МАТЕМАТИЧКИ ВЕСНИК
66, 4 (2014), 364–370
December 2014

оригинални научни рад
research paper

n-NORMAL AND n-QUASINORMAL COMPOSITION AND WEIGHTED COMPOSITION OPERATORS ON $L^2(\mu)$

Anuradha Gupta and Neha Bhatia

Abstract. An operator T is called n-normal operator if $T^nT^* = T^*T^n$ and n-quasinormal operator if $T^nT^*T = T^*TT^n$. In this paper, the conditions under which composition operators and weighted composition operators become n-normal operators and n-quasinormal operators have been obtained in terms of Radon-Nikodym derivative h_n.

1. Introduction

Let H be the infinite dimensional complex Hilbert space and $\mathbb{B}(H)$ be the algebra of all bounded linear operators on H. An operator T is called *normal* if $TT^* = T^*T$. If T is a normal operator then $\operatorname{Ker} T = \operatorname{Ker} T^*$. An operator T is called *quasinormal* if $T(T^*T) = (T^*T)T$. Every normal operator is a quasinormal operator but converse need not be true. The unilateral shift operator on $\mathbb{B}(H)$ is quasinormal but not normal. An operator T is called *n-normal* [2] if $T^nT^* = T^*T^n$ for $n \in \mathbb{N}$. Also, in [2] Alzuraiqi and Patel proved that T is *n-normal* if and only if T^n is normal. i.e., $T^nT^{*n} = T^{*n}T^n$ for $n \in \mathbb{N}$. The class of n-normal operators is denoted by $[nN]$. An operator T is called *n-quasinormal* operator [1] if $T^nT^*T = T^*TT^n$ for $n \in \mathbb{N}$. The class of n-quasinormal operators is denoted by $[nQN]$ and $[nN] \subseteq [nQN]$.

Let (X, Σ, μ) be a σ-finite measure space. A transformation T is said to be *measurable* if $T^{-1}(B) \in \Sigma$ for $B \in \Sigma$. A measurable transformation T is said to be *non-singular* if

$$\mu(T^{-1}(B)) = 0 \quad \text{whenever } \mu(B) = 0 \quad \text{for every } B \in \Sigma\,.$$

If T is a measurable transformation then T^n is also a measurable transformation. If T is non-singular, then we say that μT^{-1} is absolutely continuous with respect to μ and hence $\mu(T^{-1})^n$ becomes absolutely continuous with respect to μ. Hence,

---

2010 Mathematics Subject Classification: 47B20, 47B33, 47B38

Keywords and phrases: normal operator; quasinormal operator; n-normal operator; n-quasinormal operator; composition operator; weighted composition operator; conditional expectation operator; hyponormal operator; compact operator.

by *Radon-Nikodym theorem* there exists a unique non-negative essentially bounded measurable function h_n such that

$$\mu(T^{-1})^n(B) = \int_B h_n \, d\mu \quad \text{for } B \in \Sigma$$

and h_n is called the *Radon-Nikodym derivative* and is denoted by $d\mu(T^{-1})^n/d\mu$.

PROPOSITION 1.1. *Change of Variables: Let X be a non-empty set and let Σ be a σ-algebra on X. Let μ and μT^{-1} be measures on Σ and let $h : X \to [0, \infty]$ be a measurable function. Then the following are equivalent:*

(i) *μT^{-1} is absolutely continuous with respect to μ and h is Radon-Nikodym derivative of μT^{-1} with respect to μ.*

(ii) *For every measurable function $f : X \to [0, \infty]$, the equality*

$$\int_X f \, d\mu T^{-1} = \int_X f h \, d\mu$$

holds.

Let (X, Σ, μ) be a σ-finite measure space. Then the *conditional expectation operator* $E(\cdot \mid T^{-1}(\Sigma)) = E(f)$ is defined for each non-negative function f in L^p $(1 \leq p < \infty)$ and is uniquely determined by the following set of conditions:

(i) $E(f)$ is $T^{-1}(\Sigma)$ measurable.

(ii) If B is any $T^{-1}(\Sigma)$ measurable set for which $\int_B f \, d\mu$ converges then we have

$$\int_B f \, d\mu = \int_B E(f) \, d\mu.$$

The conditional expectation operator E has the following properties:

(i) $E(f \cdot g \circ T) = (E(f))(g \circ T)$.

(ii) E is monotonically increasing, i.e., if $f \leq g$ a.e. then $E(f) \leq E(g)$ a.e.

(iii) $E(1) = 1$.

(iv) $E(f)$ has the form $E(f) = g \circ T$ for exactly one Σ-measurable function g provided that the support of g lies in the support of h which is given by

$$\sigma(h) = \{x : h(x) \neq 0\}.$$

E is the projection operator onto the closure of the range of the composition operator C_T on $L^2(\mu)$.

Let ϕ be an essentially bounded function. The *multiplication operator* M_ϕ on the space $L^2(\mu)$ induced by ϕ is given by

$$M_\phi f = \phi f \quad \text{for } f \in L^2(\mu).$$

Let T be a measurable transformation on X. The *composition operator* C_T on the space $L^2(\mu)$ is given by

$$C_T f = f \circ T \quad \text{for } f \in L^2(\mu).$$

Let ϕ be a complex-valued measurable function then the *weighted composition operator* $W_{\phi,T}$ on the space $L^2(\mu)$ induced by ϕ and T is given by

$$W_{\phi,T} f = \phi \cdot f \circ T \quad \text{for } f \in L^2(\mu).$$

In this paper, we study n-normal composition operators, n-quasinormal composition operators and weighted composition operators in terms of Radon-Nikodym derivative and expectation operators. We have derived the condition under which the product of two n-normal composition operators is also an n-normal composition operator.

2. n-normal composition operators and n-quasinormal composition operators

Let C_T be the composition operator on $L^2(\mu)$. Then the adjoint C_T^* is given by $C_T^* f = hE(f) \circ T^{-1}$ for f in $L^2(\mu)$.

The following lemma [4, 7] plays a significant role in the subsequent results.

Lemma 2.1. *Let P be the projection of $L^2(X, \Sigma, \mu)$ onto $\overline{R(C_T)}$. Then*

(i) $C_T^* C_T f = hf$ *and* $C_T C_T^* f = (h \circ T)Pf \ \forall \ f \in L^2(\mu)$.

(ii) $\overline{R(C_T)} = \{f \in L^2(\mu) : f \text{ is } T^{-1}(\Sigma) \text{ measurable}\}$.

(iii) *If f is $T^{-1}(\Sigma)$ measurable and g and fg belong to $L^2(\mu)$, then $P(fg) = fP(g)$, (f need not be in $L^2(\mu)$).*

Also, for $k \in \mathbb{N}$,

(iv) $(C_T^* C_T)^k f = h^k f$.

(v) $(C_T C_T^*)^k f = (h \circ T)^k P(f)$.

(vi) *E is the identity operator on $L^2(\mu)$ if and only if $T^{-1}(\Sigma) = \Sigma$.*

The following theorem characterizes the n-normal composition operators.

Theorem 2.2 *Let C_T be a composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(i) *C_T is n-normal operator.*

(ii) $h_n \circ T^n E(f) = h_n f$.

Proof. For $f \in L^2(\mu)$

$$C_T^n C_T^{*n} f = C_T^n (h_n . E(f) \circ T^{-n}) = (h_n . E(f) \circ T^{-n}) \circ T^n = h_n \circ T^n . E(f).$$

Also,

$$C_T^{*n} C_T^n f = C_T^{*n} (f \circ T^n) = h_n . E(f \circ T^n) \circ T^{-n} = h_n f.$$

If C_T is n-normal composition operator then

$$C_T^n C_T^{*n} = C_T^{*n} C_T^n \iff h_n \circ T^n E(f) = h_n f. \quad \blacksquare$$

Corollary 2.3. *If $T^{-1}\Sigma = \Sigma$, then C_T is n-normal operator if and only if $h_n \circ T^n = h_n$.*

Theorem 2.4. *If C_T is a composition operator on $L^2(\mu)$, then the following statements are equivalent:*

(i) *C_T is n-normal.*

(ii) $\|f \circ T^n\| = \|h_n E(f) \circ T^{-n}\|$ *for* $f \in L^2(\mu)$.

COROLLARY 2.5. *If C_T is the composition operator and C_T^* is its adjoint, then the following statements are equivalent:*

(i) *C_T is n-normal operator.*

(ii) *C_T^* is n-normal operator.*

(iii) $\|f \circ T^n\| = \|h_n E(f) \circ T^{-n}\|$ *for $f \in L^2(\mu)$.*

COROLLARY 2.6. *If C_T is n-normal composition operator then* $\mathrm{Ker}(C_T^n) = \mathrm{Ker}(C_T^{*n})$.

The following example shows that there exists a composition operator which is quasinormal but not n-normal operator for any $n \in \mathbb{N}$.

EXAMPLE 2.7. Let $\mathbb{X} = \mathbb{Z}_+$ with μ as the counting measure. Let T^n be the transformation defined as $T^n(j) = j - n$ for all $j \in \mathbb{N}$. Then C_T^n is a unilateral shift operator on l^2 which is quasinormal but not n-normal.

In [3], it has been shown that if $h \circ T \leq h$ then C_T^n is hyponormal for each $n \in \mathbb{N}$. Also, we know that if C_T is n-hyponormal and compact then C_T is n-normal.

EXAMPLE 2.8. Let $\{e_i\}_{i=-\infty}^{+\infty}$ be an orthonormal basis of H. Define T as

$$Te_i = \begin{cases} e_{i+1}, & \text{if } i \leq 0 \\ 4e_{i+1}, & \text{if } i \geq 0 \end{cases} \quad \text{where} \quad b_i = \begin{cases} 1, & \text{if } i \leq 0 \\ 4, & \text{if } i \geq 0. \end{cases}$$

Then $T^k e_i = b_{i,k} e_{i+k}$ where $|b_{i,k}| \leq |b_{i+1,k}|$. So C_T^k is hyponormal and is not compact. Thus C_T is not n-normal operator.

LEMMA 2.9. *Let C_T, $M_h \in \mathbb{B}(L^2(\mu))$. Then $C_T^n M_h = M_h C_T^n$ if and only if $h = h \circ T^n$ a.e., where M_h is the multiplication operator induced by h.*

THEOREM 2.10. *If C_T and $C_S \in \mathbb{B}(L^2(\mu))$ are n-normal composition operators. Then the following statements are equivalent:*

(i) *$C_T^n C_S^n$ and $C_S^n C_T^n$ are normal operators.*

(ii) *$h_{S^n T^n} = h_{T^n S^n} = h_{S^n} h_{T^n}$ a.e., where $h_{S^n T^n}$, $h_{T^n S^n}$ are the Radon-Nikodym derivatives of $\mu(T^n \circ S^n)^{-1}$, $\mu(S^n \circ T^n)^{-1}$ with respect to μ, respectively.*

Proof. (1) $\Rightarrow$ (2).

For $f \in L^2(\mu)$ and using Proposition 1.1.,

$$\langle C_T^n C_S^n f, f \rangle = \int |f|^2 \circ S^n \circ T^n \, d\mu = \int |f|^2 \, d\mu (S^n \circ T^n)^{-1} = h_{S^n T^n}.$$

Also,

$$\langle C_S^n, C_T^n f, f \rangle = \int |f|^2 \circ T^n \circ S^n \, d\mu = \int |f|^2 \, d\mu (T^n \circ S^n)^{-1} = h_{T^n S^n}.$$

If $C_T^n C_S^n$ is a normal operator then

$$\begin{aligned} (C_{T^n} C_{S^n})^* (C_{T^n} C_{S^n}) &= (C_{T^n} C_{S^n})(C_{T^n} C_{S^n})^* \\ C_{S^n}^* C_{T^n}^* C_{T^n} C_{S^n} &= C_{T^n} C_{S^n} C_{S^n}^* C_{T^n}^* \\ C_{S^n}^* C_{S^n} M_{h_{T^n}} &= C_{T^n} M_{h_{S^n}} C_{T^n}^* \\ M_{h_{T^n}} M_{h_{S^n}} &= M_{h_{S^n}} M_{h_{T^n}}. \end{aligned}$$

Also,

$$\begin{aligned} M_{h_{T^n S^n}} &= (C_{T^n} C_{S^n})^* (C_{T^n} C_{S^n}) = C_{S^n}^* C_{T^n}^* C_{T^n} C_{S^n} \\ &= C_{S^n}^* M_{h_{T^n}} C_{S^n} = C_{S^n}^* C_{S^n} M_{h_{T^n}} = M_{h_{S^n}} M_{h_{T^n}}. \end{aligned}$$

Similarly, $M_{h_{S^n T^n}} = M_{h_{T^n}} M_{h_{S^n}}$.

(2) $\Rightarrow$ (1) is obvious. ■

Corollary 2.11. *If C_T is n-normal operator then any positive power of C_T is also n-normal.*

The following theorem follows from the definition of the *n-quasinormal operator*.

Theorem 2.12. *Let $C_T \in \mathbb{B}(L^2(\mu))$ be a composition operator. Then C_T is n-Quasinormal operator if and only if it commutes with the multiplication operator M_h induced by h.*

Corollary 2.13. *Let $C_T \in \mathbb{B}(L^2(\mu))$ be a composition operator. Then C_T is n-quasinormal operator if and only if it $h \circ T^n = h$ a.e. for $n \in \mathbb{N}$.*

Theorem 2.14. *Let $C_T \in \mathbb{B}(L^2(\mu))$ be a composition operator. Then C_T^* is n-quasinormal operator then $h = h \circ T^n$.*

Proof. Suppose that C_T^* is n-quasinormal. Then

$$C_T^{*n}(C_T^* C_T) = (C_T^* C_T) C_T^{*n}.$$

By taking adjoint on both the sides, we get

$$\begin{aligned} C_T^* C_T C_T^n &= C_T^n C_T^* C_T \\ M_h C_T^n &= C_T^n M_h \\ M_h C_T^n &= M_{h \circ T^n} C_T^n. \end{aligned}$$

Hence, $h = h \circ T^n$ a.e. ■

Corollary 2.15. *Let $C_T \in \mathbb{B}(L^2(\mu))$ be a composition operator. Then the following statements are equivalent:*

(i) *C_T is n-quasinormal operator.*

(ii) *C_T^* is n-quasinormal operator.*

(iii) *$h = h \circ T^n$ a.e.*

3. n-normal weighted composition operators and n-quasinormal weighted composition operators

Let (X, Σ, μ) be a σ-finite measure space and $W \equiv W_{\phi,T}$ be the weighted composition operator on $L^2(\mu)$ induced by the complex valued function ϕ and a measurable transformation T. The adjoint W^* of W is given by $W^* f = hE(\phi f) \circ T^{-1}$ for f in $L^2(\mu)$. For a natural number n, we put $\phi_n = \phi.(\phi \circ T).(\phi \circ T^2) \cdots (\phi \circ T^{(n-1)})$. For $f \in L^2(\mu)$, $W^n f = \phi_n.f \circ T^n$ and $W^{*n} f = h_n.E(\phi_n.f) \circ T^{-n}$.

Theorem 3.1. *Let W be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(i) *W is n-normal operator.*

(ii) $\phi_n(h_n \circ T^n.E(\phi_n f)) = h_n.E(\phi_n^2) \circ T^{-n} f$.

Proof. For $f \in L^2(\mu)$,

$$\begin{aligned} W^n W^{*n} f &= W^n(h_n.E(\phi_n f) \circ T^{-n}) = \phi_n(h_n.E(\phi_n f) \circ T^{-n}) \circ T^n \\ &= \phi_n(h_n \circ T^n.E(\phi_n f)). \end{aligned}$$

Also,

$$\begin{aligned} W^{*n} W^n f &= W^{*n}(\phi_n.f \circ T^n) = h_n.E(\phi_n^2.f \circ T^n) \circ T^{-n} \\ &= h_n.E(\phi_n^2) \circ T^{-n} f. \end{aligned}$$

Suppose that W is a n-normal weighted composition operator. Then

$$\begin{aligned} &\phi_n h_n.E(\phi_n f) \circ T^{-n} = h_n.E(\phi_n^2.f \circ T^n) \circ T^{-n} \\ \Longleftrightarrow \quad &\phi_n(h_n \circ T^n.E(\phi_n f)) = h_n.E(\phi_n^2) \circ T^{-n} f. \quad \blacksquare \end{aligned}$$

Corollary 3.2. *Let W be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(i) *W is n-normal operator.*

(ii) *W^* is n-normal operator.*

(iii) $\phi_n(h_n \circ T^n.E(\phi_n f)) = h_n.E(\phi_n^2) f$ *for* $f \in L^2(\mu)$.

Proposition 3.3. *For $\phi \geq 0$,*

(i) $W^* W f = hE[(\phi^2)] \circ T^{-1} f$.

(ii) $WW^* f = \phi(h \circ T)E(\phi f)$.

Theorem 3.4. *Let W be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(i) *W is n-quasinormal operator.*

(ii) $\phi_n h.E(\phi^2) \circ T^{-1}.f \circ T^n = h.E(\phi_{n+2}) \circ T^{-1}.f \circ T^n$.

Proof. For $f \in L^2(\mu)$,

$$W^n(W^* W) f = W^n(h.E(\phi^2) \circ T^{-1} f) = \phi_n h.E(\phi^2) \circ T^{-1}.f \circ T^n.$$

Also,

$$\begin{aligned} (W^* W) W^n f &= (W^* W)(\phi_n.f \circ T^n) = W^*(\phi_{n+1} f \circ T^{n+1}) \\ &= h.E(\phi_{n+2}.f \circ T^{n+1}) \circ T^{-1} = h.E(\phi_{n+2}) \circ T^{-1}.f \circ T^n. \end{aligned}$$

Suppose that W is a n-quasinormal operator. Then

$$W^n(W^* W) = (W^* W) W^n$$

$$\phi_n h.E(\phi^2) \circ T^{-1}.f \circ T^n = h.E(\phi_{n+2}) \circ T^{-1}.f \circ T^n. \quad \blacksquare$$

THEOREM 3.5. *Let W be a weighted composition operator on $L^2(\mu)$. Then the following statements are equivalent:*

(i) *W^* is n-quasinormal operator.*

(ii) $h_n.E(\phi_n.hE(\phi^2) \circ T^{-1}.f) = h \circ T^{-1}.E(\phi^2.h_n E(\phi_n.f)$.

Proof.

$$\begin{aligned} W^{*n}(W^*W)f &= W^{*n}(h.E((\phi^2) \circ T^{-1}.f)) \\ &= h_n.E(\phi_n.hE(\phi^2) \circ T^{-1}.f) \circ T^{-n}. \end{aligned}$$

Also,

$$\begin{aligned} (W^*W)W^{*n}f &= (W^*W)(h_n.E(\phi_n.f) \circ T^{-n}) \\ &= W^*(\phi.(h_n.E(\phi_n.f)) \circ T^{-n} \circ T)) \\ &= h.E(\phi^2.h_n.E(\phi_n.f) \circ T^{-n} \circ T \circ T^{-1} \\ &= h \circ T^{-1}.E(\phi^2.h_n E(\phi_n.f) \circ T^{-n}. \end{aligned}$$

Suppose that W^* is a n-quasinormal weighted composition operator. Then

$$W^{*n}(W^*W) = (W^*W)W^{*n}$$
$$h_n.E(\phi_n.hE(\phi^2) \circ T^{-1}.f) = h \circ T^{-1}.E(\phi^2.h_n E(\phi_n.f). \quad \blacksquare$$

REFERENCES

[**1**] O. Ahmed and M.S. Ahmed, *On the class of n-power quasi-normal operators on Hilbert space*, Bull. Math. Anal. Appl. **2** (2011), 213–228.

[**2**] S.A. Alzuraiqi and A.B. Patel, *On n-normal operators*, General Math. Notes **1** (2010), 61–73.

[**3**] P. Dibrell and J.T. Campbell, *Hyponormal powers of composition operators*, Proc. Amer. Math. Soc. **102** (1988), 914–918.

[**4**] D.J. Harrington and R. Whitley, *Seminormal composition operator*, J. Operator Theory **11** (1981), 125–135.

[**5**] J. Campbell and J. Jamison, *On some classes of weighted composition operators*, Glasgow Math.J. **32** (1990), 82–94.

[**6**] S. Panayappan, D. Senthilkumar and K. Thirugnanasambandam, *A note on powers of p-hyponormal and product of quasinormal composition operators*, Int. J. Math. Anal. **3** (2009), 49–54.

[**7**] R.K. Singh, *Compact and quasinormal composition operators*, Proc. Amer. Math. Soc. **45** (1974), 80–82.

(received 30.03.2013; available online 15.11.2013)

Department of Mathematics, Delhi College of Arts and Commerce, University of Delhi, Delhi 110023, India

E-mail: dishna2@yahoo.in

Department of Mathematics, University of Delhi, Delhi 110007, India

E-mail: nehaphd@yahoo.com