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ON COUNTABLY NEARLY PARACOMPACT SPACES

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Abstract. The idea of countable near paracompactness of a topological space, as a natural offshoot of the well known concept of near paracompactness, is introduced and investigated in this article. In the process, the notion of semi nearly normal spaces is initiated. Apart from its characterizations, semi near normality is used for the investigation of countably nearly paracompact spaces.

1. Introduction

The idea of countable paracompactness is well known for a long time, which was introduced independently by Dowker [2] and Katětov [4].

The concept of nearly paracompact spaces was introduced by Singal and Arya [12], followed by its numerous investigations, see, e.g., [5,7,8,11]. Although it is observed that near paracompactness of a topological space (X, τ) is nothing but the paracompactness of the semiregularization space (X, τ_s) (where τ_s is the topology on X given by the set of all regular open sets of X taken as an open base), near paracompactness possesses interesting and meaningful entity and facets of its own. In [10], near paracompactness is studied in terms of cushioned refinement, local star properties, partition of unity and above all, by use of the theory of selection.

The purpose of this paper is to introduce and study the notion of countably nearly paracompact spaces, as a natural offshoot of the idea of near paracompactness. It is seen in Section 2, that countable near paracompactness is strictly weaker than each of countable paracompactness and near paracompactness.

In Section 3, we introduce the concept of semi nearly normal space; it is shown that the class of such spaces is situated strictly between the classes of normal spaces and nearly normal spaces, the latter class of spaces being initiated and studied in [8,9]. We examine certain characterizations of semi nearly normal spaces including a parallel version of the famous Urysohn's Lemma. The condition for a semi nearly normal spaces to be countably nearly paracompact is also formulated and proved.

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In what follows, by a space X we shall mean a topological space (X, τ), and for a subset A of X, $\operatorname{cl} A$ and $\operatorname{int} A$ will stand respectively for the closure and interior of A in (X, τ). A set A in a space X is called regular open if $A = \operatorname{int} \operatorname{cl} A$, the complements of such sets are called regular closed sets. The set of all regular open sets in a space (X, τ), denoted by $RO(X)$, is known to form an open base for a topology, say τ_s, on X, called the semiregularization topology on X [1], and the space (X, τ_s) which is denoted by $(X)_s$, is called the semiregularization space of (X, τ). The members of τ_s are called δ-open sets [14], and their complements in X are called δ-closed sets. We shall sometimes write A^* for $\operatorname{int} \operatorname{cl} A$ and $\mathcal{C}^\# = \{A^* : A \in \mathcal{C}\}$, for any open cover $\mathcal{C}$ of a space (X, τ). A space X is called nearly paracompact [12] if every regular open cover of X has a locally finite open refinement.

2. Countably nearly paracompact spaces

As already proposed in the introduction, the intent of this section is to introduce and investigate countably nearly paracompact spaces. It will be revealed in the course of the future deliberation that the concept of near normality plays the same role vis-a-vis countable near paracompactness, as does the normality with regard to countable paracompactness. The definition of countable near paracompactness goes as follows.

Definition 2.1. A topological space (X, τ) is called countably nearly paracompact (CNP, for short) if every countable regular open cover of X (i.e., a cover of X consisting of countably many regular open sets) has a locally finite open refinement.

Remark 2.2. Obviously, every countably paracompact space is countably nearly paracompact and every nearly paracompact space is CNP. We now give examples to show that the converse is false in each of the cases.

Example 2.3. Let $\mathbb{R}$ be the set of all real numbers and τ the co-countable topology on $\mathbb{R}$. Let $\{x_n : n \in \mathbb{N}\}$ be an enumeration of the set $\mathbb{Q}$ of all rationals. For each $n \in \mathbb{N}$, let $U_n = (\mathbb{R} \setminus \mathbb{Q}) \cup \{x_n\}$. Then $\{U_n : n \in \mathbb{N}\}$ is a countable open cover of $(\mathbb{R}, \tau)$, which fails to possess any locally finite open refinement and hence $(\mathbb{R}, \tau)$ is not countably paracompact. On the other hand, the space $(\mathbb{R}, \tau)$ is countably nearly paracompact, as $\mathbb{R}$ and $\emptyset$ are the only regular open sets in $(\mathbb{R}, \tau)$.

Example 2.4. The space ω_1 of all countable ordinals is known to be countably compact and hence countably nearly paracompact. Now, ω_1 is regular but not paracompact. As a regular, nearly paracompact space is paracompact [12], ω_1 cannot be nearly paracompact.

The idea of near normality was initiated in [8] in the following way:

Definition 2.5. A topological space X is called nearly normal if for any pair of nonempty disjoint sets A and B in X, of which one is δ-closed and the other regular closed, there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

We now give some necessary conditions for a nearly normal space X to be countably nearly paracompact. Before that we need the following definition.

Definition 2.6. [15] An open cover $\mathcal{B} = \{B_\alpha : \alpha \in \Lambda\}$ of a space X is called shrinkable if there exists an open cover $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ of X such that $\mathrm{cl}(A_\alpha) \subseteq B_\alpha$, for all $\alpha \in \Lambda$. In this case, we say that $\mathcal{A}$ is a shrinking of $\mathcal{B}$.

In [9], it is proved that in a nearly normal space, every point-finite regular open cover is shrinkable. We now prove the following implications:

Theorem 2.7. *For a nearly normal space X, the following implications hold:*
(a) X is countably nearly paracompact.
$\Rightarrow (b)$ Every countable regular open cover of X is shrinkable.
$\Rightarrow (c)$ For every sequence $\{F_n : n \in \mathbb{N}\}$ of regular closed sets with empty intersection, there is a sequence $\{G_n : n \in \mathbb{N}\}$ of open sets with $G_n \supseteq F_n$ for each $n \in \mathbb{N}$, such that $\bigcap_{n=1}^{\infty} G_n = \emptyset$.

Proof. $(a) \Rightarrow (b)$. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable regular open cover of X. Since X is CNP, there exists a locally finite open cover $\mathcal{C} = \{A_\alpha : \alpha \in \Lambda\}$ of X such that $\mathcal{C}$ refines $\mathcal{B}$. For each $n \in \mathbb{N}$, let

$$A_n = \bigcup\{A_\alpha \in \mathcal{C} : A_\alpha \subseteq B_n \text{ and } A_\alpha \nsubseteq B_m \text{ if } m < n\} \text{ and } \mathcal{A} = \{A_n : n \in \mathbb{N}\}.$$

Obviously, $\mathcal{A}$ is a sequence of open sets in X such that $A_n \subseteq B_n$, for all $n \in \mathbb{N}$. We now show that $\mathcal{A}$ is locally finite and a cover of X. Let x be an arbitrary element of X. Since $\mathcal{C}$ is a cover of X, there exists $A_\alpha \in \mathcal{C}$ such that $x \in A_\alpha$. As $\mathcal{C}$ refines $\mathcal{B}$, there exists some $k \in \mathbb{N}$ such that $A_\alpha \subseteq B_k$. Let $\sigma = \{n \in \mathbb{N} : A_\alpha \subseteq B_n\}$. Since $k \in \sigma$, we have $\sigma \neq \emptyset$. Let n_0 be the least of such k's. Then $A_\alpha \subseteq B_{n_0}$, and $A_\alpha \nsubseteq B_n$ if $n < n_0$. This implies that $A_\alpha \subseteq A_{n_0}$, i.e., $x \in A_{n_0}$, proving $\mathcal{A}$ to be a cover of X.

We prove that $\mathcal{A}$ is locally finite. Let $x \in X$. Since $\mathcal{C}$ is locally finite, there is an open neighbourhood U of x in X which intersects at most finitely many members of $\mathcal{C}$, i.e., there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Lambda$ such that $U \cap A_{\alpha_i} \neq \emptyset$ $(i = 1, 2, \ldots, k)$ and $U \cap A_\alpha = \emptyset$ if $\alpha \neq \alpha_i$ $(i = 1, 2, \ldots, k)$. Since $\mathcal{C}$ refines $\mathcal{B}$, there exist $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that $A_{\alpha_i} \subseteq B_{n_i}$ $(i = 1, 2, \ldots, k)$. Let $n_0 = \max\,(n_1, n_2, \ldots, n_k)$. By definition, each A_n is the union of some A_α's and if $n > n_0$, none of these A_α's which constitute A_n can be equal to any of A_{α_i}'s $(i = 1, 2, \ldots, k)$. Thus, $U \cap A_n = \emptyset$, for all $n > n_0$. This shows that U can intersect at most finitely many members of $\mathcal{A}$. Thus, $\mathcal{A}$ is a locally finite open refinement of $\mathcal{B}$. Since $\mathcal{B} \subseteq RO(X)$, $\mathcal{A}^{\#} = \{A_n^* : n \in \mathbb{N}\}$ is also a locally finite open refinement of $\mathcal{B}$, as can be checked. Since every locally finite family is point finite, by near normality of X, there exists an open cover $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ of X which is a shrinking of $\mathcal{A}^{\#}$ (see [9]). Since $\mathcal{A}^{\#}$ refines $\mathcal{B}$, it follows that $\mathcal{D}$ is also a shrinking of $\mathcal{B}$. This proves '$(a) \Rightarrow (b)$'.

$(b) \Rightarrow (c)$. Let $\{F_n : n \in \mathbb{N}\}$ be a sequence of regular closed sets in X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then $\{X \setminus F_n : n \in \mathbb{N}\}$ is a regular open cover of X and hence, by

(b), it has a shrinking, i.e., there is an open cover $\{V_n : n \in \mathbb{N}\}$ of X such that $\mathrm{cl}(V_n) \subseteq X \setminus F_n$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $G_n = X \setminus \mathrm{cl}(V_n)$. Then $\{G_n : n \in \mathbb{N}\}$ is a sequence of open sets in X with $F_n \subseteq G_n$, for all $n \in \mathbb{N}$ and also $\bigcap_{n=1}^{\infty} G_n = X \setminus \bigcup_{n=1}^{\infty} \mathrm{cl}(V_n) = \emptyset$, as $\{V_n : n \in \mathbb{N}\}$ covers X. This proves '$(b) \Rightarrow (c)$'. ■

3. Semi near normality and CNP spaces

In this section, we shall introduce and study, an intermediate version of near normality, termed semi near normality. We also obtain a few sufficient conditions for countably near paracompactness. We begin by defining as follows.

DEFINITION 3.1. A topological space (X, τ) is called semi nearly normal, if for every pair of nonempty disjoint sets A and B in X, of which one is δ-closed and the other closed, there exist disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

REMARK 3.2. We immediately have the following implications for any space X:

$$X \text{ is normal} \Rightarrow X \text{ is semi nearly normal} \Rightarrow X \text{ is nearly normal.}$$

In the following two examples we show that the above implications are not reversible, in general.

EXAMPLE 3.3. Let $\mathbb{R}$ be the set of all real numbers and τ the co-countable topology on $\mathbb{R}$. Then $(\mathbb{R}, \tau)$ is not normal, but it is semi nearly normal (as $\mathbb{R}$ and $\emptyset$ are the only δ-closed sets in $(\mathbb{R}, \tau)$).

EXAMPLE 3.4. Suppose $\mathbb{R}$ is the set of all real numbers. Let τ_1 and τ_2 be respectively the Euclidean topology and co-countable topology on $\mathbb{R}$. Let τ be the smallest topology on $\mathbb{R}$ generated by $\tau_1 \cup \tau_2$. Then a set U is open in $(\mathbb{R}, \tau)$ iff $U = O \setminus A$, where O is an open set in $(\mathbb{R}, \tau_1)$ and A is a countable subset of $\mathbb{R}$. Also, a set in $(\mathbb{R}, \tau)$ is regular closed (δ-closed) iff it is regular closed (resp. δ-closed) in $(\mathbb{R}, \tau_1)$ (see page 85 of [13]). Thus, if A is a regular closed set and B a δ-closed set in $(\mathbb{R}, \tau)$ with $A \cap B = \emptyset$, then A and B are basically two disjoint closed sets in $(\mathbb{R}, \tau_1)$. Since $(\mathbb{R}, \tau_1)$ is normal, there exist disjoint open sets U and V in $(\mathbb{R}, \tau_1)$ such that $A \subseteq U$ and $B \subseteq V$. Since U and V are open in $(\mathbb{R}, \tau)$ as well, the near normality of $(\mathbb{R}, \tau)$ is established. Now we show that $(\mathbb{R}, \tau)$ is not semi nearly normal. For that, let $A = \{\sqrt{2}\}$ and $B = \mathbb{Q}$, the set of all rational numbers. Then A is δ-closed and B is closed in $(\mathbb{R}, \tau)$. We suppose that there exist disjoint open sets U and V in $(\mathbb{R}, \tau)$ such that $\sqrt{2} \in U$ and $\mathbb{Q} \subseteq V$. Since U and V are open in $(\mathbb{R}, \tau)$, there exist open sets O_1 and O_2 in $(\mathbb{R}, \tau_1)$ and two countable subsets C and D of $\mathbb{R}$ such that $U = O_1 \setminus C$ and $V = O_2 \setminus D$. Now, $U \cap V = \emptyset \Rightarrow O_1 \cap O_2 \setminus C \cup D = \emptyset \Rightarrow O_1 \cap O_2 \subseteq C \cup D \Rightarrow O_1 \cap O_2 = \emptyset$ (since $C \cup D$ is countable and $O_1 \cap O_2$ is open in $(\mathbb{R}, \tau_1)$). This shows that $\sqrt{2} \in O_1$ and $\mathbb{Q} \subseteq O_2 \subseteq \mathbb{R} \setminus O_1$ which, in turn, implies that O_1 is a nonempty open set in $(\mathbb{R}, \tau_1)$ such that $O_1 \cap \mathbb{Q} = \emptyset$. That is a contradiction. Hence $(\mathbb{R}, \tau)$ fails to be semi nearly normal.

The following characterization of semi near normality will be useful for our discussion in the sequel.

THEOREM 3.5. *A space X is semi nearly normal iff for any δ-closed set A in X and for any open set U in X with $A \subseteq U$, there exists a regular open set V in X such that $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$.*

Proof. Let A be δ-closed and U an open set in X such that $A \subseteq U$. Then $(X \setminus U)$ is closed in X and $A \cap (X \setminus U) = \emptyset$. Then, by semi near normality of X, there exist open sets G and H in X such that $A \subseteq G$, $X \setminus U \subseteq H$ and $G \cap H = \emptyset$, i.e., $\mathrm{cl}(G) \subseteq X \setminus H$. Thus we have, $A \subseteq G \subseteq \mathrm{cl}(G) \subseteq X \setminus H \subseteq U$. Since G is open in X, $G \subseteq G^*$ and $\mathrm{cl}(G) = \mathrm{cl}(G^*)$. Therefore, $A \subseteq G \subseteq G^* \subseteq \mathrm{cl}(G^*) \subseteq U$, i.e., $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$, where $V = G^*$. Since V is regular open, the necessity follows.

Conversely, let A be δ-closed and B a closed set in X with $A \cap B = \emptyset$. Then by hypothesis, there exists a regular open set V in X such that $A \subseteq V \subseteq \mathrm{cl}(V) \subseteq X \setminus B$. Let $U = X \setminus \mathrm{cl}(V)$. Then U and V are open sets in X satisfying $A \subseteq V$, $B \subseteq U$ and $U \cap V = \emptyset$ which proves the semi near normality of X. ∎

We are now going to prove an analogous version (with a similar line of proof) of the celebrated Urysohn's lemma, for semi nearly normal spaces.

THEOREM 3.6. *A space X is semi nearly normal if and only if for any pair of disjoint sets A and B in X, of which one is δ-closed and the other closed in X, there exists a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. Let A be a δ-closed set and B a closed set in a semi nearly normal space X with $A \cap B = \emptyset$, i.e., $A \subseteq X \setminus B$. We shall first define, for each $p \in \mathbb{Q}$, a regular open set U_p in X such that $\mathrm{cl}(U_p) \subseteq U_q$, whenever $p, q \in \mathbb{Q}$ with $p < q$. To do this, let $P = \mathbb{Q} \cap [0,1]$ and let $\{t_k : k \in \mathbb{N}\}$ be an enumeration of P with $t_1 = 1$ and $t_2 = 0$. Let $U_1 = X \setminus B$. Then U_1 is an open set in X with $A \subseteq U_1$. By Theorem 3.5, there exists a regular open set U_0 in X such that $A \subseteq U_0 \subseteq \mathrm{cl}(U_0) \subseteq U_1$. Let $P_n (n \geq 2)$ be the set of first n rational numbers in the sequence $\{t_k : k \in \mathbb{N}\}$ and let, for each $p \in P_n$, a regular open set U_p have been defined satisfying $\mathrm{cl}(U_p) \subseteq U_q$, whenever $p, q \in P_n$ with $p < q$. Let r be the $(n+1)$-th rational number in the sequence $\{t_k : k \in \mathbb{N}\}$. We now consider the set $P_{n+1} = P_n \cup \{r\}$. Since $r \in P_{n+1}$, $r \neq 0, 1$ and P_{n+1} is a finite set, r has an immediate predecessor p and an immediate successor q (with respect to the usual ordering of real numbers) in P_{n+1}. Since $p < r < q$, it follows that $p, q \in P_n$ and hence, by induction hypothesis, regular open sets U_p and U_q have already been defined satisfying $\mathrm{cl}(U_p) \subseteq U_q$. Since U_p is regular open in X, $\mathrm{cl}(U_p)$ is δ-closed (in fact, regular closed) in X. By Theorem 3.5, there exists a regular open set U_r in X such that $\mathrm{cl}(U_p) \subseteq U_r \subseteq \mathrm{cl}(U_r) \subseteq U_q$. We now show that $\mathrm{cl}(U_a) \subseteq U_b$, whenever $a, b \in P_{n+1}$ with $a < b$. If both a and b belong to P_n, then the requirement is satisfied by the induction hypothesis. If any one of them is r and the other is a member of P_n (say, s), then either $s \leq p$ which implies $\mathrm{cl}(U_s) \subseteq \mathrm{cl}(U_p) \subseteq U_r$; or $s \geq q$ for which $\mathrm{cl}(U_r) \subseteq U_q \subseteq U_s$. Thus for all $a, b \in P_{n+1}$, $\mathrm{cl}(U_a) \subseteq U_b$, whenever $a < b$. Hence by induction, we have defined a regular open set U_p, for each $p \in P$, satisfying $\mathrm{cl}(U_p) \subseteq U_q$, whenever $p, q \in P$ with $p < q$. If $p \in \mathbb{Q}$ is such that $p < 0$, then we define $U_p = \emptyset$; we also define $U_p = X$, whenever $p \in \mathbb{Q}$ and $p > 1$. Thus for each $r \in \mathbb{Q}$, a regular open set U_r has been defined such that $\mathrm{cl}(U_p) \subseteq U_q$, whenever $p, q \in \mathbb{Q}$ with $p < q$. For each $x \in X$, let $A(x) = \{p \in \mathbb{Q} : x \in U_p\}$. Then for any $p \in \mathbb{Q}$ with $p < 0$, $p \notin A(x)$ and for any $p \in \mathbb{Q}$ with $p > 1$, $p \in A(x)$. Thus, for each $x \in X$, the set $A(x)$ is bounded below and has its infimum in $[0, 1]$. We define $f : X \to [0, 1]$ by $f(x) = \inf A(x)$, for each $x \in X$. We now show that f is the desired function. Let x be an arbitrary element of A. Then $x \in U_p$, for all $p \geq 0$ (since $A \subseteq U_0$ and $\mathrm{cl}(U_p) \subseteq U_q$, whenever $p < q$). Thus, $A(x)$ is the set of all non-negative rational numbers and hence $\inf A(x) = 0$, i.e., $f(x) = 0$. Therefore, $f(A) = \{0\}$. Let x be an arbitrary element of B. Then $x \notin U_p$, for all $p \leq 1$ (since $U_1 = X \setminus B$ and $\mathrm{cl}(U_p) \subseteq U_q$, whenever $p < q$). Thus $A(x)$ is the set of all rational numbers which are greater than 1 and hence $\inf A(x) = 1$, i.e., $f(x) = 1$. Therefore, $f(B) = \{1\}$. It is easy to see that for any $r \in \mathbb{Q}$, the following implications hold:

(*i*) $x \in \mathrm{cl}(U_r) \Rightarrow A(x) \supseteq \{s \in \mathbb{Q} : s > r\} \Rightarrow f(x) \leq r$ and

(*ii*) $x \notin U_r \Rightarrow A(x) \subseteq \{s \in \mathbb{Q} : s > r\} \Rightarrow f(x) \geq r$.

To show that $f : X \to [0, 1]$ is continuous, let $x_0 \in X$ be arbitrary and $(f(x_0) - \epsilon, f(x_0) + \epsilon)$ an ϵ-neighbourhood of $f(x_0)$ in $\mathbb{R}$, where $\epsilon > 0$. Let us choose $p, q \in \mathbb{Q}$ satisfying $f(x_0) - \epsilon < p < f(x_0) < q < f(x_0) + \epsilon$. Let $U = U_q \setminus \mathrm{cl}(U_p)$. Then U is an open set in X and also $x_0 \in U$, because $x_0 \notin U_q \Rightarrow f(x_0) \geq q$, and $x_0 \in \mathrm{cl}(U_p) \Rightarrow f(x_0) \leq p$, which is a contradiction in each case. Also, $x \in U \Rightarrow x \in U_q$ and $x \notin \mathrm{cl}(U_p) \Rightarrow f(x) \leq q$ and $f(x) \geq p \Rightarrow f(x) \in [p, q] \subseteq (f(x_0) - \epsilon, f(x_0) + \epsilon)$. Thus U is an open set in X containing x_0 satisfying $f(U) \subseteq (f(x_0) - \epsilon, f(x_0) + \epsilon)$. Therefore, f is continuous.

Conversely, let A be a δ-closed and B a closed set in X. Then by hypothesis, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then the open sets $U = f^{-1}[0, \frac{1}{2})$ and $V = f^{-1}(\frac{1}{2}, 1]$ are disjoint and contain A and B respectively. Hence X is semi nearly normal. ∎

The following is a sufficient condition for a semi nearly normal space X to be countably nearly paracompact.

THEOREM 3.7. *If a space X is semi nearly normal and for every decreasing sequence $\{F_n : n \in \mathbb{N}\}$ of δ-closed sets in X with empty intersection, there is a sequence $\{G_n : n \in \mathbb{N}\}$ of open sets in X satisfying $G_n \supseteq F_n$, for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} G_n = \emptyset$, then X is countably nearly paracompact.*

Proof. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a regular open cover of X. For each $n \in \mathbb{N}$, let $F_n = X \setminus \bigcup_{k=1}^{n} U_k$. Then $\{F_n : n \in \mathbb{N}\}$ is a decreasing sequence of δ-closed sets in X satisfying $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then by hypothesis, there exists a sequence $\{G_n : n \in \mathbb{N}\}$ of open sets in X such that $G_n \supseteq F_n$, for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} G_n = \emptyset$. Now, $X \setminus G_1$ and F_1 are two disjoint sets in X, where $X \setminus G_1$ is closed and F_1 is δ-closed in X. By semi near normality of X, there exists an open set W_1 in X such that $X \setminus G_1 \subseteq W_1$ and $\mathrm{cl}(W_1) \cap F_1 = \emptyset$. Also, $(X \setminus G_2) \cap F_2 = \emptyset$ and

$\mathrm{cl}(W_1) \cap F_1 = \emptyset \Rightarrow [(X \setminus G_2) \cup \mathrm{cl}(W_1)] \cap F_2 = \emptyset$ (since $F_1 \supseteq F_2$). Thus, F_2 and $(X \setminus G_2) \cup \mathrm{cl}(W_1)$ are two disjoint closed sets in X with F_2 a δ-closed set. Again by semi near normality of X, there exists an open set W_2 in X such that $\mathrm{cl}(W_2) \cap F_2 = \emptyset$ and $(X \setminus G_2) \cup \mathrm{cl}(W_1) \subseteq W_2$. Continuing this process we get inductively a sequence $\{W_n : n \in \mathbb{N}\}$ of open sets in X satisfying the following conditions:

(*i*) $\mathrm{cl}(W_n) \subseteq W_{n+1}$,

(*ii*) $X \setminus G_n \subseteq W_n$ and

(*iii*) $\mathrm{cl}(W_n) \cap F_n = \emptyset$, for all $n \in \mathbb{N}$.

Since $\bigcap_{n=1}^{\infty} G_n = \emptyset$, $\{X \setminus G_n : n \in \mathbb{N}\}$ covers X. Therefore, by (*ii*) above, $\{W_n : n \in \mathbb{N}\}$ is an open cover of X. Let $S_1 = W_1$, $S_2 = W_2$ and $S_n = W_n \setminus \mathrm{cl}(W_{n-2})$, for $n > 2$. Then $\{S_n : n \in \mathbb{N}\}$ is a sequence of open sets in X. We first show that $\{S_n : n \in \mathbb{N}\}$ covers X. For that, let x be an arbitrary element of X. Since $\{W_n : n \in \mathbb{N}\}$ covers X, there exists an $n \in \mathbb{N}$ such that $x \in W_n$. Let $\sigma = \{k \in \mathbb{N} : x \in W_k\}$. Since $n \in \sigma$, $\sigma \neq \emptyset$ and hence, σ has a least element, say n_0.

If $n_0 = 1$, then $x \in W_1 = S_1$.

If $n_0 > 1$, then $x \in W_{n_0}$ and $x \notin W_n$, for $n < n_0$. Since $n_0 > 1$, there exists some $p \in \mathbb{N}$ such that $n_0 = p + 1$. If $p = 1$, then $x \in W_2 = S_2$. If $p > 1$, then $x \in W_{p+1}$ and $x \notin W_p \Rightarrow x \in W_{p+1}$ and $x \notin \mathrm{cl}(W_{p-1})$ (since by (*i*), $\mathrm{cl}(W_{p-1}) \subseteq W_p$) $\Rightarrow x \in W_{p+1} \setminus \mathrm{cl}(W_{p-1}) = S_{p+1}$. Thus $\{S_n : n \in \mathbb{N}\}$ is an open cover of X.

For each $k \in \mathbb{N}$, let $\mathcal{A}_k = \{S_k \cap U_j : j = 1, 2, \dots, k\}$. Then $\mathcal{A}_k$ consists of finitely many open subsets of X, for each $k \in \mathbb{N}$. Let $\mathcal{A} = \bigcup_{k=1}^{\infty} \mathcal{A}_k$. Then $\mathcal{A}$ is a family of open sets in X which refines $\mathcal{U}$ (note that $\mathcal{A}_k$ refines $\mathcal{U}$, for each $k \in \mathbb{N}$). We first show that $\bigcup \mathcal{A}_k = S_k$, for each $k \in \mathbb{N}$. So, fix an arbitrary $k \in \mathbb{N}$. Obviously, $S_k \subseteq W_k$ (by definition of S_k's). Also, $W_k \cap F_k = \emptyset$ (see (*iii*)) $\Rightarrow W_k \subseteq X \setminus F_k = \bigcup_{j=1}^{k} U_j \Rightarrow S_k \subseteq \bigcup_{j=1}^{k} U_j \Rightarrow S_k = S_k \cap (\bigcup_{j=1}^{k} U_j) = \bigcup_{j=1}^{k} (S_k \cap U_j) = \bigcup \mathcal{A}_k$.

We now show that $\mathcal{A}$ is a cover of X. Consider any $x \in X$. Since $\{S_n : n \in \mathbb{N}\}$ is a cover of X, there is some $k \in \mathbb{N}$ such that $x \in S_k = \bigcup \mathcal{A}_k$. Thus there is some $V \in \mathcal{A}_k \subseteq \mathcal{A}$ such that $x \in V$. Hence $\mathcal{A}$ covers X.

So it remains to show that $\mathcal{A}$ is locally finite. For this, we first show that $|k - j| > 1 \Rightarrow S_k \cap S_j = \emptyset$.

Let $k - j > 1$. For $j = 1$ or 2, $S_k \cap S_j = (W_k \cap W_j) \setminus (\mathrm{cl}(W_{k-2}) \cap W_j) = W_j \setminus W_j = \emptyset$. Again, for $j > 2$, we have $S_k \cap S_j \subseteq W_k \cap W_j \setminus \mathrm{cl}(W_{k-2} \cup W_{j-2}) = W_j \setminus \mathrm{cl}(W_{k-2})$ (since $W_j \subseteq W_k$ and $W_{j-2} \subseteq W_{k-2}$) $\subseteq W_{k-2} \setminus \mathrm{cl}(W_{k-2})$ (since $k - j > 1 \Rightarrow j \leq k - 2 \Rightarrow W_j \subseteq W_{k-2}$) $= \emptyset$.

Similarly, $j - k > 1 \Rightarrow S_j \cap S_k = \emptyset$. Thus, for every $k \in \mathbb{N}$, the members of $\mathcal{A}$ which may intersect a member of $\mathcal{A}_k$ are the members of $\mathcal{A}_{k-1}$, or $\mathcal{A}_k$, or $\mathcal{A}_{k+1}$. Let $x \in X$. Since $\mathcal{A}$ $(= \bigcup_{k=1}^{\infty} \mathcal{A}_k)$ covers X, there exists a $k \in \mathbb{N}$ and a $V \in \mathcal{A}_k$ such that $x \in V$. Then V is an open neighbourhood of x in X and V can meet no member of $\mathcal{A}$ other than the members of $\mathcal{A}_{k-1}$, $\mathcal{A}_k$ and $\mathcal{A}_{k+1}$. This shows that V meets at most finitely many members of $\mathcal{A}$. Thus $\mathcal{A}$ is a locally finite open refinement of $\mathcal{U}$ and hence X is CNP. ■

Finally we prove another sufficient condition for countably near paracompactness.

Theorem 3.8. *A space X is countably nearly paracompact if $X \times [0,1]$ is semi nearly normal.*

Proof. We first show that X is semi nearly normal. Let A be a δ-closed set and B a closed set in X such that $A \cap B = \emptyset$. Then $A \times [0,1]$ and $B \times [0,1]$ are two disjoint closed sets in $X \times [0,1]$, where $A \times [0,1]$ is δ-closed (note that for two spaces X and Y, $(X \times Y)_s = X_s \times Y_s$ [3]). Since $X \times [0,1]$ is semi nearly normal, there exist disjoint open sets G and H in $X \times [0,1]$ such that $A \times [0,1] \subseteq G$ and $B \times [0,1] \subseteq H$. Then $U = \{x \in X : (x,0) \in G\}$ and $V = \{x \in X : (x,0) \in H\}$ are open sets in X; and also U and V satisfy $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$, which proves that X is semi nearly normal. Now, let $\{F_n : n \in \mathbb{N}\}$ be a decreasing sequence of δ-closed sets in X such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Let $W_n = X \setminus F_n$, for each $n \in \mathbb{N}$. Note that $W_n \times [0, \frac{1}{n})$ is δ-open in $X \times [0,1]$, for each $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} W_n \times [0, \frac{1}{n})$ is also δ-open in $X \times [0,1]$. Let $C = X \times [0,1] \setminus \bigcup_{n=1}^{\infty} W_n \times [0, \frac{1}{n})$. Then C is δ-closed in $X \times [0,1]$ and $D = X \times \{0\}$ is closed in $X \times [0,1]$. Also, $t \in D \Rightarrow t = (x,0)$, for some $x \in X$. Since $\bigcap_{n=1}^{\infty} F_n = \emptyset$, $\{W_n : n \in \mathbb{N}\}$ covers X and hence there is some $n \in \mathbb{N}$ such that $x \in W_n \Rightarrow (x,0) \in W_n \times [0, \frac{1}{n}) \Rightarrow t \notin C$. Thus $C \cap D = \emptyset$. By semi near normality of $X \times [0,1]$, there is an open set O in $X \times [0,1]$ such that $C \subseteq O$ and $\mathrm{cl}(O) \cap D = \emptyset$. For each $n \in \mathbb{N}$, we define $f_n : X \to X \times [0,1]$ by $f_n(x) = (x, \frac{1}{n})$, for all $x \in X$. Let $G_n = f_n^{-1}(O)$, for each $n \in \mathbb{N}$. Then G_n is open in X (since f_n is continuous) and $G_n = \{x \in X : (x, \frac{1}{n}) \in O\}$, for each $n \in \mathbb{N}$. We now show that $G_n \supseteq F_n$, for all $n \in \mathbb{N}$. Let $x \in F_n$. Then $x \in F_m$ for all $m \leq n$ (since $\{F_n : n \in \mathbb{N}\}$ is decreasing) $\Rightarrow x \notin W_m$ for all $m \leq n$. Also, $\frac{1}{n} \notin [0, \frac{1}{m})$, for all $m > n$. Thus $(x, \frac{1}{n}) \notin \bigcup_{m=1}^{\infty} W_m \times [0, \frac{1}{m}) \Rightarrow (x, \frac{1}{n}) \in C \subseteq O \Rightarrow x \in G_n$. Therefore, $G_n \supseteq F_n$, for all $n \in \mathbb{N}$. We suppose that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$. Then there exists some $x \in X$ such that $x \in G_n$, for all $n \in \mathbb{N} \Rightarrow (x, \frac{1}{n}) \in O$ for all $n \in \mathbb{N}$, $\Rightarrow$ $(x,0)$ is a limit point of $O \Rightarrow (x,0) \in \mathrm{cl}(O)$, which is contradictory to the fact that $\mathrm{cl}(O) \cap D = \emptyset$. Therefore, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and hence, by Theorem 3.7, X is CNP. ∎

Note 3.9. In fact, the above theorem follows from a result of J. Mack (Theorem 1 of [6]) which states that a space X is countably paracompact iff $X \times [0,1]$ is δ-normal. However, we have retained the proof as it provides an alternative and direct proof of the result of Mack in our particular setting.

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REFERENCES

[**1**] D. E. Cameron, *Maximal QHC-spaces*, Rocky Mountain Jour. Math. **7**(2) (1977), 313–322.

[**2**] C. H. Dowker, *On countably paracompact spaces*, Can. Jour. Math **3** (1951), 219–224.

[**3**] L. L. Herrington, *Properties of nearly-compact spaces*, Proc. Amer. Math. Soc. **45** (1974), 431–436.

[**4**] M. Katětov, *On real valued functions in topological spaces*, Fund. Math. **38** (1951), 85–91.

[**5**] I. Kovačević, *On nearly paracompact spaces*, Publ. Inst. Math. **25(39)** (1979), 63–69.

[**6**] J. Mack, *Countable paracompactness and weak normality properties*, Trans. Amer. Math. Soc. **148** (1970), 265–272.

[**7**] M. N. Mukherjee and A. Debray, *On nearly paracompact spaces via regular even covers*, Mat. Vesnik **50** (1998), 23–29.

[**8**] M. N. Mukherjee and A. Debray, *On nearly paracompact spaces and nearly full normality*, Mat. Vesnik **50** (1998), 99–104.

[**9**] M. N. Mukherjee and A. Debray, *On nearly normal spaces*, Southeast Asian Bull. Math. **29** (2005), 1087–1093.

[**10**] M. N. Mukherjee and D. Mandal, *On some new characterizations of near paracompactness and associated results*, Mat. Vesnik **65**(3) (2013), 334–345.

[**11**] T. Noiri, *A note on nearly-paracompact spaces*, Mat. Vesnik **5(18)(33)** (1981), 103–108.

[**12**] M. K. Singal and S. P. Arya, *On nearly paracompact spaces*, Mat. Vesnik **6(21)** (1969), 3–16.

[**13**] L. A. Steen and J. A. Seebach, *Counterexamples in Topology*, Holt, Rinehart and Winston, Inc. New York, 1970.

[**14**] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. Ser. 2, **78** (1968), 103–118.

[**15**] S. Willard, *General Topology*, Addison-Wesley, Reading, Mass., 1970.

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