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POWER MEAN INEQUALITY OF GENERALIZED TRIGONOMETRIC FUNCTIONS

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Abstract. The author here studies the convexity and concavity properties of the generalized p-trigonometric functions in the sense of P. Lindqvist with respect to the Power Mean.

1. Introduction

The generalized trigonometric and hyperbolic functions depending on a parameter $p > 1$ were studied by P. Lindqvist in 1995 [16]. For the case when $p = 2$, these functions coincide with elementary functions. Later on numerous authors have extended this work in various directions, see [8–10, 13, 17]. The generalized trigonometric function $\sin_p$, know as eigenfunction has been a tool in the analysis of more complicated equations, see [6, 7, 11] and the bibliography of these papers. Here we study the Power Mean inequality of $\sin_p$ and other generalized trigonometric functions.

We introduce some notation and terminology for the statement of the main results.

Given complex numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit place $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \qquad |z| < 1.$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the *shifted factorial function* or the *Appell symbol*

$$(a,n) = a(a+1)(a+2)\cdots(a+n-1)$$

for $n \in \mathbb{Z}_+$, see [1]. The integral representation of the hypergeometric function is given as follows [21, p. 20]

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(c)(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt \tag{1.1}$$

$\mathrm{Re}(c) > \mathrm{Re}(a) > 0, |\arg(1-z)| < \pi$.

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Let us start the discussion of eigenfunctions of one-dimensional p-Laplacian Δ_p on $(0,1)$, $p \in (1,\infty)$. The eigenvalue problem [13]

$$-\Delta_p u = -\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u, \quad u(0)=u(1)=0,$$

has eigenvalues $\lambda_n = (p-1)(n\pi_p)^p$, and eigenfunctions $\sin_p(n\pi_p\, t)$, $n \in \mathbb{N}$, where $\sin_p$ is the inverse function of arcsin_p, which is defined below, and

$$\pi_p = \frac{2}{p}\int_0^1 (1-s)^{-1/p} s^{1/p-1} ds = \frac{2}{p} B\left(1-\frac{1}{p}, \frac{1}{p}\right) = \frac{2\pi}{p\,\sin(\pi/p)},$$

with $\pi_2 = \pi$.

Let us consider the following homeomorphisms

$$\sin_p : (0,a_p) \to I, \quad \cos_p : (0,a_p) \to I, \quad \tan_p : (0,b_p) \to I,$$
$$\sinh_p : (0,\infty) \to I, \quad \tanh_p : (0,\infty) \to I,$$

where $I = (0,1)$ and

$$a_p = \frac{\pi_p}{2},\, b_p = 2^{-1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; \frac{1}{2}\right).$$

From integral formula (1.1) and by using the change of variables we define the inverse functions of the above homeomorphisms for $x \in I$,

$$\begin{aligned}
\operatorname{arcsin}_p x &= \int_0^x (1-t^p)^{-1/p} dt = x\, F\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; x^p\right)\\
&= x(1-x^p)^{(p-1)/p} F\left(1, 1; 1+\frac{1}{p}; x^p\right),\\
\operatorname{arctan}_p x &= \int_0^x (1+t^p)^{-1} dt = x\, F\left(1, \frac{1}{p}; 1+\frac{1}{p}; -x^p\right)\\
&= \left(\frac{x^p}{1+x^p}\right)^{1/p} F\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; \frac{x^p}{1+x^p}\right),\\
\operatorname{arsinh}_p x &= \int_0^x (1+t^p)^{-1/p} dt = x\, F\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; -x^p\right)\\
&= \left(\frac{x^p}{1+x^p}\right)^{1/p} F\left(1, \frac{1}{p}; 1+\frac{1}{p}; \frac{x^p}{1+x^p}\right),\\
\operatorname{artanh}_p x &= \int_0^x (1-t^p)^{-1} dt = x\, F\left(1, \frac{1}{p}; 1+\frac{1}{p}; x^p\right),
\end{aligned}$$

and by [10, Prop. 2.2] $\operatorname{arccos}_p x = \operatorname{arcsin}_p((1-x^p)^{1/p})$. The special case of above functions for $p=2$ is defined in terms of hypergeometric functions in [2, p. 8]. In particular, these functions reduce to the familiar functions for the case $p=2$.

For $t \in \mathbb{R}$ and $x, y > 0$, the Power Mean M_t of order t is defined by

$$M_t = \begin{cases} \left(\dfrac{x^t + y^t}{2}\right)^{1/t}, & t \neq 0,\\ \sqrt{x\,y}, & t = 0.\end{cases}$$

The main results of the paper are the following theorems:

THEOREM 1.1. *For $p>1$, $t\geq 0$ and $r,s\in(0,1)$, we have*

(1) $\mathrm{arcsin}_p(M_t(r,s))\leq M_t(\mathrm{arcsin}_p(r),\mathrm{arcsin}_p(s))$,
(2) $\mathrm{artanh}_p(M_t(r,s))\leq M_t(\mathrm{artanh}_p(r),\mathrm{artanh}_p(s))$,
(3) $\mathrm{arctan}_p(M_t(r,s))\geq M_t(\mathrm{arctan}_p(r),\mathrm{arctan}_p(s))$,
(4) $\mathrm{arsinh}_p(M_t(r,s))\geq M_t(\mathrm{arsinh}_p(r),\mathrm{arsinh}_p(s))$.

THEOREM 1.2. *For $p>1$, $t\geq 1$ and $r,s\in(0,1)$, the following relations hold*

(1) $\sin_p(M_t(r,s))\geq M_t(\sin_p(r),\sin_p(s))$,
(2) $\cos_p(M_t(r,s))\leq M_t(\cos_p(r),\cos_p(s))$,
(3) $\tan_p(M_t(r,s))\leq M_t(\tan_p(r),\tan_p(s))$,
(4) $\tanh_p(M_t(r,s))\geq M_t(\tanh_p(r),\tanh_p(s))$,
(5) $\sinh_p(M_t(r,s))\leq M_t(\sinh_p(r),\mathrm{arsinh}_p(s))$.

The above results also generalize some results of [4] (Theorem 2.5, Lemma 2.9), which are the special cases of the above theorems when $t=0$ and $t=2$.

Generalized convexity/concavity with respect to general mean values has been studied recently in [3].

Let $f\colon I\to(0,\infty)$ be continuous, where I is a subinterval of $(0,\infty)$. Let M_t be a Power Mean. We say that f is M_tM_t-convex (concave) if

$$f(M_t(x,y))\leq(\geq)M_t(f(x),f(y))\quad\text{for all } x,y\in I.$$

In conclusion, we see that the above results are (M_t,M_t)-convexity or (M_t,M_t)-concavity properties of the functions involved. In view of [3], it is natural to expect that similar results might also hold for some other pairs (M,N) of mean values.

2. Preliminaries and proofs

For easy reference we record the following lemma from [2], which is sometimes called the *monotone l'Hospital rule*.

LEMMA 2.1. [2, Theorem 1.25] *For $-\infty<a<b<\infty$, let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b). Let $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$[f(x)-f(a)]/[g(x)-g(a)]\quad\text{and}\quad[f(x)-f(b)]/[g(x)-g(b)].$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

For the next two lemmas see [4, Theorems 1.1, 1.2, 2.5 & Lemma 3.6].

LEMMA 2.2. *For $p>1$ and $x\in(0,1)$, we have*

(1) $\left(1+\frac{x^p}{p(1+p)}\right)x < \operatorname{arcsin}_p x < \frac{\pi_p}{2}\,x$,

(2) $\left(1+\frac{1-x^p}{p(1+p)}\right)(1-x^p)^{1/p} < \operatorname{arccos}_p x < \frac{\pi_p}{2}\,(1-x^p)^{1/p}$,

(3) $\frac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}} < \operatorname{arctan}_p x < 2^{1/p}\; b_p\left(\frac{x^p}{1+x^p}\right)^{1/p}$,

(4) $z\left(1+\frac{\log(1+x^p)}{1+p}\right) < \operatorname{arsinh}_p x < z\left(1+\frac{1}{p}\log(1+x^p)\right),\; z=\left(\frac{x^p}{1+x^p}\right)^{1/p}$,

(5) $x\left(1-\frac{1}{1+p}\log(1-x^p)\right) < \operatorname{artanh}_p x < x\left(1-\frac{1}{p}\log(1-x^p)\right)$.

Lemma 2.3. *For $p,q>1$ and $r,s\in(0,1)$, the following inequalities hold:*

(1) $\operatorname{arcsin}_p(\sqrt{r\,s}) \le \sqrt{\operatorname{arcsin}_p(r)\operatorname{arcsin}_p(s)}$,

(2) $\operatorname{artanh}_p(\sqrt{r\,s}) \le \sqrt{\operatorname{artanh}_p(r)\operatorname{artanh}_p(s)}$,

(3) $\sqrt{\operatorname{arsinh}_p(r)\operatorname{arsinh}_p(s)} \le \operatorname{arsinh}_p(\sqrt{r\,s})$,

(4) $\sqrt{\operatorname{arctan}_p(r)\operatorname{arctan}_p(s)} \le \operatorname{arctan}_p(\sqrt{r\,s})$,

(5) $\pi_{\sqrt{p\,q}} \le \sqrt{\pi_p\,\pi_q}$.

Lemma 2.4. *For $m\ge -1$, $p>1$, the following functions*

(1) $f_1(x)=\left(\frac{\operatorname{arcsin}_p x}{x}\right)^m \frac{d}{dx}(\operatorname{arcsin}_p x)$,

(2) $f_2(x)=\left(\frac{\operatorname{artanh}_p x}{x}\right)^m \frac{d}{dx}(\operatorname{artanh}_p x)$

are increasing in $x\in(0,1)$, and

(3) $f_3(x)=\left(\frac{\operatorname{arctan}_p x}{x}\right)^m \frac{d}{dx}(\operatorname{arctan}_p x)$,

(4) $f_4(x)=\left(\frac{\operatorname{arsinh}_p x}{x}\right)^m \frac{d}{dx}(\operatorname{arsinh}_p x)$

are decreasing in $x\in(0,1)$.

Proof. By definition,

$$f_1(x)=\left(\frac{\operatorname{arcsin}_p x}{x}\right)^m \frac{1}{(1-x^p)^{1/p}}.$$

For $m\ge 0$, $\left(\frac{\operatorname{arcsin}_p x}{x}\right)^m$ is increasing by Lemma 2.1, and clearly $(1-x^p)^{1/p}$ is increasing. For the case $m\in[-1,0)$, we define

$$h_1(x)=\left(\frac{x}{\operatorname{arcsin}_p x}\right)^s \frac{1}{(1-x^p)^{1/p}},\qquad s\in(0,1].$$

We get

$$
\begin{aligned}
h_1'(x) &= \frac{\xi}{1-x^p}((1-x^p)^{1/p}(x^p+s(1-x^p))F_1(x)-s(1-x^p)) \\
&> \frac{\xi}{1-x^p}\left((1-x^p)^{1/p}(x^p+s(1-x^p))(1+\frac{x^p}{p(1+p)})-s(1-x^p)\right)>0,
\end{aligned}
$$

by Lemma 2.2(1), where

$$
\xi = \frac{(1-x^p)^{-(1+2/p)}}{x}\left(\frac{1}{F_1(x)}\right)^{1+s} \quad \text{and} \quad F_1(x)=F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};x^p\right).
$$

For (2), clearly f_2 is increasing for $m \geq 0$. For the case when $m \in [-1,0)$, we define

$$
h_2(x) = \left(\frac{x}{\operatorname{artanh}_p x}\right)^s \frac{1}{1-x^p}, \quad s \in (0,1].
$$

Differentiating with respect to x, we get

$$
h_2'(x) = \frac{(F_2(x))^{-(1+s)}\left((px^p-sx^p+s)\ F_2(x)-s\right)}{x\left(x^p-1\right)^2} > 0,
$$

where $F_2(x)=F\left(1,\frac{1}{p};1+\frac{1}{p};x^p\right)$.

For (3), the proof for the case when $m \geq 0$ follows similarly from Lemma 2.1. For the case $m \in [-1,0)$, let

$$
h_3(x) = \left(\frac{\arctan_p x}{x}\right)^{-s} \frac{d}{dx}(\arctan_p x), \quad s \in (0,1].
$$

We have

$$
\begin{aligned}
h_3'(x) &= \frac{F_3(x)^{-(1+s)}}{r(1+r^p)^2}((s+s\,r^p-p\,r^p)F_3(x)-s) \\
&< \frac{F_3(x)^{-(1+s)}}{r(1+r^p)^2}((s+s\,r^p-s\,r^p)F_3(x)-s) \\
&= -\frac{s\,F_3(x)^{-(1+s)}}{r(1+r^p)^2}(1-F_3(x))<0,
\end{aligned}
$$

where $F_3(x)=F\left(1,\frac{1}{p};1+\frac{1}{p};-x^p\right)$

For (4), when $m \geq 0$, the proof follows from Lemma 2.1. For $m \in [-1,0)$, let

$$
h_4(x) = \left(\frac{x}{\operatorname{arsinh}_p x}\right)^s \frac{1}{(1+x^p)^{1/p}}, \quad s \in (0,1].
$$

We have

$$
\begin{aligned}
h_4'(x) &= \gamma((1-x^p)^{1/p}(s(1-x^p)-x^p)F_4(x)-s(1-x^p)) \\
&< \gamma\Big(s(1+x^p)\Big(1+\frac{1}{p}\log(1+x^p)\Big)-s(1+x^p)-x^p\Big(1+\frac{1}{1+p}\log(1+x^p)\Big)\Big) \\
&= \frac{\gamma}{p(1+p)}(s(1+x^p)(1+p)\log(1+x^p)-p(1+p)x^p-p\,x^p\log(1+x^p))<0,
\end{aligned}
$$

by Lemma 2.2(4), where

$$\gamma = \frac{(1-x^p)^{-(1+2/p)}}{x}\left(\frac{1}{F_4(x)}\right)^{1+s} \quad \text{and} \quad F_4(x) = F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p};-x^p\right). \quad \blacksquare$$

Proof of Theorem 1.1. Let $0<x<y<1$, and $u=((x^t+y^t)/2)^{1/t}>x$. We denote $\operatorname{arcsin}(x)$, $\operatorname{artanh}(x)$, $\arctan(x)$, $\operatorname{arsinh}(x)$ by $g_i(x)$, $i=1,\ldots,4$ respectively, and define

$$g(x) = g_i(u)^t - \frac{g_i(x)^t + g_i(y)^t}{2}.$$

Differentiating with respect to x, we get $du/dx = (1/2)(x/u)^{t-1}$ and

$$\begin{aligned} g'(x) &= \frac{1}{2}\, t\, g_i(x)^{t-1}\frac{d}{du}(g_i(u))\left(\frac{x}{u}\right)^{t-1} - \frac{1}{2}\, t\, g_i(x)^{t-1}\frac{d}{dx}(g_i(x)) \\ &= \frac{t}{2}x^{t-1}(f_i(u)-f_i(x)), \end{aligned}$$

where

$$f_i(x) = \left(\frac{g_i(x)}{x}\right)^{t-1}\frac{d}{dx}(g_i(x)),\ i=1,\ldots,4.$$

By Lemma 2.4, g' is positive and negative for $f_{i=1,2}$ and $f_{i=3,4}$, respectively. This implies that

$$g(x) < (>) g(y) = 0,$$

for $g_{i=1,2}$ and $g_{i=3,4}$, respectively. The case when $t=0$ follows from Lemma 2.3. This completes the proof. ∎

Lemma 2.5. *For $p>1$ and $s\in(0,1)$, the function*

$$f(p) = \left(\frac{\pi_p}{p}\right)^{-s}\frac{\Big(p-\pi\cot\big(\pi/p\big)\Big)\csc(\pi/p)}{p^3}$$

is decreasing in $p\in(1,\infty)$.

Proof. We have

$$f'(p) = \xi\left[2p^2(1-s) + \pi^2(1-s)\cot^2\left(\frac{\pi}{p}\right) - \pi p(4-3s)\cot\left(\frac{\pi}{p}\right) + \pi^2\csc^2\left(\frac{\pi}{p}\right)\right],$$

where

$$\xi = -\frac{(2\pi)^{-s}}{p^3}\left(\frac{\csc(\pi/2)}{p^2}\right)^{1-s},$$

which is negative. ∎

Lemma 2.6. [14, Thm 2, p.151] *Let $J\subset\mathbb{R}$ be an open interval, and let $f\colon J\to\mathbb{R}$ be strictly monotonic function. Let $f^{-1}\colon f(J)\to J$ be the inverse to f then*

(1) *if f is convex and increasing, then f^{-1} is concave,*
(2) *if f is convex and decreasing, then f^{-1} is convex,*
(3) *if f is concave and increasing, then f^{-1} is convex,*
(4) *if f is concave and decreasing, then f^{-1} is concave.*

LEMMA 2.7. *For $m \geq 1$, $p > 1$ and $x \in (0,1)$, the following functions*

(1) $h_1(x) = \left(\frac{\sin_p x}{x}\right)^{m-1} \frac{d}{dx}(\sin_p x)$,

(2) $h_2(x) = \left(\frac{\tanh_p x}{x}\right)^{m-1} \frac{d}{dx}(\tanh_p x)$,

are decreasing in x, and

(3) $h_3(x) = \left(\frac{\cos_p x}{x}\right)^{m-1} \frac{d}{dx}(\cos_p x)$,

(4) $h_4(x) = \left(\frac{\tan_p x}{x}\right)^{m-1} \frac{d}{dx}(\tan_p x)$,

(5) $h_5(x) = \left(\frac{\sinh_p x}{x}\right)^{m-1} \frac{d}{dx}(\sinh_p x)$,

are increasing in x.

Proof. Let $f(x) = \operatorname{arcsin}_p x$, $x \in (0,1)$. We get

$$f'(x) = \frac{1}{(1-x^p)^{1/p}},$$

which is positive and increasing, hence f is convex. Clearly $\sin_p x$ is increasing, and by Lemma 2.6 is concave, this implies that $\frac{d}{dx}\sin_p x$ is decreasing, and $(\sin_p x)/x$ is decreasing also by Lemma 2.1. Similarly we get that $\frac{d}{dx}\tanh_p x$ is decreasing and $\frac{d}{dx}\cos_p x$, $\frac{d}{dx}\tan_p x$, $\frac{d}{dx}\sinh_p x$ are increasing, and the rest of proof follows from Lemma 2.1. ■

Proof of Theorem 1.2. The proof is similar to the proof of Theorem 1.1 and follows from Lemma 2.7. ■

PROPOSITION 2.8. *For $p, q > 1$ and $t < 1$, we have*

$$\pi_{M_t(p,q)} \leq M_t(\pi_p, \pi_q).$$

Proof. Let $1 < p < q < \infty$, and $w = ((p^t + q^t)/2)^{1/t} > p$. We define

$$g(p) = (\pi_p)^t - \frac{(\pi_p)^t + (\pi_q)^t}{2}.$$

Differentiating with respect to p, we get $dw/dp = (1/2)(p/w)^{t-1}$ and

$$\begin{aligned} g'(p) &= \frac{1}{2}\, t\, (\pi_p)^{t-1} \frac{d}{dx}(\pi_w) \left(\frac{p}{w}\right)^{t-1} - \frac{1}{2}\, t\, (\pi_p)^{t-1} \frac{d}{dx}(\pi_p) \\ &= \frac{t}{2} p^{t-1}(f(w) - f(p)), \end{aligned}$$

where

$$f(p) = \left(\frac{\pi_p}{p}\right)^{t-1} \frac{d}{dp}\pi_p.$$

Clearly π_p is decreasing, hence $(\pi_p/p)^{t-1}$ is increasing for $t < 1$ and $d/dp(\pi_p)$ is increasing by the proof of Lemma [4, Lemma 3.6]. This implies that $f(p)$ is increasing, and it follows that g is increasing. Hence $g(p) < g(q) = 0$. The case when $t = 0$ follows from Lemma 2.3(5). This completes the proof. ■

The following corollary follows immediately from Lemma 2.7.

COROLLARY 2.9. *For $p > 1$ and $r, s \in (0, 1)$ with $r \leq s$, we have*

(1) $\dfrac{\sin_p r}{r} \geq \dfrac{\sin_p s}{s}$,

(2) $\dfrac{\cos_p r}{r} \geq \dfrac{\cos_p s}{s}$,

(3) $\dfrac{\tan_p r}{r} \leq \dfrac{\tan_p s}{s}$,

(4) $\dfrac{\sinh_p r}{r} \leq \dfrac{\sinh_p s}{s}$,

(5) $\dfrac{\tanh_p r}{r} \geq \dfrac{\tanh_p s}{s}$.

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