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ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. In this paper we use the principle of subordination between analytic functions and the convolution to introduce the class $\tilde{S}(f,g;A,B;\alpha,\beta)$. We obtain coefficient inequalities, distortion theorems, extreme points, radii of close to convexity, starlikeness and convexity for this class and modified Hadamard product of several functions belonging to it. Also, we investigate several distortion inequalities involving fractional calculus. Finally, we obtain integral means for functions belonging to this class.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $g(z) \in \mathcal{A}$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k; \tag{1.2}$$

then the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$. Futhermore, if the function g is univalent in U, then we have the following equivalence (see [11]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

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For $0 \le \alpha < 1$, $\beta \ge 0$, $-1 \le B < A \le 1$, $-1 \le B < 0$ and $g(z)$ given by (1.2) with $b_k > 0$ $(k \ge 2)$, we denote by $S(f, g; A, B; \alpha, \beta)$ the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\frac{z(f*g)'(z)}{(f*g)(z)} - \beta\left|\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right| \prec (1-\alpha)\frac{1+Az}{1+Bz} + \alpha.$$

In other words, $f(z) \in S(f, g; A, B; \alpha, \beta)$ if and only if there exists a function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that

$$\left|\frac{\frac{z(f*g)'(z)}{(f*g)(z)} - \beta\left|\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right| - 1}{B\left[\frac{z(f*g)'(z)}{(f*g)(z)} - \beta\left|\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right|\right] - [B + (A-B)(1-\alpha)]}\right| < 1.$$

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \ge 0). \tag{1.3}$$

Further, we define the class $\tilde{S}(f, g; A, B; \alpha, \beta)$ as follows:

$$\tilde{S}(f, g; A, B; \alpha, \beta) = S(f, g; A, B; \alpha, \beta) \cap \mathcal{T}.$$

We note that for suitable choices of $g(z), A, B, \alpha$ and β, we obtain the following subclasses:

(1) $\tilde{S}(f, \frac{z}{1-z}; 1, -1; \alpha, 0) = T^*(\alpha)$ and $\tilde{S}(f, \frac{z}{(1-z)^2}; 1, -1; \alpha, 0) = K(\alpha)$ $(0 \le \alpha < 1)$ (see Silverman [15]);

(2) $\tilde{S}(f, \frac{z}{1-z}; A, B; \alpha, 0) = T^*(A, B, \alpha)$ and $\tilde{S}(f, \frac{z}{(1-z)^2}; A, B; \alpha, 0) = C(A, B, \alpha)$ $(-1 \le A < B \le 1,\ 0 < B \le 1,\ 0 \le \alpha < 1)$ (see Aouf [1, with $p = 1$]);

(3) $\tilde{S}(f, g; A, B; \alpha, \beta) = \tilde{E}_{m,n}(\Phi, \Psi; A, B, \alpha, \beta)$ $(0 \le \alpha < 1,\ \beta \ge 0,\ -1 \le B < A \le 1,\ -1 \le B < 0)$ (see Srivastava et al. [21] with $\Phi = \Psi = g$, $m = 1$ and $n = 0$);

(4) $\tilde{S}(f, \frac{z}{1-z}; \gamma, -\gamma; \alpha, 0) = S^*(\alpha, \gamma)$ and $\tilde{S}(f, \frac{z}{(1-z)^2}; \gamma, -\gamma; \alpha, 0) = C^*(\alpha, \gamma)$ $(0 \le \alpha < 1$ and $0 < \gamma \le 1)$ (see Gupta and Jain [8]);

(5) $\tilde{S}(f, \frac{z}{1-z}; A, B; 0, \beta) = \tilde{\mathcal{U}}(\beta, A, B)$ (see Li and Tang [9] with $m = 1$ and $n = 0$).

Also, we note that:

(1) $\tilde{S}\left(f, z + \sum_{k=2}^{\infty}\left[\frac{\ell+1+\lambda(k-1)}{\ell+1}\right]^m z^k; A, B; \alpha, \beta\right) = \left\{f \in \mathcal{T} : \frac{z(I^m(\lambda,\ell)f(z))'}{I^m(\lambda,\ell)f(z)} - \beta\left|\frac{z(I^m(\lambda,\ell)f(z))'}{I^m(\lambda,\ell)f(z)} - 1\right| \prec (1-\alpha)\frac{1+Az}{1+Bz} + \alpha\right\}$ $(0 \le \alpha < 1;\ \beta \ge 0;\ -1 \le B < A \le 1;\ -1 \le B < 0;\ m \in \mathbb{Z};\ \lambda \ge 0;\ \ell \ge 0$ and $z \in U)$, where the operator

$$I^m(\lambda, \ell)(z) = z + \sum_{k=2}^{\infty}\left[\frac{\ell+1+\lambda(k-1)}{\ell+1}\right]^m z^k,$$

was introduced and studied by Prajapat [13] (see also El-Ashwah and Aouf [7] and Catas [4]);

(2) $\tilde{S}(f, z+\sum\limits_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k; A, B; \alpha, \beta) = \left\{ f \in \mathcal{T} : \dfrac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - \right.$ $\left. \beta\left|\dfrac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - 1\right| \prec (1-\alpha)\dfrac{1+Az}{1+Bz} + \alpha \right\}$, $(0 \le \alpha < 1;\ \beta \ge 0;\ -1 \le B < A \le 1;\ -1 \le B < 0;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ q \le s+1$ and $z \in U)$, where the operator

$$H_{q,s}(\alpha_1)(z) = z + \sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k,$$
$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1}\dots(\alpha_q)_{k-1}}{(\beta_1)_{k-1}\dots(\beta_s)_{k-1}}\frac{1}{(k-1)!}\ ,$$

for real parameters $\alpha_1, \dots, \alpha_q$ and $\beta_1, \dots, \beta_s$, $\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$, $j = 1, 2, \dots, s$, was introduced and studied by Dziok and Srivastava [6].

In our present paper, we shall make use of the familiar integral operator $(J_c f)(z)$ defined by (see [3])

$$(J_c f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \quad (f \in \mathcal{A};\ c > -1).$$

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this paper that

$$0 \le \alpha < 1,\ \ \beta \ge 0,\ \ -1 \le B < A \le 1,\ \ -1 \le B < 0$$

and the function $g(z)$ is given by (1.2) with $b_k \ge b_2 > 0$ $(k \ge 2)$.

THEOREM 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $S(f, g; A, B; \alpha, \beta)$. Then*

$$\sum_{k=2}^{\infty}\left[(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)\right]b_k\left|a_k\right| \le (A-B)(1-\alpha). \tag{2.1}$$

Proof. Let the condition (2.1) hold true. Then we have

$$\begin{aligned}&\left|z(f*g)'(z) - \beta e^{i\theta}\left|z(f*g)'(z) - (f*g)(z)\right| - (f*g)(z)\right| - |(A-B)(1-\alpha)\times \\ &\quad\times(f*g)(z) - B[z(f*g)'(z) - \beta e^{i\theta}\left|z(f*g)'(z) - (f*g)(z)\right| - (f*g)(z)]| \\ &= \left|\sum_{k=2}^{\infty}(k-1)a_k b_k z^k - \beta e^{i\theta}\left|\sum_{k=2}^{\infty}(k-1)a_k b_k z^k\right|\right| - |(A-B)(1-\alpha)z + (A-B)\times \\ &\quad\times(1-\alpha)\sum_{k=2}^{\infty}a_k b_k z^k - B\left[\sum_{k=2}^{\infty}(k-1)a_k b_k z^k - \beta e^{i\theta}\left|\sum_{k=2}^{\infty}(k-1)a_k b_k z^k\right|\right]\Bigg| \\ &\le (1+\beta)\sum_{k=2}^{\infty}(k-1)b_k\left|a_k\right|\left|z\right|^k - (A-B)(1-\alpha)\left|z\right| + (A-B)(1-\alpha)\times\end{aligned}$$

$$\times \sum_{k=2}^{\infty} b_k |a_k| |z|^k + |B| (1+\beta) \sum_{k=2}^{\infty} (k-1) b_k |a_k| |z|^k$$

$$\leq (1-B)(1+\beta) \sum_{k=2}^{\infty} (k-1) b_k |a_k| + (A-B)(1-\alpha) \sum_{k=2}^{\infty} b_k |a_k| - (A-B)(1-\alpha)$$

$$\leq 0.$$

On simplification we easily arrive at the inequality (2.1). This completes the proof of Theorem 1. ■

Theorem 2. *The function $f(z)$ defined by* (1.3) *is in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$ if and only if*

$$\sum_{k=2}^{\infty} [(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)] a_k b_k \leq (A-B)(1-\alpha). \tag{2.2}$$

Proof. We only need to prove the "only if" part of Theorem 2. For functions $f(z) \in \mathcal{T}$, we can write

$$\left| \frac{\dfrac{z(f*g)'(z)}{(f*g)(z)} - \beta \left| \dfrac{z(f*g)'(z)}{(f*g)(z)} - 1 \right| - 1}{B\left[\dfrac{z(f*g)'(z)}{(f*g)(z)} - \beta \left| \dfrac{z(f*g)'(z)}{(f*g)(z)} - 1 \right|\right] - [B + (A-B)(1-\alpha)]} \right|$$

$$= \left| \frac{z(f*g)'(z) - \beta e^{i\theta} \left| z(f*g)'(z) - (f*g)(z) \right| - (f*g)(z)}{B[z(f*g)'(z) - \beta e^{i\theta} |z(f*g)'(z) - (f*g)(z)|] - [B + (A-B)(1-\alpha)](f*g)(z)} \right|$$

$$\leq \left| \frac{(1+\beta e^{i\theta}) \sum_{k=2}^{\infty} (k-1) a_k b_k z^{k-1}}{(A-B)(1-\alpha) - (A-B)(1-\alpha) \sum_{k=2}^{\infty} a_k b_k z^{k-1} + (1+\beta e^{i\theta}) B \sum_{k=2}^{\infty} (k-1) a_k b_k z^{k-1}} \right|.$$

Since $\mathrm{Re}\{z\} \leq |z| \ (z \in U)$, we thus find that

$$\mathrm{Re} \left\{ \frac{(1+\beta e^{i\theta}) \sum_{k=2}^{\infty} (k-1) a_k b_k z^{k-1}}{(A-B)(1-\alpha) - (A-B)(1-\alpha) \sum_{k=2}^{\infty} a_k b_k z^{k-1} + (1+\beta e^{i\theta}) B \sum_{k=2}^{\infty} (k-1) a_k b_k z^{k-1}} \right\} < 1.$$

If we now choose z to be real and let $z \to 1^-$, we get

$$\sum_{k=2}^{\infty} [(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)] a_k b_k \leq (A-B)(1-\alpha),$$

which is equivalent to (2.2). ■

Corollary 1. *Let the function $f(z)$ defined by* (1.3) *be in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$. Then*

$$a_k \leq \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)] b_k}.$$

The result is sharp for the function

$$f(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)]\, b_k} z^k. \tag{2.3}$$

3. Distortion theorems

THEOREM 3. *Let the function $f(z)$ defined by* (1.3) *be in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$; then for $z \in U$, we have*

$$|z| - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta) + (A-B)(1-\alpha)]\, b_2} |z|^2 \le |f(z)| \le |z| + \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta) + (A-B)(1-\alpha)]\, b_2} |z|^2. \tag{3.1}$$

Furthermore,

$$1 - \frac{2(A-B)(1-\alpha)}{[(1-B)(1+\beta) + (A-B)(1-\alpha)]\, b_2} |z| \le |f'(z)| \le 1 + \frac{2(A-B)(1-\alpha)}{[(1-B)(1+\beta) + (A-B)(1-\alpha)]\, b_2} |z|. \tag{3.2}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta) + (A-B)(1-\alpha)]\, b_2} z^2. \tag{3.3}$$

Proof. It is easy to see from Theorem 2 that

$$\begin{aligned} &[(1-B)(1+\beta) + (A-B)(1-\alpha)] b_2 \sum_{k=2}^{\infty} a_k \\ &\quad \le \sum_{k=2}^{\infty} [(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)]\, a_k b_k \\ &\quad \le (A-B)(1-\alpha). \end{aligned}$$

Then

$$\sum_{k=2}^{\infty} a_k \le \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta) + (A-B)(1-\alpha)]\, b_2}. \tag{3.4}$$

Making use of (3.4), we have

$$\begin{aligned} |f(z)| &\ge |z| - |z|^2 \sum_{k=2}^{\infty} a_k \\ &\ge |z| - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta) + (A-B)(1-\alpha)]\, b_2} |z|^2, \end{aligned}$$

and

$$\begin{aligned}|f(z)| &\leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \\ &\leq |z| + \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2} |z|^2 ,\end{aligned}$$

which proves the assertion (3.1).

From (3.4) and Theorem 2, it follows also that

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2}.$$

Consequently, we have

$$\begin{aligned}|f'(z)| &\geq 1 - |z| \sum_{k=2}^{\infty} k a_k \\ &\geq 1 - \frac{2(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2} |z| ,\end{aligned}$$

and

$$\begin{aligned}|f'(z)| &\leq 1 + |z| \sum_{k=2}^{\infty} k a_k \\ &\leq 1 + \frac{2(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2} |z| ,\end{aligned}$$

which proves the assertion (3.2). Since each of equalities in (3.1) and (3.2) is satisfied by the function $f(z)$ given by (3.3), our proof of Theorem 3 is thus completed. ■

4. Closure theorems

We will consider the functions $f_j(z)$ defined, for $j = 1, 2, \ldots, m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0). \tag{4.1}$$

THEOREM 4. *Let the functions $f_j(z)$ be in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$. Then the function $h(z)$ defined by*

$$h(z) = z - \sum_{k=2}^{\infty} \Big(\frac{1}{m} \sum_{j=1}^{m} a_{k,j} \Big) z^k,$$

also belongs to the class $\tilde{S}(f, g; A, B; \alpha, \beta)$.

Proof. Since $f_j(z)$ $(j = 1, 2, \ldots, m)$ are in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$, it follows from Theorem 2 that

$$\sum_{k=2}^{\infty} [(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)]\, a_{k,j} b_k \leq (A-B)(1-\alpha),$$

for every $j = 1, 2, \ldots, m$. Hence

$$\sum_{k=2}^{\infty} [(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)]\, b_k \Big(\frac{1}{m}\sum_{j=1}^{m} a_{k,j}\Big)$$
$$= \frac{1}{m}\sum_{j=1}^{m}\Big(\sum_{k=2}^{\infty} [(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)]\, a_{k,j} b_k\Big)$$
$$\leq (A-B)(1-\alpha).$$

From Theorem 2, it follows that $h(z) \in \tilde{S}(f, g; A, B; \alpha, \beta)$. This completes the proof of Theorem 4. ∎

Corollary 2. *The class $\tilde{S}(f, g; A, B; \alpha, \beta)$ is closed under convex linear combinations.*

Proof. Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$. Then it is sufficient to show that the function

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \ (0 \leq \lambda \leq 1),$$

is in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$. Since for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{k=2}^{\infty} [\lambda a_{k,1} + (1-\lambda) a_{k,2}] z^k,$$

with the aid of Theorem 2, we have

$$\sum_{k=2}^{\infty} [(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)]\, [\lambda a_{k,1} + (1-\lambda) a_{k,2}] b_k$$
$$\leq \lambda (A-B)(1-\alpha) + (1-\lambda)(A-B)(1-\alpha)$$
$$= (A-B)(1-\alpha),$$

which implies that $h(z) \in \tilde{S}(f, g; A, B; \alpha, \beta)$. ∎

Theorem 5. *Let $f_1(z) = z$ and*

$$f_k(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)]\, b_k} z^k.$$

Then $f(z)$ is in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{4.2}$$

where $\mu_k \geq 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$$
$$= z - \sum_{k=2}^{\infty} \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)(k-1) + (A-B)(1-\alpha)]\, b_k} \mu_k z^k.$$

Then it follows that

$$\sum_{k=2}^{\infty} \tfrac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)} \cdot \tfrac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k} \mu_k$$
$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1.$$

which implies that $f(z) \in \tilde{S}(f, g; A, B; \alpha, \beta)$.

Conversely, assume that the function $f(z)$ defined by (1.3) be in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$. Then

$$a_k \leq \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]\, b_k}.$$

Setting

$$\mu_k = \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]\, b_k}{(A-B)(1-\alpha)} a_k,$$

where $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$, we can see that $f(z)$ can be expressed in the form (4.2). This completes the proof of Theorem 5. ■

Corollary 3. *The extreme points of the class $\tilde{S}(f, g; A, B; \alpha, \beta)$ are the functions $f_1(z) = z$ and*

$$f_k(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]\, b_k} z^k.$$

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 6. *Let the function $f(z)$ defined by* (1.3) *be in the class $\tilde{S}(f, g; A, B; \alpha, \beta)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in $|z| \leq r_1$, where*

$$r_1 = \inf_{k \geq 2} \left\{ \frac{(1-\delta)\,[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]\, b_k}{k(A-B)(1-\alpha)} \right\}^{\frac{1}{k-1}}. \tag{5.1}$$

The result is sharp, the extremal function given by (2.3).

Proof. We must show that

$$|f'(z) - 1| \leq 1 - \delta \quad \text{for} \quad |z| \leq r_1,$$

where r_1 is given by (5.1). Indeed we find from (1.3) that

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k a_k \, |z|^{k-1}.$$

Thus $|f'(z) - 1| \leq 1 - \delta$ if

$$\sum_{k=2}^{\infty} \frac{k}{(1-\delta)} a_k \, |z|^{k-1} \leq 1. \tag{5.2}$$

But by using Theorem 2, (5.2) will be true if

$$\frac{k}{(1-\delta)}\left|z\right|^{k-1} \leq \frac{\left[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)\right]b_k}{(A-B)(1-\alpha)}.$$

Then

$$\left|z\right| \leq \left\{\frac{(1-\delta)\left[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)\right]b_k}{k(A-B)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{5.3}$$

The result follows easily from (5.3). This completes the proof of Theorem 6. ■

THEOREM 7. *Let the function* $f(z)$ *defined by* (1.3) *be in the class* $\tilde{S}(f,g;A,B;\alpha,\beta)$. *Then* $f(z)$ *is starlike of order* δ $(0 \leq \delta < 1)$ *in* $|z| \leq r_2$, *where*

$$r_2 = \inf_{k\geq 2}\left\{\frac{(1-\delta)\left[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)\right]b_k}{(k-\delta)(A-B)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{5.4}$$

The result is sharp, the extremal function given by (2.3).

Proof. We must show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\delta \text{ for } |z| \leq r_2,$$

where r_2 is given by (5.4). Indeed we find from (1.3) that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k\left|z\right|^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k\left|z\right|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\delta$, if

$$\sum_{k=2}^{\infty}\left(\frac{k-\delta}{1-\delta}\right)a_k\left|z\right|^{k-1} \leq 1. \tag{5.5}$$

But by using Theorem 2, (5.5) will be true if

$$\left(\frac{k-\delta}{1-\delta}\right)\left|z\right|^{k-1} \leq \frac{\left[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)\right]b_k}{(A-B)(1-\alpha)}.$$

Then

$$\left|z\right| \leq \left\{\frac{(1-\delta)\left[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)\right]b_k}{(k-\delta)(A-B)(1-\alpha)}\right\}^{\frac{1}{k-1}}. \tag{5.6}$$

The result follows easily from (5.6). This completes the proof of Theorem 7. ■

Corollary 4. *Let the function $f(z)$ defined by* (1.3) *be in the class* $\tilde{S}(f,g;A,B;\alpha,\beta)$. *Then $f(z)$ is convex of order δ $(0 \le \delta < 1)$ in $|z| \le r_3$, where*

$$r_3 = \inf_{k\ge 2}\left\{\frac{(1-\delta)\left[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)\right]b_k}{k(k-\delta)(A-B)(1-\alpha)}\right\}^{\frac{1}{k-1}}.$$

The result is sharp, with the extremal function given by (2.3).

Remark 1. Putting $g(z)=\frac{z}{1-z}$ and $\alpha = 0$ in our results, we obtain the results obtained by Li and Tang [9, with $m=1$ and $n=0$].

6. Definitions and applications of fractional calculus

Many essentially equivalent definitions of fractional calculus (that is, fractional derivatives and fractional integrals) have been given in the literature (cf., e.g. [2], [18] and [20]. We find it to be convenient to recall here the following definitions which were used recently by Owa [12] and by Srivastava and Owa [19]).

Definition 1. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)}\int_0^z \frac{f(t)}{(z-t)^{1-\mu}}\,dt \quad (\mu > 0),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

Definition 2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z \frac{f(t)}{(z-t)^{\mu}}\,dt \quad (0 \le \mu < 1),$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane containing the origin and the multiplicity of $(z-t)^{-\mu}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $n+\mu$ is defined by

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \quad (0 \le \mu < 1;\;\; n \in \mathbb{N}_0).$$

In order to derive our results, we need the following lemma given by Chen et al. [5].

Lemma 1 (see Chen et al. [5 with $p=1$]). *Let the function $f(z)$ be defined by* (1.3). *Then*

$$D_z^{\mu}\{(J_c f)(z)\} = \frac{1}{\Gamma(2-\mu)}z^{1-\mu} - \sum_{k=2}^{\infty}\frac{(c+1)\Gamma(k+1)}{(c+k)\Gamma(k-\mu+1)}a_k z^{k-\mu} \;\; (\mu\in\mathbb{R}; c>-1), \tag{6.1}$$

and

$$J_c(D_z^{\mu}\{f(z)\}) = \frac{(c+1)}{(c-\mu+1)\Gamma(2-\mu)} z^{1-\mu} - \sum_{k=2}^{\infty} \frac{(c+1)\Gamma(k+1)}{(k-\mu+c)\Gamma(k-\mu+1)} a_k z^{k-\mu} \tag{6.2}$$

$(\mu \in \mathbb{R};\ c > -1)$, *provided that no zeros appear in the denominators in* (6.1) *and* (6.2).

THEOREM 8. *Let the function* $f(z)$ *defined by* (1.3) *be in the class* $\tilde{S}(f, g; A, B; \alpha, \beta)$. *Then we have*

$$\left|D_z^{-\mu}\{(J_c f)(z)\}\right| \geq \tfrac{1}{\Gamma(2+\mu)} |z|^{\mu} \left\{|z| - \tfrac{2(A-B)(1-\alpha)(c+1)}{(2+\mu)(c+2)[(1-B)(1+\beta)+(A-B)(1-\alpha)]b_2} |z|^2\right\},$$

and

$$\left|D_z^{-\mu}\{(J_c f)(z)\}\right| \leq \tfrac{1}{\Gamma(2+\mu)} |z|^{\mu} \left\{|z| + \tfrac{2(A-B)(1-\alpha)(c+1)}{(2+\mu)(c+2)[(1-B)(1+\beta)+(A-B)(1-\alpha)]b_2} |z|^2\right\},$$

for $\mu > 0$ *and* $z \in U$. *The result is sharp.*

Proof. Let

$$F(z) = \Gamma(2+\mu) z^{-\mu} D_z^{-\mu}\{(J_c f)(z)\} = z - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)(c+1)}{\Gamma(k+\mu+1)(k+c)} a_k z^k.$$

Then

$$F(z) = z - \sum_{k=2}^{\infty} \Phi(k) a_k z^k, \tag{6.3}$$

where $\Phi(k) = \dfrac{\Gamma(k+1)\Gamma(2+\mu)(c+1)}{\Gamma(k+\mu+1)(k+c)}$ $(\mu > 0)$. Since $\Phi(k)$ is a decreasing function of k $(k \geq 2)$, then

$$0 < \Phi(k) \leq \Phi(2) = \frac{2(c+1)}{(2+\mu)(c+2)}. \tag{6.4}$$

From (6.3) and (6.4), we have

$$|F(z)| \geq |z| - \Phi(2) |z|^2 \sum_{k=2}^{\infty} a_k. \tag{6.5}$$

In view of (3.4) and (6.5), we have

$$\begin{aligned} |F(z)| &= \left|\Gamma(2+\mu) z^{-\mu} D_z^{-\mu}\{(J_c f)(z)\}\right| \\ &\geq |z| - \frac{2(A-B)(1-\alpha)(c+1)}{(2+\mu)(c+2)\left[(1-B)(1+\beta)+(A-B)(1-\alpha)\right] b_2} |z|^2, \end{aligned}$$

and

$$\begin{aligned} |F(z)| &= \left|\Gamma(2+\mu) z^{-\mu} D_z^{-\mu}\{(J_c f)(z)\}\right| \\ &\leq |z| + \frac{2(A-B)(1-\alpha)(c+1)}{(2+\mu)(c+2)\left[(1-B)(1+\beta)+(A-B)(1-\alpha)\right] b_2} |z|^2. \end{aligned}$$

which proves the inequalities of Theorem 8. Further, equalities are attained for the function

$$D_z^{-\mu}\{(J_cf)(z)\} = \frac{1}{\Gamma(2+\mu)} z^{\mu}\left\{z - \frac{2(A-B)(1-\alpha)(c+1)}{(2+\mu)(c+2)[(1-B)(1+\beta)+(A-B)(1-\alpha)]b_2} z^2\right\},$$

or by $f(z)$ given by (3.3). ∎

Using similar arguments to those in the proof of Theorem 8, we obtain the following theorem.

Theorem 9. *Let the function* $f(z)$ *defined by* (1.3) *be in the class* $\tilde{S}(f,g;A,B;\alpha,\beta)$. *Then we have*

$$|D_z^{\mu}\{(J_cf)(z)\}| \geq \frac{1}{\Gamma(2-\mu)} |z|^{-\mu}\left\{|z| - \frac{2(A-B)(1-\alpha)(c+1)}{(2-\mu)(c+2)[(1-B)(1+\beta)+(A-B)(1-\alpha)]b_2} |z|^2\right\},$$

and

$$|D_z^{\mu}\{(J_cf)(z)\}| \leq \frac{1}{\Gamma(2-\mu)} |z|^{-\mu}\left\{|z| + \frac{2(A-B)(1-\alpha)(c+1)}{(2-\mu)(c+2)[(1-B)(1+\beta)+(A-B)(1-\alpha)]b_2} |z|^2\right\},$$

for $0 \leq \mu < 1$ *and* $z \in U$. *The result is sharp for the function* $f(z)$ *given by* (3.3).

7. Modified Hadamard products

Let the functions $f_j(z)$ $(j = 1,2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=2}^{\infty} a_{k,1}a_{k,2}z^k = (f_2 * f_1)(z).$$

Theorem 10. *Let the functions* $f_j(z)$ $(j = 1,2)$ *be in the class* $\tilde{S}(f,g;A,B;\alpha,\beta)$. *Then* $(f_1 * f_2)(z) \in \tilde{S}(f,g;A,B;\eta,\beta)$, *where*

$$\eta = 1 - \frac{(A-B)(1-\alpha)^2(1-B)(1+\beta)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]^2 b_2 - (A-B)^2(1-\alpha)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2)$ *given by*

$$f_j(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]b_2} z^2 \quad (j = 1,2). \tag{7.1}$$

Proof. Employing the technique used earlier by Schild and Silverman [14], we need to find the largest η such that

$$\sum_{k=2}^{\infty} \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\eta)]b_k}{(A-B)(1-\eta)} a_{k,1}a_{k,2} \leq 1.$$

Since $f_j(z) \in \tilde{S}(f,g;A,B;\alpha,\beta)$ $(j = 1,2)$, we readily see that

$$\sum_{k=2}^{\infty} \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)} a_{k,1} \leq 1,$$

and
$$\sum_{k=2}^{\infty} \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)} a_{k,2} \leq 1.$$
By the Cauchy-Schwarz inequality we have
$$\sum_{k=2}^{\infty} \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{7.2}$$
Thus it is sufficient to show that
$$\frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\eta)]b_k}{(A-B)(1-\eta)} a_{k,1}a_{k,2} \leq \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)} \sqrt{a_{k,1}a_{k,2}},$$
or, equilvalently, that
$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)](1-\eta)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\eta)](1-\alpha)}.$$
Hence, in the light of inequality (7.2), it is sufficient to prove that
$$\frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k} \leq \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)](1-\eta)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\eta)](1-\alpha)}. \tag{7.3}$$
It follows from (7.3) that
$$\eta \leq 1 - \frac{(A-B)(1-\alpha)^2(1-B)(1+\beta)(k-1)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]^2 b_k - (A-B)^2(1-\alpha)^2}.$$
Now defining the function $D(k)$ by
$$D(k) = 1 - \frac{(A-B)(1-\alpha)^2(1-B)(1+\beta)(k-1)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]^2 b_k - (A-B)^2(1-\alpha)^2},$$
we see that $D(k)$ is an increasing function of k $(k \geq 2)$. Therefore, we conclude that
$$\eta \leq D(2) = 1 - \frac{(A-B)(1-\alpha)^2(1-B)(1+\beta)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]^2 b_2 - (A-B)^2(1-\alpha)^2},$$
which evidently completes the proof of Theorem 10. ■

Using similar arguments to those in the proof of Theorem 10, we obtain the following theorem.

Theorem 11. *Let the function $f_1(z)$ defined by (4.1) be in the class $\tilde{S}(f,g;A,B;\alpha,\beta)$. Suppose also that the function $f_2(z)$ defined by (4.1) be in the class $\tilde{S}(f,g;A,B;\phi,\beta)$. Then $(f_1 * f_2)(z) \in \tilde{S}(f,g;A,B;\zeta,\beta)$, where*
$$\zeta = 1 - \frac{(A-B)(1-\alpha)(1-\phi)(1-B)(1+\beta)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)][(1-B)(1+\beta)+(A-B)(1-\phi)]b_2-(A-B)^2(1-\alpha)(1-\phi)},$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ *given by*

$$f_1(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]b_2}z^2,$$
$$f_2(z) = z - \frac{(A-B)(1-\phi)}{[(1-B)(1+\beta)+(A-B)(1-\phi)]b_2}z^2.$$

Theorem 12. *Let the functions* $f_j(z)$ $(j=1,2)$ *defined by* (4.1) *be in the class* $\tilde{S}(f,g;A,B;\alpha,\beta)$. *Then the function*

$$h(z) = z - \sum_{k=2}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k$$

belongs to the class $\tilde{S}(f,g;A,B;\varphi,\beta)$, *where*

$$\varphi = 1 - \frac{2(A-B)(1-\alpha)^2(1-B)(1+\beta)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]^2 b_2 - 2(A-B)^2(1-\alpha)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ *defined by* (7.1).

Proof. By using Theorem 2, we obtain

$$\sum_{k=2}^{\infty}\left\{\frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)}\right\}^2 a_{k,1}^2 \le \left\{\sum_{k=2}^{\infty}\frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)}a_{k,1}\right\}^2 \le 1, \quad (7.4)$$

and

$$\sum_{k=2}^{\infty}\left\{\frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)}\right\}^2 a_{k,2}^2 \le \left\{\sum_{k=2}^{\infty}\frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)}a_{k,2}\right\}^2 \le 1. \quad (7.5)$$

It follows from (7.4) and (7.5) that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left\{\frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)}\right\}^2 (a_{k,1}^2 + a_{k,2}^2) \le 1.$$

Therefore, we need to find the largest φ such that

$$\frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\varphi)]b_k}{(A-B)(1-\varphi)} \le \frac{1}{2}\left\{\frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]b_k}{(A-B)(1-\alpha)}\right\}^2,$$

that is

$$\varphi \leq 1 - \frac{2(A-B)(1-\alpha)^2(1-B)(1+\beta)(k-1)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]^2 b_k - 2(A-B)^2(1-\alpha)^2}.$$

Since

$$G(k) = 1 - \frac{2(A-B)(1-\alpha)^2(1-B)(1+\beta)(k-1)}{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]^2 b_k - 2(A-B)^2(1-\alpha)^2},$$

is an increasing function of k $(k \geq 2)$, we obtain

$$\varphi \leq G(2) = 1 - \frac{2(A-B)(1-\alpha)^2(1-B)(1+\beta)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]^2 b_2 - 2(A-B)^2(1-\alpha)^2},$$

and hence the proof of Theorem 12 is completed. ■

8. Integral means

In this section integral means for functions belonging to the class $\tilde{S}(f, g; A, B; \alpha, \beta)$ are obtained. In [15], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$. He applied this function to resolve his integral means inequality, conjectured in [16] and settled in [17], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta \, d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta \, d\theta,$$

for all $f \in \mathcal{T}, \eta > 0$ and $0 < r < 1$.

In 1925, Littlewood [10] proved the following lemma.

Lemma 2. *If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$ and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta \, d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta \, d\theta.$$

Applying Lemma 2, Theorem 2, and Corollary 3, we prove the following theorem.

Theorem 13. *Suppose $f(z) \in \tilde{S}(f, g; A, B; \alpha, \beta)$, $\eta > 0$, $0 < r < 1$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2} z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{8.1}$$

Proof. For $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ $(a_k \geq 0)$, (8.1) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^{\infty} a_k z^{k-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2} z\right|^\eta d\theta.$$

Using Lemma 2, it suffices to show that

$$1-\sum_{k=2}^{\infty} a_k z^{k-1} \prec 1-\frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2} z.$$

Setting

$$1-\sum_{k=2}^{\infty} a_k z^{k-1} = 1-\frac{(A-B)(1-\alpha)}{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2} w(z),$$

and using Theorem 2, we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{k=2}^{\infty} \frac{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2}{(A-B)(1-\alpha)} a_k z^{k-1} \right| \\ &\le |z| \sum_{k=2}^{\infty} \frac{[(1-B)(1+\beta)+(A-B)(1-\alpha)]\, b_2}{(A-B)(1-\alpha)} a_k \\ &\le |z| \sum_{k=2}^{\infty} \frac{[(1-B)(1+\beta)(k-1)+(A-B)(1-\alpha)]\, b_k}{(A-B)(1-\alpha)} a_k \\ &\le |z| . \end{aligned}$$

This completes the proof of Theorem 13. ∎

Remark 2.

(1) Putting $g(z)=\frac{z}{1-z}$ and $\beta=0$ in our results, we obtain some analogous results for Aouf [1, with $p=1$];

(2) Putting $g(z)=\frac{z}{(1-z)^2}$ and $\beta=0$ in our results, we obtain some analogous results for Aouf [1, with $p=1$];

(3) Putting $\Phi=\Psi=g$ in our results, we obtain some analogous results for Srivastava et al. [21, with $m=1$ and $n=0$];

(4) Putting $g(z)=\frac{z}{1-z}$, $A=\gamma$, $B=-\gamma$ and $\beta=0$ in our results, we obtain some analogous results for Gupta and Jain [8];

(5) Putting $g(z)=\frac{z}{(1-z)^2}$, $A=\gamma$, $B=-\gamma$ and $\beta=0$ in our results, we obtain some analogous results for Gupta and Jain [8].

Applications.

We can derive new results for the class $\tilde{S}(f,g;A,B;\alpha,\beta)$ by taking $g(z)$ as follows:

(1) $g(z)=z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)z^k$ (see [6]), where $\Gamma_k(\alpha_1)$ is given by (1.9);

(2) $g(z)=z+\sum_{k=2}^{\infty}\left[\frac{\ell+1+\lambda(k-1)}{\ell+1}\right]^m z^k$ (see [4], [7], [13]), where $\lambda\ge 0$, $\ell\ge 0$, $m\in\mathbb{Z}$.

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