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FIXED POINT FOR FUZZY CONTRACTION MAPPINGS SATISFYING AN IMPLICIT RELATION

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Abstract. We prove a common fixed point theorem for generalized fuzzy contraction mappings satisfying an implicit relation.

1. Introduction and preliminaries

Heilpern [8] introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler [11]. Afterwards several fixed point theorems for fuzzy contractive mappings have appeared in the literature (see, [1–5, 12, 13, 15]). In this paper, we prove a common fixed point theorem for fuzzy mappings satisfying an implicit relations. Our results generalize and extend results in Rashwan and Ahmed [14], Arora and Sharma [1, Lemma 3.1] and Lee and Cho [10, Proposition 3.2].

Let (X, d) be a metric linear space [8]. A *fuzzy set* in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the *grade of membership* of x in A. The collection of all fuzzy sets in X is denoted by $\Im(X)$. A fuzzy mapping on a set X is a usual mapping from X into $\Im(X)$.

Let $A \in \Im(X)$ and $\alpha \in [0, 1]$. The *α-level set* of A, denoted by A_α, is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \quad \text{if} \quad \alpha \in (0, 1], \quad A_0 = \overline{\{x : A(x) > 0\}},$$

whenever $\overline{B}$ is the closure of set (nonfuzzy) B.

Definition 1.1. [8] A fuzzy set A in X is an *approximate quantity* if and only if its α-level set is a nonempty compact convex subset (nonfuzzy) of X for each $\alpha \in [0, 1]$.

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The set of all approximate quantities, denoted by $W(X)$, is a subcollection of $\Im(X)$.

Definition 1.2. [11] Let $A, B \in W(X)$, $\alpha \in [0,1]$ and $CP(X)$ be a set of all nonempty compact subsets of X. Then

$$p_\alpha(A,B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x,y), \qquad \delta_\alpha(A,B) = \sup_{x \in A_\alpha, y \in B_\alpha} d(x,y)$$
$$\text{and} \qquad D_\alpha(A,B) = H(A_\alpha, B_\alpha),$$

where H is the *Hausdorff metric* between two sets in the collection $CP(X)$.

We also define the following functions

$$p(A,B) = \sup_\alpha p_\alpha(A,B), \qquad \delta(A,B) = \sup_\alpha \delta_\alpha(A,B)$$
$$\text{and} \qquad D(A,B) = \sup_\alpha D_\alpha(A,B).$$

It is noted that p_α is a nondecreasing function of α.

Definition 1.3. [11] Let $A, B \in W(X)$. Then A is said to be *more accurate* than B (or that B includes A), denoted by $A \subset B$, if and only if $A(x) \leq B(x)$ for each $x \in X$.

The relation $\subset$ induces a partial order on $W(X)$.

Definition 1.4. [4] Let X be an arbitrary set and Y be a metric linear space. The mapping T is said to be a *fuzzy mapping* if and only if T is a mapping from the set X into $W(Y)$, i.e., $T(x) \in W(Y)$ for each $x \in X$.

The following proposition is used in the sequel.

Proposition 1.5. [11] *If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a,b) \leq H(A,B)$.*

Let Ψ be the family of real valued lower semi-continuous functions $F : [0,\infty)^6 \to \mathbb{R}$, satisfying the following conditions:

(ψ_1) F is non-decreasing in 1^{st} coordinate and F is non-increasing in 3^{rd}, 4^{th}, 5^{th}, 6^{th} coordinate variable,

(ψ_2) there exists $h \in (0,1)$ such that for every $u, v \geq 0$ with

(ψ_{21}) $F(u,v,v,u,u+v,0) \leq 0$ or

(ψ_{22}) $F(u,v,u,v,0,u+v) \leq 0$,

we have $u \leq hv$, and

(ψ_3) $F(u,u,0,0,u,u) > 0$ for all $u > 0$.

Conditions ψ_i $(i = 1,2,3)$ are called implicit conditions and we refer for examples and their applications in fixed point theory to Beg and Butt [6, 7].

2. Main results

Let (X, d) be a metric space. We consider a subcollection of $\Im(X)$ denoted by $W^*(X)$; for any $A \in W^*(x)$, its α-level set is a nonempty compact subset (nonfuzzy) of X for each $\alpha \in [0, 1]$. It is obvious that each element $A \in W(X)$ leads to $A \in W^*(X)$ but the converse is not true.

Next, we introduce the improvements of the lemmas in Heilpern [8] as follows.

LEMMA 2.1. *If $\{x_0\} \subset A$ for each $A \in W^*(X)$ and $x_0 \in X$, then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in W^*(X)$.*

LEMMA 2.2. *$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for all $x, y \in X$ and $A \in W^*(X)$.*

LEMMA 2.3. *Let $x \in X$, $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.*

Proof. If $\{x\} \subset A$, then $x \in A_\alpha$ for each $\alpha \in [0, 1]$. It implies that $p_\alpha(x, A) = \inf_{y \in A_\alpha} d(x, y) = 0$ for each $\alpha \in [0, 1]$.

Conversely, if $p_\alpha(x, A) = 0$, then $\inf_{y \in A_\alpha} d(x, y) = 0$. It follows that $x \in \overline{A_\alpha} = A_\alpha$ for each $\alpha \in [0, 1]$. Thus $\{x\} \subset A$. ∎

Next, we state and prove a new lemma.

LEMMA 2.4. *Let (X, d) be a complete metric space, $T : X \to W^*(X)$ be a fuzzy map and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$.*

Proof. For $n \in N$, $((T(x_0))_{n/(n+1)})$ is a decreasing sequence of nonempty compact subsets of X. Thus we have from [16, Prop. 11.4 and Remark 11.5 on page 495-496] that $\bigcap_{n=1}^{\infty}(F(x_0))_{n/(n+1)}$ is nonempty and compact. Let $x_1 \in \bigcap_{n=1}^{\infty}(T(x_0))_{n/(n+1)}$. Then $\frac{n}{n+1} \leq (T(x_0))(x_1) \leq 1$. As $n \to \infty$, we get that $(T(x_0))(x_1) = 1$. It implies that $\{x_1\} \subset T(x_0)$. ∎

REMARK 2.5. Lemma 2.4 is a generalization of Arora and Sharma [1, Lemma 3.1] and Lee and Cho [10, Prop.3.2].

Now, we prove our main theorem.

THEOREM 2.6. *Let (X, d) be a complete metric space and T_1, T_2 be fuzzy mappings from X into $W^*(X)$. If there is an $F \in \Psi$ such that for all $x, y \in X$,*

$$F(D(T_1(x), T_2(y)), d(x, y), p(x, T_1(x)), p(y, T_2(y)), p(x, T_2(y)), p(y, T_1(x))) \leq 0,$$

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Proof. Let $x_0 \in X$. Then by Lemma 2.4, there exists an element $x_1 \in X$ such that $\{x_1\} \subset T_1(x_0)$. For $x_1 \in X$, $(T_2(x_1))_1$ is a nonempty compact subset of X. Since $(T_1(x_0))_1, (T_2(x_1))_1 \in CP(X)$ and $x_1 \in (T_1(x_0))_1$, then Proposition 1.5

asserts that there exists $x_2 \in (T_2(x_1))_1$ such that $d(x_1, x_2) \leq D_1(T_1(x_0), T_2(x_1))$. So, we have from Lemma 2.3 and the property (ψ_1) of F that

$$\begin{aligned} &F(d(x_1,x_2), d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), d(x_0,x_1)+d(x_1,x_2), 0) \\ &\quad \leq F(D_1(T_1(x_0),T_2(x_1)), d(x_0,x_1), p(x_0,T_1(x_0)), p(x_1,T_2(x_1)), \\ &\qquad p(x_0,T_2(x_1)), p(x_1,T_1(x_0))) \\ &\quad \leq F(D(T_1(x_0),T_2(x_1)), (d(x_0,x_1), p(x_0,T_1(x_0)), p(x_1,T_2(x_1)), \\ &\qquad p(x_0,T_2(x_1)), p(x_1,T_1(x_0))) \leq 0. \end{aligned}$$

From the property (ψ_{21}) of $F \in \Psi$, there exists $h \in (0,1)$ such that $d(x_1, x_2) \leq hd(x_0, x_1)$. Similarly, one can deduce from the property (ψ_{22}) of $F \in \Psi$ that there exists $h \in (0,1)$ such that $d(x_2, x_3) \leq hd(x_1, x_2)$. By induction, we have a sequence (x_n) of points in X such that, for all $n \in N \cup \{0\}$,

$$\{x_{2n+1}\} \subset T_1(x_{2n}), \quad \{x_{2n+2}\} \subset T_2(x_{2n+1}).$$

It follows by induction that $d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)$. Since

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m) \\ &\leq h^n d(x_0,x_1) + h^{n+1} d(x_0,x_1) + \cdots + h^{m-1} d(x_0,x_1) \leq \frac{h^n}{1-h} d(x_0,x_1), \end{aligned}$$

then $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Therefore, (x_n) is a Cauchy sequence. Since X is a complete metric space, then there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Next, we show that $\{z\} \subset T_i(z)$, $i = 1, 2$. We get from Lemma 2.1 and Lemma 2.2 that

$$p_\alpha(z,T_2(z)) \leq d(z,x_{2n+1}) + p_\alpha(x_{2n+1},T_2(z)) \leq d(z,x_{2n+1}) + D_\alpha(T_1(x_{2n}),T_2(z)),$$

for each $\alpha \in [0,1]$. Taking supremum on α in the last inequality, we obtain from the property (ψ_1) of F that

$$\begin{aligned} &F(p(x_{2n+1},T_2(z)), d(x_{2n},z), d(x_{2n},x_{2n+1}), p(z,T_2(z)), p(x_{2n},T_2(z)), d(z,x_{2n+1})) \\ &\quad \leq F(D_1(T_1(x_{2n}),T_2(z)), d(x_{2n},z), p(x_{2n},T_1(x_{2n})), p(z,T_2(z)), p(x_{2n},T_2(z)), \\ &\qquad p(z,T_1(x_{2n}))) \\ &\quad \leq F(D(T_1(x_{2n}),T_2(z)), d(x_{2n},z), p(x_{2n},T_1(x_{2n})), p(z,T_2(z)), p(x_{2n},T_2(z)), \\ &\qquad p(z,T_1(x_{2n}))) \leq 0. \end{aligned}$$

As $n \to \infty$, we have

$$F(p(z,T_2(z)), 0, 0, p(z,T_2(z)), p(z,T_2(z)), 0) \leq 0.$$

From the property (ψ_3) of $F \in \Psi$, it yields that $p(z, T_2(z)) = 0$. So, we get from Lemma 2.3 that $\{z\} \subset T_2(z)$. Similarly, it can be shown that $\{z\} \subset T_1(z)$. ∎

EXAMPLE 2.7. Let $X = [0,1]$ be endowed with the metric d defined by $d(x,y) = |x-y|$. It is clear that (X,d) is a complete metric space. Assume that $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \frac{3}{4}t_2$ for every $t_1,t_2,t_3,t_4,t_5,t_6 \in [0,\infty)$. It is obvious that $F \in \Psi$. Let $T_1 = T_2 = T$. Define a fuzzy mapping T on X such that for all $x \in X$, $T(x)$ is the characteristic function for $\{\frac{3}{4}x\}$. For each $x,y \in X$,

$$F(D(F(x),F(y)),d(x,y),p(x,F(x)),p(y,F(y)),p(x,F(y)),p(y,F(x))) \\ = D(F(x),F(y)) - \frac{3}{4}d(x,y) = \frac{3}{4}d(x,y) - \frac{3}{4}d(x,y) = 0.$$

The characteristic function for $\{0\}$ is the fixed point of T.

REMARK 2.8. (I) If there is an $F \in \Psi$ such that, for each $x,y \in X$,

$$F(\delta(T_1(x),T_2(y)),d(x,y),p(x,T_1(x)),p(y,T_2(y)),p(x,T_2(y)),p(y,T_1(x)))) \leq 0,$$

then the conclusion of Theorem 2.6 remains valid. This result is considered as a special case of Theorem 2.6 because $D(T_1(x),T_2(y)) \leq \delta(T_1(x),T_2(y))$ [9, page 414].

(II) Park and Jeong [12, Theorems 3.1 and 3.4] and Rashwan and Ahmed [14, Theorem 2.1] are special cases of Theorem 2.6.

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