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COUPLED FIXED POINT THEOREMS IN G_b-METRIC SPACES

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Abstract. T. G. Bhaskar and V. Lakshmikantham [Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 1379–1393], V. Lakshmikantham and Lj. B. Ćirić [Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009) 4341–4349] introduced the concept of a coupled coincidence point of a mapping F from $X \times X$ into X and a mapping g from X into X. In this paper we prove a coupled coincidence fixed point theorem in the setting of a generalized b-metric space. Three examples are presented to verify the effectiveness and applicability of our main result.

1. Introduction

Mustafa and Sims [25] introduced a new notion of generalized metric space called a G-metric space. Mustafa, Sims and others studied fixed point theorems for mappings satisfying different contractive conditions [1, 2, 6, 10, 11, 19, 22, 23, 25, 27, 28, 32, 35, 36, 39]. Abbas and Rhoades [1] obtained some common fixed point theorems for non-commuting maps without continuity satisfying different contractive conditions in the setting of generalized metric spaces. Lakshmikantham et al. in [7, 21] introduced the concept of a coupled coincidence point for a mapping F from $X \times X$ into X and a mapping g from X into X, and studied coupled fixed point theorems in partially ordered metric spaces. In [33], Sedghi et al. proved a coupled fixed point theorem for contractive mappings in complete fuzzy metric spaces. On the other hand, the concept of b-metric space was introduced by Czerwik in [13]. After that, several interesting results for the existence of fixed point for single-valued and multivalued operators in b-metric spaces have been obtained [3, 5, 8, 9, 12, 14, 15, 16, 18, 20, 30, 31, 34, 37, 38]. Pacurar [29] proved some results on sequences of almost contractions and fixed points in b-metric spaces. Recently, Hussain and Shah [17] obtained results on KKM mappings in cone b-metric spaces.

Aghajani et al., in a submitted paper [4], extended the notion of G-metric space to the concept of G_b-metric space. Very recently, Mustafa et al. [24] have obtained

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some coupled coincidence point theorems for nonlinear (ψ, φ)-weakly contractive mappings in partially ordered G_b-metric spaces.

In this paper, we prove a coupled coincidence fixed point theorem in the setting of a generalized b-metric space. First, we present some basic properties of G_b-metric spaces.

Following is the definition of generalized b-metric spaces or G_b-metric spaces.

Definition 1.1. [24] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$,

(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(G_b5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized b-metric and the pair (X, G) is called a generalized b-metric space or G_b-metric space.

It should be noted that the class of G_b-metric spaces is effectively larger than that of G-metric spaces given in [25]. Indeed, each G-metric space is a G_b-metric space with $s = 1$. The following example shows that a G_b-metric on X need not be a G-metric on X.

Example 1.1. [24] Let (X, G) be a G-metric space, and $G_*(x, y, z) = G^p(x, y, z)$, where $p > 1$ is a real number. Note that G_* is a G_b-metric with $s = 2^{p-1}$. In [24], it is proved that (X, G_*) is not necessarily a G-metric space.

Example 1.2. [24] Let $X = \mathbb{R}$ and $d(x, y) = |x - y|^2$. We know that (X, d) is a b-metric space with $s = 2$. Let $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, then (X, G) is not a G_b-metric space.

However, $G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ is a G_b-metric on $\mathbb{R}$ with $s = 2$. Similarly, if $d(x, y) = |x - y|^p$ is selected with $p \geq 1$, then $G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ is a G_b-metric on $\mathbb{R}$ with $s = 2^{p-1}$.

Now we present some definitions and propositions in G_b-metric spaces.

Definition 1.2. [24] A G_b-metric G is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 1.3. [24] Let (X, G) be a G_b-metric space. Then, for $x_0 \in X$, $r > 0$, the G_b-ball with center x_0 and radius r is

$$B_G(x_0, r) = \{y \in X \mid G(x_0, y, y) < r\}.$$

Definition 1.4. [24] Let X be a G_b-metric space and let $d_G(x, y) = G(x, y, y) + G(x, x, y)$. Then d_G defines a b-metric on X, which is called the b-metric associated with G.

Proposition 1.2. [24] *Let X be a G_b-metric space. For any $x_0 \in X$ and $r > 0$, if $y \in B_G(x_0, r)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.*

From the above proposition the family of all G_b-balls

$$\Lambda = \{B_G(x, r) \mid x \in X,\ r > 0\}$$

is a base of a topology $\tau(G)$ on X, which is called the G_b-metric topology.

Definition 1.5. [24] Let X be a G_b-metric space. A sequence (x_n) in X is said to be:

(1) G_b-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n, l \geq n_0, G(x_n, x_m, x_l) < \varepsilon$;

(2) G_b-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0, G(x_n, x_m, x) < \varepsilon$.

Using the above definitions, one can easily prove the following proposition.

Proposition 1.4. [24] *Let X be a G_b-metric space and (x_n) be a sequence in X. Then the following are equivalent:*

(1) *the sequence (x_n) is G_b-Cauchy;*

(2) *for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq n_0$.*

Definition 1.6. [24] A G_b-metric space X is called complete if every G_b-Cauchy sequence is G_b-convergent in X.

Mustafa and Sims proved that each G-metric function $G(x, y, z)$ is jointly continuous in all three of its variables (see [26, Proposition 8]). But in general a G_b-metric function $G(x, y, z)$ for $s > 1$ is not jointly continuous in all three of its variables. Now we recall an example of a discontinuous G_b-metric.

Example 1.3. [24] Let $X = \mathbb{N} \cup \{\infty\}$ and let $D : X \times X \to \mathbb{R}^+$ be defined by

$$D(m, n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \\ & \qquad \text{or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m, p) \leq \frac{5}{2}(D(m, n) + D(n, p)).$$

Thus, (X, D) is a b-metric space with $s = \frac{5}{2}$ (see [16, Example 2]). Let $G(x, y, z) = \max\{D(x, y), D(y, z), D(z, x)\}$. It is easy to see that G is a G_b-metric with $s = \frac{5}{2}$. In [24], it is proved that $G(x, y, z)$ is not a continuous function.

Definition 1.7. Let (X, G) and (X', G') be G_b-metric spaces, and let $f : X \to X'$ be a mapping. Then f is said to be continuous at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is continuous at X if and only if it is continuous at all $a \in X$.

Definition 1.8. [7] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.9. [21] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.10. [21] Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $gF(x, y) = F(gx, gy)$.

2. Common fixed point results

Let Φ denote the class of all functions $\phi : [0, \infty) \to [0, \infty)$ such that ϕ is increasing, continuous, $\phi(t) < \frac{t}{2}$ for all $t > 0$ and $\phi(0) = 0$. It is easy to see that for every $\phi \in \Phi$ we can choose a $0 < k < \frac{1}{2}$ such that $\phi(t) \leq kt$.

We start our work by proving the following two crucial lemmas.

Lemma 2.1. *Let (X, G) be a G_b-metric space with $s \geq 1$, and suppose that (x_n) is G_b-convergent to x. Then we have*

$$\frac{1}{s} G(x, y, y) \leq \liminf_{n\to\infty} G(x_n, y, y) \leq \limsup_{n\to\infty} G(x_n, y, y) \leq sG(x, y, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} G(x_n, y, y) = 0$.

Proof. Using the rectangle inequality in (X, G), it is easy to see that

$$G(x_n, y, y) \leq sG(x_n, x, x) + sG(x, y, y),$$

and

$$\frac{1}{s} G(x, y, y) \leq G(x, x_n, x_n) + G(x_n, y, y).$$

Taking the upper limit as $n \to \infty$ in the first inequality and the lower limit as $n \to \infty$ in the second inequality we obtain the desired result. ∎

Lemma 2.2. *Let (X, G) be a G_b-metric space and let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$G(F(x, y), F(u, v), F(z, w)) \leq \phi(G(gx, gu, gz) + G(gy, gv, gw)) \tag{1}$$

for some $\phi \in \Phi$ and for all $x, y, z, w, u, v \in X$. Assume that (x, y) is a coupled coincidence point of the mappings F and g. Then

$$F(x, y) = gx = gy = F(y, x).$$

Proof. Since (x, y) is a coupled coincidence point of the mappings F and g, we have $gx = F(x, y)$ and $gy = F(y, x)$. Assume $gx \neq gy$. Then by (1), we get

$$G(gx, gy, gy) = G(F(x, y), F(y, x), F(y, x)) \leq \phi(G(gx, gy, gy) + G(gy, gx, gx)).$$

Also by (1), we have

$$G(gy, gx, gx) = G(F(y, x), F(x, y), F(x, y)) \leq \phi(G(gy, gx, gx) + G(gx, gy, gy)).$$

Therefore

$$G(gx, gy, gy) + G(gy, gx, gx) \leq 2\phi(G(gx, gy, gy) + G(gy, gx, gx)).$$

Since $\phi(t) < \frac{t}{2}$, we get

$$G(gx, gy, gy) + G(gy, gx, gx) < G(gx, gy, gy) + G(gy, gx, gx),$$

which is a contradiction. So $gx = gy$, and hence $F(x, y) = gx = gy = F(y, x)$. ■

The following is the main result of this section.

THEOREM 2.1. *Let (X, G) be a complete G_b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$G(F(x, y), F(u, v), F(z, w)) \leq \frac{1}{s^2}\phi(G(gx, gu, gz) + G(gy, gv, gw)) \tag{2}$$

for some $\phi \in \Phi$ and all $x, y, z, w, u, v \in X$. Assume that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$,*
2. *$g(X)$ is complete, and*
3. *g is continuous and commutes with F.*

Then there is a unique x in X such that $gx = F(x, x) = x$.

Proof. Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. For $n \in \mathbb{N} \cup \{0\}$, by (2) we have

$$\begin{aligned} G(gx_{n-1}, gx_n, gx_n) &= G(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})) \\ &\leq \frac{1}{s^2}\phi(G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})). \end{aligned}$$

Similarly, by (2) we have

$$\begin{aligned} G(gy_{n-1}, gy_n, gy_n) &= G(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})) \\ &\leq \frac{1}{s^2}\phi(G(gy_{n-2}, gy_{n-1}, gy_{n-1}) + G(gx_{n-2}, gx_{n-1}, gx_{n-1})). \end{aligned}$$

Hence, we have that

$$\begin{aligned} a_n &:= G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n) \\ &\le \frac{2}{s^2}\phi(G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})) \\ &= \frac{2}{s^2}\phi(a_{n-1}) \end{aligned}$$

holds for all $n \in \mathbb{N}$. Thus, we get a k, $0 < k < \frac{1}{2}$ such that

$$a_n \le \frac{2}{s^2}\phi(a_{n-1}) \le \frac{2k}{s^2}a_{n-1} \le \frac{2k}{s}a_{n-1} = qa_{n-1},$$

for $q = \frac{2k}{s}$. Hence we have

$$a_n \le \frac{2k}{s}a_{n-1} \le \cdots \le (\frac{2k}{s})^n a_0.$$

Let $m, n \in \mathbb{N}$ with $m > n$. By Axiom G_b5 of definition of G_b-metric spaces, we have

$$\begin{aligned} &G(gx_{n-1}, gx_m, gx_m) + G(gy_{n-1}, gy_m, gy_m) \\ &\quad \le s(G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_m, gx_m)) \\ &\qquad + s(G(gy_{n-1}, gy_n, gy_n) + G(gy_n, gy_m, gy_m)) \\ &\quad = s(G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)) \\ &\qquad + s(G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m)) \\ &\quad \le \\ &\quad \vdots \\ &\quad \le sa_n + s^2a_{n+1} + s^3a_{n+2} + \cdots + s^{m-n}a_{m-1} + s^{m-n}a_m \\ &\quad \le sq^na_0 + s^2q^{n+1}a_0 + \cdots + s^{m-n}q^{m-1}a_0 + +s^{m-n}q^ma_0 \\ &\quad \le sq^na_0(1 + sq + s^2q^2 + \cdots) \\ &\quad \le \frac{sq^na_0}{1 - sq} \longrightarrow 0, \end{aligned}$$

since $sq = 2k < 1$. Thus (gx_n) and (gy_n) are G_b-Cauchy in $g(X)$. Since $g(X)$ is complete, we get (gx_n) and (gy_n) are G_b-convergent to some $x \in X$ and $y \in X$ respectively. Since g is continuous, we have that (ggx_n) is G_b-convergent to gx and (ggy_n) is G_b-convergent to gy. Also, since g and F commute, we have

$$ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n),$$

and

$$ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

Thus

$$\begin{aligned} G(ggx_{n+1}, F(x,y), F(x,y)) &= G(F(gx_n, gy_n), F(x,y), F(x,y)) \\ &\le \frac{1}{s^2}\phi(G(ggx_n, gx, gx) + G(ggy_n, gy, gy)). \end{aligned}$$

Letting $n \to \infty$, and using Lemma 2.1, we get that

$$\begin{aligned}\frac{1}{s}G(gx, F(x,y), F(x,y)) &\leq \limsup_{n\to\infty} G(F(gx_n, gy_n), F(x,y), F(x,y)) \\ &\leq \limsup_{n\to\infty} \frac{1}{s^2}\phi(G(ggx_n, gx, gx) + G(ggy_n, gy, gy)) \\ &\leq \frac{1}{s^2}\phi(s(G(gx, gx, gx) + G(gy, gy, gy)) = 0.\end{aligned}$$

Hence, $gx = F(x, y)$. Similarly, we may show that $gy = F(y, x)$. By Lemma 2.2, (x, y) is a coupled fixed point of the mappings F and g, i.e.,

$$gx = F(x,y) = F(y,x) = gy.$$

Thus, using Lemma 2.1 we have

$$\begin{aligned}\frac{1}{s}G(x, gx, gx) &\leq \limsup_{n\to\infty} G(gx_{n+1}, gx, gx) \\ &= \limsup_{n\to\infty} G(F(x_n, y_n), F(x,y), F(x,y)) \\ &\leq \limsup_{n\to\infty} \frac{1}{s^2}\phi(G(gx_n, gx, gx) + G(gy_n, gy, gy)) \\ &\leq \frac{1}{s^2}\phi(s(G(x, gx, gx) + G(y, gy, gy))).\end{aligned}$$

Hence, we get

$$G(x, gx, gx) \leq \frac{1}{s}\phi(s(G(x, gx, gx) + G(y, gy, gy))).$$

Similarly, we may show that

$$G(y, gy, gy) \leq \frac{1}{s}\phi(s(G(x, gx, gx) + G(y, gy, gy))).$$

Thus,

$$\begin{aligned}G(x, gx, gx) + G(y, gy, gy) &\leq \frac{2}{s}\phi(s(G(x, gx, gx) + G(y, gy, gy))) \\ &\leq 2kG(x, gx, gx) + G(y, gy, gy).\end{aligned}$$

Since $2k < 1$, the last inequality happens only if $G(x, gx, gx) = 0$ and $G(y, gy, gy) = 0$. Hence $x = gx$ and $y = gy$. Thus we get

$$gx = F(x,x) = x.$$

To prove the uniqueness, let $z \in X$ with $z \neq x$ such that

$$z = gz = F(z,z).$$

Then

$$\begin{aligned}G(x,z,z) = G(F(x,x), F(z,z), F(z,z)) &\leq \frac{1}{s^2}\phi(2G(gx, gz, gz)) \\ &< \frac{1}{s^2}2kG(x,z,z) \leq 2kG(x,z,z).\end{aligned}$$

Since $2k < 1$, we get $G(x,z,z) < G(x,z,z)$, which is a contradiction. Thus, F and g have a unique common fixed point. ■

Corollary 2.1. *Let (X,G) be a G_b-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$G(F(x,y), F(u,v), F(u,v)) \leq \frac{k}{s^2}(G(gx,gu,gu) + G(gy,gv,gv)) \tag{3}$$

for all $x,y,u,v \in X$. Assume F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$,*
2. *$g(X)$ is complete, and*
3. *g is continuous and commutes with F.*

If $k \in (0, \frac{1}{2})$, then there is a unique x in X such that $gx = F(x,x) = x$.

Proof. Follows from Theorem 2.1 by taking $z = u$, $v = w$ and $\phi(t) = kt$. ■

Corollary 2.2. *Let (X,G) be a complete G_b-metric space. Let $F : X \times X \to X$ be a mapping such that*

$$G(F(x,y), F(u,v), F(u,v)) \leq \frac{k}{s^2}(G(x,u,u) + G(y,v,v))$$

for all $x,y,u,v \in X$. If $k \in [0, \frac{1}{2})$, then there is a unique x in X such that $F(x,x) = x$.

Remark 2.1. Since every G_b-metric is a G-metric when $s = 1$, so our results can be viewed as generalizations and extensions of corresponding results in [35] and several other comparable results.

Now, we introduce some examples for Theorem 2.1.

Example 2.1. Let $X = [0,1]$. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x,y,z) = (|x-y| + |x-z| + |y-z|)^2$$

for all $x,y,z \in X$. Then (X,G) is a complete G_b-metric space with $s = 2$, according to Example 1.1. Define a map $F : X \times X \to X$ by $F(x,y) = \dfrac{x}{128} + \dfrac{y}{256}$ for $x,y \in X$. Also, define $g : X \to X$ by $g(x) = \dfrac{x}{4}$ for $x \in X$ and $\phi(t) = \dfrac{t}{4}$ for $t \in \mathbb{R}^+$. We have that

$$\begin{aligned}
&G(F(x,y),F(u,v),F(z,w))\\
&\quad=(|F(x,y)-F(u,v)|+|F(u,v)-F(z,w)|+|F(z,w)-F(x,y)|)^2\\
&\quad=(|\frac{x}{128}+\frac{y}{256}-\frac{u}{128}-\frac{v}{256}|+|\frac{u}{128}+\frac{v}{256}-\frac{z}{128}-\frac{w}{256}|\\
&\qquad+|\frac{z}{128}+\frac{w}{256}-\frac{x}{128}-\frac{y}{256}|)^2\\
&\quad\le(\frac{1}{128}|x-u|+\frac{1}{256}|y-v|+\frac{1}{128}|u-z|+\frac{1}{256}|v-w|+\frac{1}{128}|z-x|\\
&\qquad+\frac{1}{256}|w-y|)^2\\
&\quad=(\frac{1}{32}(\left|\frac{x}{4}-\frac{u}{4}\right|+\left|\frac{u}{4}-\frac{z}{4}\right|+\left|\frac{z}{4}-\frac{x}{4}\right|)+\frac{1}{64}(\left|\frac{y}{4}-\frac{v}{4}\right|+\left|\frac{v}{4}-\frac{w}{4}\right|+\left|\frac{w}{4}-\frac{y}{4}\right|))^2\\
&\quad\le\frac{2}{32^2}(\left|\frac{x}{4}-\frac{u}{4}\right|+\left|\frac{u}{4}-\frac{z}{4}\right|+\left|\frac{z}{4}-\frac{x}{4}\right|)^2+\frac{2}{64^2}(\left|\frac{y}{4}-\frac{v}{4}\right|+\left|\frac{v}{4}-\frac{w}{4}\right|+\left|\frac{w}{4}-\frac{y}{4}\right|)^2\\
&\quad=\frac{2}{32^2}G(gx,gu,gz)+\frac{2}{64^2}G(gy,gv,gw)\\
&\quad\le\frac{2}{32^2}(G(gx,gu,gz)+G(gy,gv,gw))\\
&\quad\le\frac{1}{4}\frac{G(gx,gu,gz)+G(gy,gv,gw)}{4}\\
&\quad=\frac{1}{2^2}\phi(G(gx,gu,gz)+G(gy,gv,gw))
\end{aligned}$$

holds for all $x,y,u,v,z,w\in X$. It is easy to see that F and g satisfy all the hypothesis of Theorem 2.1. Thus F and g have a unique common fixed point. Here $F(0,0)=g(0)=0$. ∎

EXAMPLE 2.2. Let X and G be as in Example 2.1. Define a map

$$F:X\times X\to X \quad\text{by}\quad F(x,y)=\frac{1}{16}x^2+\frac{1}{16}y^2+\frac{1}{8}$$

for $x,y\in X$. Then $F(X\times X)=[\frac{1}{8},\frac{1}{4}]$. Also,

$$\begin{aligned}
&G(F(x,y),F(u,v),F(u,v))\\
&\quad=(2|F(x,y)-F(u,v)|)^2=\frac{1}{64}(|x^2-u^2+y^2-v^2|)^2\\
&\quad\le\frac{1}{64}(\left|x^2-u^2\right|+\left|y^2-v^2\right|)^2\le\frac{1}{32}(|x^2-u^2|^2+|y^2-v^2|^2)\\
&\quad\le\frac{1}{32}(4|x-u|^2+4\left|y-v\right|^2)=\frac{1}{32}(G(x,u,u)+G(y,v,v))\\
&\quad\le\frac{\frac{1}{8}}{2^2}(G(x,u,u)+G(y,v,v))
\end{aligned}$$

Then by Corollary 2.2, F has a unique fixed point. Here $x=4-\sqrt{15}$ is the unique fixed point of F, that is, $F(x,x)=x$. ∎

Now we present an example for the main result in an asymmetric G_b-metric space.

EXAMPLE 2.3. Let $X = \{0, 1, 2\}$ and let

$$A = \{(2,0,0),(0,2,0),(0,0,2)\}, \qquad B = \{(2,2,0),(2,0,2),(0,2,2)\}$$
$$\text{and} \quad C = \{(x,x,x) : x \in X\}.$$

Define $G : X^3 \to \mathbb{R}^+$ by

$$G(x,y,z) = \begin{cases} 1, & \text{if } (x,y,z) \in A \\ 3, & \text{if } (x,y,z) \in B \\ 4, & \text{if } (x,y,z) \in X^3 - (A \cup B \cup C) \\ 0, & \text{if } x = y = z. \end{cases}$$

It is easy to see that (X, G) is an asymmetric G_b-metric space with coefficient $s = \frac{3}{2}$. Also, (X, G) is complete. Indeed, for each (x_n) in X such that $G(x_n, x_m, x_m) \to 0$, then there is a $k \in \mathbb{N}$ such that for each $n \geq k$, $x_n = x_m = x$ for an $x \in X$, so $G(x_n, x_n, x) \to 0$.

Define mappings F and g by

$$F = \begin{pmatrix} (0,0) & (0,1) & (1,0) & (1,1) & (1,2) & (2,1) & (2,2) & (2,0) & (0,2) \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$
$$g = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

We see that $F(X \times X) \subseteq gX$, g is continuous and commutes with F, and $g(X)$ is complete.

Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = \frac{27}{4} \ln(\frac{2t}{27} + 1)$. Since

$$(F(x,y), F(u,v), F(z,w)), (gx, gu, gz), (gy, gv, gw) \in A \cup B,$$

we have

$$G(F(x,y), F(u,v), F(z,w)), G(gx, gu, gz), G(gy, gv, gw) \in \{0, 1, 3\}.$$

Hence, one can easily check that the contractive condition (2) is satisfied for every $x, y, z, u, v, w \in X$.

Thus, all the conditions of Theorem 2.1 are fulfilled and F and g have a unique common fixed point. Here $F(0, 0) = g(0) = 0$. ■

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