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COMPACT FAMILIES AND CONTINUITY OF THE INVERSE

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Abstract. We generalize a topological game introduced by Kenderov, Kortezov and Moors and use it to establish conditions under which a paratopological group is a topological group.

1. Introduction

A *paratopological group* is a group equipped with a topology which makes multiplication continuous. In this paper we consider the much studied question of when a paratopological group is a topological group. At this juncture a number of recent results should be mentioned, including [1–4, 13, 14].

A natural starting point for our current research is [9], where Ellis proved the following theorem.

THEOREM 1.1. *Let X be an paratopological group with a locally compact Hausdorff topology. Then X is a topological group.*

Subsequent results from [7] and [16] were summarized by Pfister in [15], wherein new proofs were constructed that rely on the following lemma.

LEMMA 1.2. *Let X be a regular paratopological group and $\mathcal{U}$ be the neighbourhood filter of the neutral element.*

(i) For each $U \in \mathcal{U}$ there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$ such that $\overline{U}_1^2 \subset U$, $\overline{U}_{n+1}^2 \subset U_n$ for each $n \in \mathbb{N}$.

(ii) Suppose $(U_n)_{n\in\mathbb{N}}$ is chosen as in (i), $x_n \in U_n$, $y_k := x_1 \cdot x_2 \cdots x_k$ and $(y_k)_{k\in\mathbb{N}}$ has a cluster point. Then for each $n \in \mathbb{N}$ there exists $k > n$ such that $x_k^{-1} \in U_n$.

Later, in [10], Kenderov, Kortezov and Moors defined a topological game and use it to construct *strongly Baire* spaces which, by way of the above lemma, are then used to improve upon the earlier results of Pfister et al. This topological

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game implicitly defines a class of families that are, in the language of [8], countably compact relative to their intersection. In what follows, we establish conditions under which a paratopological group is a topological group. In particular, we show that Pfister's lemma, together with its assumption of regularity, is superfluous.

2. Compact families

Let (X,τ) be a topological space. If $A \subset X$, then $\overline{A}$ is the closure of A, $A^\circ = X \setminus \overline{X \setminus A}$ is the interior of A, and $A' = X \setminus A$. We use script to denote families of nonempty subsets of X and blackboard bold to denote classes of families.

The *adherence of* a family $\mathcal{B}$ is defined by

$$adh\mathcal{B} = \bigcap \overline{\mathcal{B}} = \bigcap \{\overline{B} : B \in \mathcal{B}\}.$$

If $\mathcal{B}$ is a filter, its *adherent points* (i.e., those in $adh\mathcal{B}$) are called *cluster points*. If $\mathcal{B}$ is a filter base, its adherent points are the cluster points of the filter generated by $\mathcal{B}$. A subbase of a filter may have adherent points that are not the cluster points of the filter it generates.

Let $\mathcal{A}$ be another family of subsets of X. We write $\mathcal{A}\#\mathcal{B}$, and say that $\mathcal{A}$ *meshes with* $\mathcal{B}$, if $A \cap B \neq \emptyset$ for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$. We write $\mathcal{A} \geq \mathcal{B}$, and say that $\mathcal{A}$ is *finer than* $\mathcal{B}$ or that $\mathcal{B}$ is *coarser than* $\mathcal{A}$, if for each $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $A \subset B$.

Let $\mathcal{P}$ be a family of subsets of X and $A \subset X$. We say that $\mathcal{P}$ is a *cover* (resp. *overcover* of A) if $A \subset \bigcup \mathcal{P}$ (resp. $A \subset \bigcup \mathcal{P}^\circ = \bigcup \{P^\circ : P \in \mathcal{P}\}$).

Denote by $\mathbb{P}_0$ the class that consists of all finite subfamilies of all the members $\mathcal{P}$ of a class $\mathbb{P}$. The family $\mathcal{P} \in \mathbb{P}$ is said to be *additive* if it is closed under finite unions, that is, if

$$\mathcal{P} = \mathcal{P}^\cup := \{\bigcup \mathcal{R} : \mathcal{R} \subset \mathcal{P}, \mathcal{R} \in \mathbb{P}_0\}.$$

The class $\mathbb{P}$ is said to be *additively saturated* if, for each $\mathcal{P} \in \mathbb{P}$, $\mathcal{P}^\cup \in \mathbb{P}$.

We refrain from using the 'prime' notation at the level of classes to denote complements. This makes it possible to write $\mathbb{P}' = \{\mathcal{P}' : \mathcal{P} \in \mathbb{P}\}$ without any risk of confusion.

We adopt the following notation:

- The symbol $\mathcal{P}_*$ is used to denote the filter generated by the family of complements $\mathcal{P}'$ *only* if $\mathcal{P}'$ is a filter base (of $\mathcal{P}_*$). Only such filters are considered to be the filters *dual* to the class $\mathbb{P}$.
- $\mathbb{P}_*$ denotes the class of dual filters. Note that if $\mathbb{P}$ is additively saturated, this is the class of *all* filters generated by the families $\mathcal{P}'$ in $\mathbb{P}'$. This explains the significance of the condition of additive saturation of $\mathbb{P}$.

The family $\mathcal{B}$ is said to be $\mathbb{P}$-*compact*[1] *relative to* A if it satisfies the condition (i) or (ii) of the following theorem.

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[1] Or $\mathbb{P}$-*compactoid*, see [11].

Duality 2.1. *Consider the following conditions.*

(i) *For each overcover $\mathcal{P} \in \mathbb{P}$ of A there exists $B \in \mathcal{B}$ and a finite subset $\mathcal{R}$ of $\mathcal{P}$ covering B (i.e., such that $B \subset \bigcup \mathcal{R}$).*

(ii) *For each $\mathcal{P} \in \mathbb{P}$ such that for each finite subset $\mathcal{R}$ of $\mathcal{P}$ the family $\mathcal{B}$ meshes with $\bigcap \mathcal{R}'$, $\mathcal{P}'$ has an adherent point in A.*

(iii) *For each filter $\mathcal{P}_* \in \mathbb{P}_*$ meshing with $\mathcal{B}$, $\mathcal{P}_*$ has a cluster point in A.*

Then (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii) and, if $\mathbb{P}$ is additively saturated, all conditions are equivalent.

Let $\mathbb{F}_\omega$ be the class of filters that is dual to the class of countable families of subsets of X. Clearly, these are the filters on X admitting a countable base. This class is additively saturated so that a family $\mathcal{B}$ is $\mathbb{F}_\omega$-*compact relative to its intersection* if and only if each filter $\mathcal{J} \in \mathbb{F}_\omega$ meshing with $\mathcal{B}$ has a cluster point in $\{\bigcap B : B \in \mathcal{B}\}$. The interested reader is referred to [8] for the proof of the above theorem and further results on compact families.

3. Continuity of the inverse

We define the $\mathcal{G}(X)$ game played between players α and β on a topological space (X, τ) as follows: Player β goes first and chooses a non-empty open subset $B_1 \subseteq X$. Player α responds by choosing an non-empty open subset $A_1 \subseteq B_1$. Player β now chooses $B_2 \subseteq A_1$. Continuing in this manner, the players produce a sequence $(A_n, B_n)_{n\in\mathbb{N}}$ of open sets called a play of the $\mathcal{G}(X)$ game. Player α wins the play if the filter generated by $\{A_n : n \in \mathbb{N}\}$ is $\mathbb{F}_\omega$-compact relative to its intersection. Otherwise, β wins the play.

A strategy $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ for player β is a sequence of τ-valued functions that specifies the choice of player β for all possible outcomes. A strategy σ is a *winning strategy* for player β if each play of the game according to the strategy results in a win for player β. The interesting case occurs when β does not have a winning strategy in the $\mathcal{G}(X)$ game.

Theorem 3.1. *Let X be a paratopological group in which β does not have a winning strategy in the $\mathcal{G}(X)$ game. Then X is a topological group.*

Proof. It suffices to show that the inverse operation is *quasi-continuous* at the neutral element [10, Lemma 4]. Suppose not and let $U \in \mathcal{U}$ such that $V_\alpha^{-1} \nsubseteq U$ for each open set $V_\alpha \subseteq U$. We proceed by induction to define a strategy σ for the player β in the $\mathcal{G}(X)$ game.

Base step. Set $x_1 = \mathrm{e}$, $U_1 = U$, and let $\sigma_1(\emptyset) = x_1 \cdot U_1$.

Inductive step. Let $y_k = x_1 \cdot x_2 \cdots x_k$ and suppose that for each sequence $(A_1, \cdots, A_{j-1})$, $1 \leq j \leq n$, we have defined x_j, U_j and $\sigma_j(A_1, A_2, \cdots, A_{j-1}) = (y_j) \cdot U_j$ such that:

(i) $x_j \in (y_{j-1})^{-1} \cdot A_{j-1}$ and $x_j^{-1} \notin U$,

(ii) $(y_j) \cdot U_j \subseteq A_{j-1}$,

(iii) $U_j \cdot U_j \subseteq U_{j-1}$.

For each sequence $(A_1, A_2, \cdots, A_n)$ of length n we choose x_{n+1} and $U_{n+1} \subseteq U$ such that:

(i) $x_{n+1} \in (y_n)^{-1} \cdot A_n$ and $x_{n+1}^{-1} \notin U$,

(ii) $(y_{n+1}) \cdot U_{n+1} \subseteq A_n$,

(iii) $U_{n+1} \cdot U_{n+1} \subseteq U_n$.

We define $\sigma_{n+1}(A_1, A_2, \cdots, A_n) = y_{n+1} \cdot U_{n+1}$. Since player β does not have a winning strategy in the $\mathcal{G}(X)$ game there is a play $(A_n, B_n)_{n\in\mathbb{N}}$ where player α wins, so that the filter generated by $\{A_n : n \in \mathbb{N}\}$ is $\mathbb{F}_\omega$-compact relative to its intersection. Note that the family $\{y_n \cdot U_n : n \in \mathbb{N}\}$ is finer than $\{A_n : n \in \mathbb{N}\}$ and also generates a filter that is $\mathbb{F}_\omega$-compact relative to its intersection.

The sequence $(y_n)_{n\in\mathbb{N}}$ satisfies $y_{n+1} \in y_n \cdot U_n$ so the filter (base) associated with the sequence, and hence the sequence itself[2], must have a cluster point $y \in \{\bigcap y_n \cdot U_n : n \in \mathbb{N}\}$. Choose $U_j \in \{U_n : n \in \mathbb{N}\}$ and $k > j$ such that $y_k \in y \cdot U_j$. Then $x_{k+1}^{-1} = y_{k+1}^{-1} \cdot y_k \in y_{k+1}^{-1} \cdot y \cdot U_j$.

We claim $y_{k+1}^{-1} \cdot y \in U_{k+1}$. Indeed, $y \in \{\bigcap y_n \cdot U_n : n \in \mathbb{N}\} \Rightarrow y \in y_n \cdot U_n$ for each $n \in \mathbb{N}$ so that $y_{k+1}^{-1} \cdot y \in y_{k+1} \cdot y_n \cdot U_n$ for each $n \in \mathbb{N}$ as well. It follows that $y_{k+1}^{-1} \cdot y \in y_{k+1}^{-1} \cdot y_{k+1} \cdot U_{k+1} = (y_{k+1}^{-1} \cdot y_{k+1}) \cdot U_{k+1} = U_{k+1}$.

Clearly, $x_{k+1}^{-1} \in y_{k+1}^{-1} \cdot y \cdot U_j = (y_{k+1}^{-1} \cdot y) \cdot U_j \subseteq U_{k+1} \cdot U_j \subseteq U_j \cdot U_j \subseteq U$. This contradiction establishes the continuity of the inverse. ■

4. Strongly Baire spaces

Let D be a subset of X. Consider the $\mathcal{G}_S(D)$ game, played between players α and β on a topological space (X, τ), defined as follows: Player β goes first and chooses a non-empty open subset $B_1 \subseteq X$. Player α responds by choosing an non-empty open subset $A_1 \subseteq B_1$. Player β now chooses $B_2 \subseteq A_1$. Continuing in this manner, the players produce a sequence $(A_n, B_n)_{n\in\mathbb{N}}$ of open sets called a play of the $\mathcal{G}_S(D)$ game. Player α wins the play if $\bigcap A_n \neq \emptyset$ and each sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A_n \cap D$ has a cluster point in X. Otherwise, β wins the play.

A topological space (X, τ) is *strongly Baire* [10, p. 158] if it is *regular* and there exists a dense subset D of X such that player β does not have a winning strategy in the $\mathcal{G}_S(D)$ game played on X.

We define the class $\mathbb{F}_\omega \vee D$ as the class of filters having a countable base consisting of elements of D and modify the $\mathcal{G}(X)$ game of Section 3 as follows: Player β wins a play $(A_n, B_n)_{n\in\mathbb{N}}$ *of the* $\mathcal{G}(D)$ *game* if the filter generated by $\{A_n : n \in \mathbb{N}\}$ is $\mathbb{F}_\omega \vee D$-compact relative to its intersection.

Proposition 4.1. *Suppose X is a topological space and $(A_n)_{n\in\mathbb{N}}$ is a sequence of neighbourhoods linearly ordered by containment. The following are equivalent:*

(i) Each sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A_n \cap D$ has a cluster point in X.

(ii) The filter generated by $\{A_n : n \in \mathbb{N}\}$ is $\mathbb{F}_\omega \vee D$-compact relative to its adherence.

[2] See the article by Bartle, [5], for a rigorous examination of the relationship between cluster points of a sequence and the cluster points of its associated filter base.

Proof. Assume each sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A_n \cap D$ has a cluster point in X and let $\mathcal{J} \in \mathbb{F}_\omega \vee D$ be a filter meshing with the filter generated by $\{A_n : n \in \mathbb{N}\}$. Let $\{J_n : n \in \mathbb{N}\}$ be a countable base of $\mathcal{J}$ and choose a_n so that $a_n \in A_n \cap J_n \cap D$ for each n. By assumption, $(a_n)_{n\in\mathbb{N}}$ has a cluster point. Clearly, this cluster point is contained in the adherence of the filter generated by $\{A_n : n \in \mathbb{N}\}$.

Now assume the filter generated by $\{A_n : n \in \mathbb{N}\}$ is $\mathbb{F}_\omega \vee D$-compact relative to its adherence and let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $a_n \in A_n \cap D$ for each n. The filter associated with this sequence is an element of $\mathbb{F}_\omega \vee D$ meshing with the filter generated by $\{A_n : n \in \mathbb{N}\}$. ∎

Recall that a point $x \in X$ is a *q-point* [12] if there exists a countable subfamily $\{A_n : n \in \mathbb{N}\}$ of the neighbourhood filter $\mathcal{U}(x)$ such that each sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A_n$ has a cluster point in X. With $D = X$ we obtain the following.

Corollary 4.2. *Let $\mathcal{U}(x)$ be the neighbourhood filter of $x \in X$. The point x is a q-point if and only if there exists filter $\mathcal{J} \in \mathbb{F}_\omega$, $\mathcal{J} \leq \mathcal{U}(x)$ such that $\mathcal{J}$ is $\mathbb{F}_\omega$-compact relative to its adherence.*

Proposition 4.1 allows us to reformulate the definition of a strongly Baire space. In particular, player α wins a play of the $\mathcal{G}_S(D)$ game if $\bigcap A_n \neq \emptyset$ and the filter generated by $\{A_n : n \in \mathbb{N}\}$ is $\mathbb{F}_\omega \vee D$-compact relative to its adherence.

Proposition 4.3. *If X is a strongly Baire space, then there exists a dense subset D of X such that player β does not have a winning strategy in the $\mathcal{G}(D)$ game played on X.*

Proof. Suppose X is regular and let $\sigma = (\sigma_n)_{n\in\mathbb{N}}$ be a winning strategy for β in the $\mathcal{G}(D)$ game played on X. It suffices to show that player β has a winning strategy $\gamma = (\gamma_n)_{n\in\mathbb{N}}$ in the $\mathcal{G}_S(D)$ game played on X.

Base step. Player β chooses $\sigma_1(\emptyset) = B_1 \subseteq X$ according to the winning strategy of the $\mathcal{G}(D)$ game and we define $\gamma_1(\emptyset) = \sigma_1(\emptyset) = B_1^S$.

Inductive step. Suppose that for each sequence of plays $(A_1^S, \cdots, A_{j-1}^S)$, $1 \leq j \leq n$ we have defined $\gamma_j(A_1^S, A_2^S, \cdots, A_{j-1}^S) = B_j^S$ such that $\sigma_j(A_1, \cdots, A_{j-1}) = B_j$ and $\overline{A_j} \subseteq A_j^S \subseteq B_j = B_j^S$.

The space X is regular, so for each sequence of plays $(A_1^S, A_2^S, \cdots, A_n^S)$ of length n we choose A_n such that $\overline{A_n} \subseteq A_n^S \subseteq B_n = B_n^S$ and define $B_{n+1}^S = \gamma_{n+1}(A_1^S, A_2^S, \cdots, A_n^S)$ to be $B_{n+1} = \sigma_{n+1}(A_1, A_2, \cdots, A_n)$.

Each play $(A_n, B_n)_{n\in\mathbb{N}}$ of the $\mathcal{G}(D)$ game results in a filter generated by $\{A_n : n \in \mathbb{N}\}$ that is not $\mathbb{F}_\omega \vee D$-compact relative to $\bigcap_{n\in\mathbb{N}} A_n$. The filter generated by $\{A_n : n \in \mathbb{N}\}$ is finer than the filter generated by $\{A_n^S : n \in \mathbb{N}\}$ so that this filter also is not $\mathbb{F}_\omega \vee D$-compact relative to $\bigcap_{n\in\mathbb{N}} A_n$.

Clearly, $\bigcap_{n\in\mathbb{N}} A_n \subseteq \bigcap_{n\in\mathbb{N}} \overline{A_n^S}$. Now suppose x is an element of $\bigcap_{n\in\mathbb{N}} \overline{A_n^S}$ but is not an element of $\bigcap_{n\in\mathbb{N}} A_n$. In this case, some element A_k of the family $\{A_n : n \in \mathbb{N}\}$ does not contain x. But $x \in \overline{A_{k+1}^S} \subseteq B_{k+1}^S = B_{k+1} \subseteq A_k$. This contradiction establishes the equality of $\bigcap_{n\in\mathbb{N}} \overline{A_n^S}$ and $\bigcap_{n\in\mathbb{N}} A_n$ and shows that the filter generated by $\{A_n^S : n \in \mathbb{N}\}$ is not $\mathbb{F}_\omega \vee D$-compact relative to $\bigcap_{n\in\mathbb{N}} \overline{A_n^S}$. We conclude that the strategy we have constructed for β in the $\mathcal{G}_S(D)$ game is a winning one. ■

The following result appears without proof in [10]. We choose to take a more pedestrian approach.

Lemma 4.4. *Suppose U and V are neighborhoods of the neutral element e in a paratopological group X such that $U \cdot U \subseteq V$ and D is any subset of X such that $D^{-1} \subseteq U$. Then $(\overline{D})^{-1} \subseteq V$.*

Proof. Assume the result does not hold and choose $x \in \overline{D}$ such that $x^{-1} \notin V$. Then $x^{-1} \notin V \Rightarrow x^{-1} \notin U \Rightarrow x \cdot x^{-1} \notin x \cdot U \Rightarrow e \notin x \cdot U$. Now suppose there exists $y \in D \cap x \cdot U$. Then $y \in D \Rightarrow y^{-1} \in U \Rightarrow y \cdot y^{-1} \in x \cdot U \cdot U \Rightarrow e \in x \cdot U \cdot U \Rightarrow e \in x \cdot V$, a contradiction. But $D \cap x \cdot U = \emptyset \Rightarrow x \notin \overline{D}$. ■

Theorem 4.5. *Let $(X, \cdot, \tau)$ be a paratopological group and suppose there exists a dense subset D of X such that player β does not have a winning strategy in the $\mathcal{G}(D)$ game played on X. Then X is a topological group.*

Proof. As before, it suffices to show that the inverse is quasi-continuous at the neutral element. Suppose not and find $U \in \mathcal{U}$ so that $D \cap V_\alpha^{-1} \nsubseteq U$ for each open $V_\alpha \subseteq U$ and player β does not have a winning strategy in the $\mathcal{G}(D)$ game. The above lemma makes this selection possible. Continue in a manner similar to the proof of Theorem 3.1 to obtain the result. ■

Corollary 4.6. *Let $(X, \cdot, \tau)$ be a regular paratopological group.*

(i) If X is locally compact, then X is a topological group [9].

(ii) If X is completely metrizable, then X is a topological group [16].

(iii) If X is locally Čech-complete, then X is a topological group [7].

(iv) If X is a strongly Baire space, then X is a topological group [10].

We conclude by providing an example of a topological space in which player β does not have a winning strategy in the $\mathcal{G}(D)$ game played on X and such that X is not strongly Baire.

Example 4.7. A non-regular topological space in which player β does not have a winning strategy in the $\mathcal{G}(D)$ game played on X.

Let $(\mathbf{R}, \tau)$ be the set of real numbers with its usual topology. Let $A = \{1 \setminus n : n \in \mathbb{N}\}$, $D = \mathbf{R} \setminus A$, and define a topology τ^* on the set $\mathbf{R}$ such that $\tau \cup D$ is a subbase of τ^*. Note that $(\mathbf{R}, \tau^*)$ is not regular[3]. We claim that player β does not have a winning strategy in the $\mathcal{G}(D)$ game played on $(\mathbf{R}, \tau^*)$. Indeed, for each choice $B_n \subseteq A_{n-1}$ for player β, player α responds by choosing $A_n \subseteq B_n$ such that

[3] The set $D = \mathbf{R} \setminus A$ is dense in $(\mathbf{R}, \tau)$. See problem 20, page 138 in [6].

$A_n \in \tau$, $\overline{A_n} \subseteq A_{n-1}$ and diam $A_n < 1 \setminus n$. The filter generated by $\{A_n : n \in \mathbb{N}\}$ is $\mathbb{F}_\omega \vee D$-compact relative to its intersection, as required.

REFERENCES

[**1**] A. V. Arhangel'skii, M. M. Choban and P. S. Kenderov,*Topological games and topologies on groups*, Math. Maced. **8** (2010), 1–19.

[**2**] A. V. Arhangel'skii, M. M. Choban and P. S. Kenderov, *Topological games and continuity of group operations*, Topology Appl. **157** (2010), 2542–2552.

[**3**] A. V. Arhangel'skii and E. A. Reznichenko, *Paratopological and semitopological groups versus topological groups*, Topology Appl. **151** (2005), 107–119.

[**4**] A. V. Arhangel'skii and M. G. Tkachenko, *Topological Groups and Related Structures*, Atlantic Press, Amsterdam/Paris, 2008.

[**5**] Bartle, *Nets to filters*, Amer. Math. Monthly **62** (1955), 551–557.

[**6**] N. Bourbaki, *General Topology*, Part 1, Addison-Wesley, Reading, 1966.

[**7**] N. Brand, *Another note on the continuity of the inverse*, Arch. Math. **39** (1982), 241–245.

[**8**] B.L. Davis and I. Labuda, *Unity of compactness*, Quaestiones Math. **30** (2007), 193–206.

[**9**] R. Ellis, *A note on the continuity of the inverse*, Proc. Amer. Math. Soc. **8** (1957), 372–373.

[**10**] P. Kenderov, I. Kortezov and W. Moors, *Topological games and topological groups*, Topology Appl. **109** (2001), 157–165.

[**11**] I. Labuda, *Compactoidness*, Rocky Mountain J.Math. **36** (2) (2006), 555–574.

[**12**] E. Michael, *A note on closed maps and compact sets*, Israel J. Math. **2** (1964), 173–176.

[**13**] W. B. Moors, *Any semitopological group that is homeomorphic to a product of Čech-complete spaces is a topological group* Set-Valued Var. Anal., 2013.

[**14**] W. B. Moors, *Semitopological groups, Bouziad spaces and topological groups*, Topology Appl. **160** (2013), 2038–2048.

[**15**] H. Pfister, *Continuity of the inverse*, Proc. Amer. Math. Soc. **96** (1985), 312–314.

[**16**] W. Zelasko, *A theorem on B_0 division algebras*, Bull. Acad. Polon. Sci. **8** (1960), 373–375.

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