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ON THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let $P(z)$ be a polynomial of degree n having no zeros in $|z|<k$ where $k\geq 1$. Then it is known that for every real or complex number α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha P(z)|\leq n\left(\frac{|\alpha|+k}{1+k}\right)\max_{|z|=1}|P(z)|,$$

where $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to a point $\alpha\in\mathbb{C}$. In this paper, by a simple method, a refinement of the above inequality and other related results are obtained.

1. Introduction and statement of results

If $P(z)$ is a polynomial of degree n, then concerning the estimate of the maximum of $|P'(z)|$ on the unit disk $|z|=1$, we have

$$\max_{|z|=1}|P'(z)|\leq n\max_{|z|=1}|P(z)|. \tag{1.1}$$

Inequality (1.1) is an immediate consequence of Bernstein's inequality on the derivative of a trigonometric polynomial (for reference, see [8–10]). Equality in (1.1) holds for $P(z)=az^n$, $a\neq 0$.

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then inequality (1.1) can be replaced by

$$\max_{|z|=1}|P'(z)|\leq \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.2}$$

Inequality (1.2) was conjectured by Erdös and later verified by Lax [5]. The result is sharp and equality holds for $P(z)=\alpha z^n+\beta$, $|\alpha|=|\beta|\neq 0$.

As an extension of (1.2), Malik [7] proved that if $P(z)$ is a polynomial of degree n which does not vanish in $|z|<k$ where $k\geq 1$, then

$$\max_{|z|=1}|P'(z)|\leq \frac{n}{1+k}\max_{|z|=1}|P(z)|. \tag{1.3}$$

The result is best possible and equality in (1.3) holds for $P(z)=(z+k)^n$.

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Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree n with respect to a point $\alpha \in \mathbb{C}$. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

A. Aziz [1] extended inequality (1.3) to the polar derivative and proved that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < k$ where $k \geq 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left(\frac{|\alpha| + k}{1 + k} \right) \max_{|z|=1} |P(z)|. \tag{1.4}$$

The result is best possible and equality in (1.4) holds for the polynomial $P(z) = (z+1)^n$.

The bound in (1.4) depends only upon the modulus of the zero of smallest modulus and not on the moduli of other zeros. It is of interest to obtain a bound which depends upon the location of all the zeros rather than just on the location of the zero of smallest modulus.

In this paper, by a simple method, we first present the following result which is a refinement of inequality (1.4).

THEOREM 1.1 *Let $P(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ be a polynomial of degree n. If $|z_\nu| \geq k_\nu \geq 1$ where $1 \leq \nu \leq n$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \left(\frac{|\alpha| + t_0}{1 + t_0} \right) \max_{|z|=1} |P(z)|, \tag{1.5}$$

where

$$t_0 = \begin{cases} 1 + \dfrac{n}{\sum_{\nu=1}^{n} \frac{1}{k_\nu - 1}} & \text{if} \quad k_\nu > 1 \quad \text{for all} \quad \nu,\ 1 \leq \nu \leq n \\ 1 & \text{if} \quad k_\nu = 1 \quad \text{for some} \quad \nu,\ 1 \leq \nu \leq n. \end{cases} \tag{1.6}$$

REMARK 1.2. If $k_\nu \geq k$, $k \geq 1$ for $1 \leq \nu \leq n$, then $t_0 \geq k$ which implies

$$\frac{|\alpha| + t_0}{1 + t_0} \leq \frac{|\alpha| + k}{1 + k} \quad \text{for} \quad |\alpha| \geq 1.$$

This shows that (1.5) is a refinement of inequality (1.4).

REMARK 1.3. If we divide the two sides of inequality (1.5) by $|\alpha|$ and let $|\alpha| \to \infty$, we get a result due to Govil et al. [4].

Next, as an application of Theorem 1.1, we present the following result.

Theorem 1.4. *Let $P(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ be a polynomial of degree n with $P(0) \neq 0$. If $|z_\nu| \leq k_\nu \leq 1$, $1 \leq \nu \leq n$, then for $\delta \in \mathbb{C}$ with $|\delta| \leq 1$,*

$$\max_{|z|=1} |D_\delta P(z)| \leq n \left(\frac{1 + |\delta|\, s_0}{1 + s_0} \right) \max_{|z|=1} |P(z)|, \tag{1.7}$$

where

$$s_0 = \begin{cases} 1 + \dfrac{n}{\sum_{\nu=1}^{n} \frac{k_\nu}{1-k_\nu}} & \text{if} \quad k_\nu < 1 \quad \text{for all} \quad \nu,\ 1 \leq \nu \leq n \\ 1 & \text{if} \quad k_\nu = 1 \quad \text{for some} \quad \nu,\ 1 \leq \nu \leq n. \end{cases} \tag{1.8}$$

Remark 1.5. If $k_\nu \leq k \leq 1$ for $1 \leq \nu \leq n$, then $1/k \leq s_0$ which implies

$$\frac{1 + |\delta|\, s_0}{1 + s_0} \leq \frac{|\delta| + k}{1 + k} \quad for \quad |\delta| \leq 1.$$

Therefore, it follows that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| < k$ where $k \leq 1$, then for $\delta \in \mathbb{C}$ with $|\delta| \leq 1$,

$$\max_{|z|=1} |D_\delta P(z)| \leq n \left(\frac{|\delta| + k}{1 + k} \right) \max_{|z|=1} |P(z)|. \tag{1.9}$$

The result is sharp.

2. Lemmas

For the proof of these theorems, we need the following lemmas. The first lemma is due to Gardner and Govil [2].

Lemma 2.1. *Let $P(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ be a polynomial of degree n. If $|z_\nu| \geq k_\nu \geq 1$, $1 \leq \nu \leq n$, then for $|z| = 1$,*

$$|Q'(z)| \geq t_0 |P'(z)|, \tag{2.1}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$ and t_0 is given by (1.6).

Lemma 2.2. *Let $P(z)$ be the polynomial of degree n and $Q(z) = z^n \overline{P(1/\overline{z})}$. Then for $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{2.2}$$

This is a special case of a result due to Govil and Rahman [6].

3. Proof of theorems

Proof of Theorem 1.1. Let $Q(z) = z^n \overline{P(1/\overline{z})}$. Then for $|z| = 1$, it can be easily verified that

$$|P'(z)| = |nQ(z) - zQ'(z)| \quad \text{and} \quad |Q'(z)| = |nP(z) - zP'(z)|. \tag{3.1}$$

Now, for every real or complex number α and $|z|=1$, we have by using (3.1),

$$\begin{aligned}|D_\alpha P(z)| &= |nP(z)+(\alpha-z)P'(z)|\\ &\le |\alpha||P'(z)|+|nP(z)-zP'(z)|\\ &= (|\alpha|-1)|P'(z)|+|P'(z)|+|Q'(z)|. \end{aligned} \tag{3.2}$$

Multiplying the two sides of inequality (3.2) by t_0 and using Lemma 2.1, we obtain for $|\alpha|\ge 1$,

$$\begin{aligned}t_0|D_\alpha P(z)| &\le (|\alpha|-1)t_0|P'(z)|+t_0(|P'(z)|+|Q'(z)|)\\ &\le (|\alpha|-1)|Q'(z)|+t_0(|P'(z)|+|Q'(z)|) \quad \text{for} \quad |z|=1.\end{aligned} \tag{3.3}$$

Adding (3.2), (3.3) and using Lemma 2.2, we get for $|\alpha|\ge 1$ and $|z|=1$,

$$\begin{aligned}(1+t_0)|D_\alpha P(z)| &\le (|\alpha|+t_0)(|P'(z)|+|Q'(z)|)\\ &\le n(|\alpha|+t_0)\max_{|z|=1}|P(z)|,\end{aligned} \tag{3.4}$$

which gives

$$|D_\alpha P(z)| \le n\left(\frac{|\alpha|+t_0}{1+t_0}\right)\max_{|z|=1}|P(z)| \quad for \quad |z|=1. \tag{3.5}$$

This completes the proof of Theorem 1.1. ■

Proof of Theorem 1.4. Since $P(z)=a_n\prod_{\nu=1}^{n}(z-z_\nu)$ where $|z_\nu|\le k_\nu\le 1$, $\nu=1,2,\cdots,n$ with $P(0)\ne 0$, $Q(z)=z^n\overline{P(1/\overline{z})}=\overline{a_n}\prod_{\nu=1}^{n}(1-\bar{z}_\nu z)$ is a polynomial of degree n with $\left|\frac{1}{z_\nu}\right|\ge\frac{1}{k_\nu}\ge 1$. Applying Theorem 1.1 to the polynomial $Q(z)$ and noting that $|P(z)|=|Q(z)|$ for $|z|=1$, we obtain for $\alpha\in\mathbb{C}$ with $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha Q(z)| \le n\left(\frac{|\alpha|+s_0}{1+s_0}\right)\max_{|z|=1}|P(z)|, \tag{3.6}$$

where

$$s_0=\begin{cases}1+\dfrac{n}{\sum_{\nu=1}^{n}\frac{k_\nu}{1-k_\nu}} & \text{if} \quad k_\nu<1 \quad \text{for all} \quad \nu,\ 1\le\nu\le n\\ 1 & \text{if} \quad k_\nu=1 \quad \text{for some} \quad \nu,\ 1\le\nu\le n.\end{cases}$$

Again, for $|z|=1$ so that $z\overline{z}=1$, we have

$$\begin{aligned}|D_\alpha Q(z)| &= |nQ(z)+(\alpha-z)Q'(z)|\\ &= \left|nz^n\overline{P(1/\overline{z})}+(\alpha-z)\left\{nz^{n-1}\overline{P(1/\overline{z})}-z^{n-2}\overline{P'(1/\overline{z})}\right\}\right|\\ &= \left|\alpha\left\{nz^{n-1}\overline{P(1/\overline{z})}-z^{n-2}\overline{P'(1/\overline{z})}\right\}+z^{n-1}\overline{P'(1/\overline{z})}\right|\\ &= \left|\alpha\left(n\overline{P(z)}-\overline{z}\overline{P'(z)}\right)+\overline{P'(z)}\right|\\ &= |\overline{\alpha}nP(z)+(1-\overline{\alpha}z)P'(z)| = |\overline{\alpha}||D_{1/\overline{\alpha}}P(z)|.\end{aligned}$$

Hence, from (3.6), we get

$$|\alpha| \max_{|z|=1} |D_{1/\bar{\alpha}} P(z)| \leq n \left(\frac{|\alpha| + s_0}{1 + s_0} \right) \max_{|z|=1} |P(z)|. \tag{3.7}$$

Replacing $1/\overline{\alpha}$ by δ, we obtain for every real or complex number δ with $|\delta| \leq 1$,

$$\max_{|z|=1} |D_\delta P(z)| \leq n \left(\frac{1 + |\delta|\, s_0}{1 + s_0} \right) \max_{|z|=1} |P(z)|. \tag{3.8}$$

This completes the proof of Theorem 1.4. ∎

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REFERENCES

[**1**] A. Aziz, *Inequalities for the polar derivative of a polynomial*, J. Approx. Theory **55** (1988), 183–193.

[**2**] R. B. Gardner and N. K. Govil, *Inequalities concerning L^p norm of a polynomial and its derivative*, J. Math. Anal. Appl. **179** (1993), 208–213.

[**3**] R. B. Gardner and N. K. Govil, *An L^p inequality for a polynomial and its derivative*, J. Math. Anal. Appl. **193** (1995), 490–496.

[**4**] N. K. Govil and G. Labelle, *On Bernstein's inequality*, J. Math. Anal. Appl. **126**, (1987), 494–500.

[**5**] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.

[**6**] N. K. Govil and Q. I. Rahman, *Functions of exponential type not vanishing in a half plane and related polynomials*, *Trans. Amer. Math. Soc.* **137** (1969), 501–517.

[**7**] M. A. Malik, *On the derivative of a polynomial*, J. Lond. Math. Soc., Second Ser. **1** (1969), 57–60.

[**8**] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*, World Scientific Publishing Co., Singapore, (1994).

[**9**] Q. I. Rahman and G. Schmessier, *Analytic Theory of Polynomials,* Claredon Press, Oxford, 2002.

[**10**] A. C. Schaffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc. **47** (1941), 565–579.

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