. Therefore, (6) is satisfied with $\lambda \in [\frac{2}{5}, 1)$. Similarly, (6) is satisfied in all possible cases with $\lambda \in [\frac{2}{5}, 1)$. Thus, all the conditions of Corollary 2 are satisfied and f has a fixed point $0 \in X$.

On the other hand, as (X, d_u) and (X, d) (where d_u is usual and d is the induced metric on X) are not complete spaces, we cannot conclude the existence of fixed point of f with the metric version of Corollary 2. Also, it is easy to see that f is not a Prešić-Ćirić contraction in both the spaces (X, d_u) and (X, d). Indeed, at $x_1 = x_2 = 1, x_3 = \frac{9}{10}$ the mapping f fails to be a Prešić-Ćirić contraction in these metric spaces. Thus we can say that the class of Prešić-Ćirić contractions in partial metric spaces is wider than that in metric spaces.

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REFERENCES

[**1**] T. Abdeljawad, *Fixed points of generalized weakly contractive mappings in partial metric spaces*, Math. Comput. Model. **54** (2011), 2923–2927.

[**2**] T. Abdeljawad, E. Karapinar, K. Taş, *Existence and uniqueness of a common fixed point on partial metric spaces*, Appl. Math. Lett. **24** (2011), 1900–1904.

[**3**] A.G.B. Ahmad, Z.M. Fadail, V.Ć. Rajić, S. Radenović, *Nonlinear contractions in* 0*-complete partial metric spaces*, Abstract Appl. Anal. **2012**, Article ID 451239, 13 p., (2012).

[**4**] I. Altun, F. Sola, H. Simsek, *Generalized contractions on partial metric spaces*, Topology Appl. **157** (2010) 2778–2785.

[**5**] H. Aydi, M. Abbas, M. Postolache, *Coupled coincidence points for hybrid pair of mappings via mixed monotone property*, J. Adv. Math. Stud. **5** (2012), 118–126.

[**6**] H. Aydi, M. Abbas, C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology Appl. **159** (2012), 3234–3242.

[**7**] C. Di Bari, Z. Kadelburg, H.K. Nashine, S. Radenović, *Common fixed points of g-quasi-contractions and related mappings in* 0*-complete partial metric spaces*, Fixed Point Theory Appl. **2012**:113 (2012).

[**8**] M. Berinde, V. Berinde, *On a general class of multi-valued weakly Picard mappings*, J. Math. Anal. Appl. **326** (2007), 772–782.

[**9**] M. Bukatin, R. Kopperman, S. Matthews, H. Pajoohesh, *Partial metric spaces*, Am. Math. Mon. **116** (2009), 708–718.

[**10**] P. Chaipunya, C. Mongkolkeha, W. Sintunavarat, P. Kumam, *Fixed point theorems for multivalued mappings in modular metric spaces*, Abstr. Appl. Anal. **2012**, Article ID 503504, 14 p. (2012).

[**11**] Y.Z. Chen, *A Prešić type contractive condition and its applications*, Nonlinear Anal. **71** (2009), 2012–2017.

[**12**] S.H. Cho, J.S. Bae, *Fixed point theorems for multi-valued maps in cone metric spaces*, Fixed Point Theory Appl. **2011**:87 (2011).

[**13**] Lj.B. Ćirić, S.B. Prešić, *On Prešić type generalisation of Banach contraction principle*, Acta. Math. Univ. Com. **76** (2007), 143–147.

[**14**] Lj.B. Ćirić, B. Samet, H. Aydi, C. Vetro, *Common fixed points of generalized contractions on partial metric spaces and an application*, Appl. Math. Comput. **218** (2011), 2398–2406.

[**15**] G.M. Eshaghi, H. Baghani, H. Khodaei, M. Ramezani, *A generalization of Nadler's fixed point theorem*, J. Nonlinear Sci. Appl. **3** (2010), 148–151.

[**16**] R. George, K.P. Reshma, R. Rajagopalan, *A generalized fixed point theorem of Prešić type in cone metric spaces and application to Markov process*, Fixed Point Theory Appl. **2011**:85 (2011).

[**17**] D. Ilić, V. Pavlović, V. Rakočević, *Extensions of Zamfirescu theorem to partial metric spaces*, Math. Comput. Model. **55** (2012), 801–809.

[**18**] Z. Kadelburg, H.K. Nashine, S. Radenović, *Fixed point results under various contractive conditions in partial metric spaces*, Rev. Real Acad. Cienc. Exac., Fis. Nat., Ser. A, Mat. **107** (2013), 241–256.

[**19**] M.S. Khan, M. Berzig, B. Samet, *Some convergence results for iterative sequences of Prešić type and applications*, Adv. Difference Equ. **2012**:38 (2012).

[**20**] S.K. Malhotra, S. Shukla, R. Sen, *A generalization of Banach contraction principle in ordered cone metric spaces*, J. Adv. Math. Stud. **5** (2012), 59–67.

[**21**] S.K. Malhotra, S. Shukla, R. Sen, *Some coincidence and common fixed point theorems for Prešić-Reich type mappings in cone metric spaces*, Rend. Sem. Mat. Univ. Pol. Torino, **70** (2013) (to appear).

[**22**] S.G. Matthews, *Partial metric topology*, In: Proc. 8th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., vol. 728, (1994) pp. 183–197.

[**23**] N. Mizoguchi, W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces*, J. Math. Anal. Appl. **141** (1989), 177–188.

[**24**] S.B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[**25**] M. Păcurar, *A multi-step iterative method for approximating common fixed points of Prešić-Rus type operators on metric spaces*, Studia Univ. “Babeş-Bolyai”, Mathematica, **15** (2010).

[**26**] M. Păcurar, *Approximating common fixed points of Prešić-Kannan type operators by a multi-step iterative method*, An. Şt. Univ. Ovidius Constanţa **17** (2009), 153–168.

[**27**] M. Păcurar, *Common fixed points for almost Prešić type operators*, Carpathian J. Math. **28** (2012), 117–126.

[**28**] S.B. Prešić, *Sur la convergence des suites*, Compt. Rendus Acad. Paris **260** (1965), 3828–3830.

[**29**] S.B. Prešić, *Sur une classe d'inequations aux differences finite et sur la convergence de certaines suites*, Publ. Inst. Math. Belgrade, **5(19)** (1965), 75–78.

[**30**] S. Romaguera, *A Kirk type characterization of completeness for partial metric spaces*, Fixed Point Theory Appl. **2010**, Article ID 493298, 6 p. (2010).

[**31**] S. Romaguera, *Fixed point theorems for generalized contractions on partial metric spaces*, Topology Appl. **159** (2012), 194–199.

[**32**] S. Romaguera, *Matkowski's type theorems for generalized contractions on (ordered) partial metric spaces*, Appl. Gen. Topology **12** (2011), 213–220.

[**33**] S. Shukla, *Prešić type results in 2-Banach spaces*, Afrika Mat. (2013) DOI 10.1007/s13370-013-0174-2.

[**34**] S. Shukla, I. Altun, R. Sen, *Fixed point theorems and asymptotically regular mappings in partial metric spaces*, ISRN Computat. Math. **2013**, Article ID 602579, 6 p. (2013).

[**35**] S. Shukla, B. Fisher, *A generalization of Prešić type mappings in metric-like spaces*, J. Operators **2013**, Article ID 368501, 5 p. (2013).

[**36**] S. Shukla, S. Radenović, *A generalization of Prešić type mappings in* 0*-complete ordered partial metric spaces*, Chinese J. Math. **2013**, Article ID 859531, 8 p. (2013).

[**37**] S. Shukla, S. Radenović, *Some common fixed point theorems for F-contraction type mappings in* 0*-complete partial metric spaces*, J. Mathematics **2013**, Article ID 878730, 7 p. (2013).

[**38**] S. Shukla, S. Radenović, S. Pantelić, *Some fixed point theorems for Prešić-Hardy-Rogers type contractions in metric spaces*, J. Mathematics **2013**, Article ID 295093, 8 p. (2013).

[**39**] S. Shukla, R. Sen, S. Radenović, *Set-valued Prešić type contraction in metric spaces*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) (2013) (to appear).

[**40**] S. Shukla, R. Sen, *Set-valued Prešić-Reich type mappings in metric spaces*, Rev. Real Acad. Cienc. Exac., Fis. Nat., Ser. A, Mat. DOI 10.1007/s13398-012-0114-2.

[**41**] D. Wardowski, *On set-valued contractions of Nadler type in cone metric spaces*, Appl. Math. Lett. **24** (2011), 275–278.

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