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DING PROJECTIVE MODULES WITH RESPECT TO A SEMIDUALIZING MODULE

Chaoling Huang and Peihua Zhong

Abstract. In this paper, for a fixed semidualizing module C, we introduce the notion of D_C-projective modules which are the special setting of G_C-projective modules introduced by White [D. White, Gorenstein projective dimension with respect to a semidualizing module, J. Commut. Algebra 2(1) (2010) 111–137]. Then we investigate the properties of D_C-projective modules and dimensions, in particular, we give descriptions of the finite D_C-projective dimensions.

1. Introduction

Auslander and Bridger in [1], introduced the notion of so-called G-dimension for finitely generated modules over commutative Noetherian rings. Enochs and Jenda defined in [4] a homological dimension, namely the Gorenstein projective dimension, $Gpd_R(-)$, for any R-module as an extension of G-dimension. Let R be any associative ring. Recall that an R-module M is said to be Gorenstein projective (for short G-projective; see [4]) if there is an exact sequence

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective modules with $M = Ker(P^0 \to P^1)$ such that $Hom(P, Q)$ is exact for each projective R-module Q. Such exact sequence is called a complete projective resolution. We use $\mathcal{GP}(R)$ to denote the class of all G-projective R-modules. We say that M has Gorenstein projective dimension at most n, denoted $Gpd_R(M) \leq n$, if there is a Gorenstein projective resolution, i.e., there is an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$, where all G_i are G-projective R-modules, and say $Gpd_R(M) = n$ if there is not a shorter Gorenstein projective resolution.

In [3], an R-module M is called strongly Gorenstein flat if there is an exact sequence

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective modules with $M = Ker(P^0 \to P^1)$ such that $Hom(\mathbf{P},\ Q)$ is exact for each flat R-module Q. It is clear that strongly Gorenstein flat R-modules are

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Gorenstein projective. But no one knows whether there is a Gorenstein projective R-module which is not strongly Gorenstein flat. Following [8, 19], the strongly Gorenstein flat R-modules are called Ding projective, since strongly Gorenstein flat R-modules are not necessarily Gorenstein flat [3, Example 2.19] and strongly Gorenstein flat R-modules were first introduced by Ding and his coauthors. In [3], the authors gave a lot of wonderful results about Ding projective R-modules over coherent rings. Mahdou and Tamekkante in [14], generalized some of these results over arbitrary associative rings. In this paper, we use $\mathcal{DP}(R)$ to denoted the class of all Ding projective R-modules.

In [7], the author initiated the study of semidualizing modules; see Definition 2.1. Over a noetherian ring R, Vasconcelos [17] studied them too. Golod [9] used these to define G_C-dimension for finitely generated modules, which is a refinement of projective dimension. Holm and Jørgensen [11] have extended this notion to arbitrary modules over a noetherian ring. Moreover, for semi-dualizing R-module C and the trivial extension of R by C $R \ltimes C$; that is, the ring $R \oplus C$ equipped with the product: $(r,\ c)(r',\ c') = (rr',\ rc' + r'c)$, they considered the ring changed Gorenstein dimensions, $Gpd_{R\ltimes C}M$ and proved that M is G_C-projective R-module if and only if M is G-projective $R \ltimes C$-module [11, Theorem 2.16]. In [18], White unified and generalized treatment of this concept over any commutative rings and showed many excellent G_C-projective properties shared by G-projectives. Recall that an R-module M is called G_C-projective if there exists a complete PC-resolution of M, which means that

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

is an exact complex such that $M \cong Coker(P_1 \to P_0)$ and each P_i and P^i is projective and such that the complex $Hom_R(P,\ C\otimes_R Q)$ is exact for every projective R-module Q. We use $\mathcal{G}_C\mathcal{P}(R)$ to denote the class of all G_C-projective R-modules. Motivated by the above, in this paper, we define the concept of Ding projective R-modules with respect to a fixed semidualizing module C, for short, D_C-projective and show properties of D_C-projective modules and dimensions. It is organized as follows:

Section 2 is devoted to the study of the D_C-projective modules and dimensions. White proved that every module that is either projective or C-projective is G_C-projective [18, Proposition 2.6]. Moreover, we show that they are also D_C-projective, see Proposition 2.7. Further, we give homological descriptions of the D_C-projective dimension, see Proposition 2.11. And then characterize modules with the finite D_C-projective dimension as follows,

THEOREM 1.1. *Let M be an R-module and n be a non-negative integer. Then the following are equivalent,*

(1) D_C-$pd_R(M) \leq n$;

(2) For some integer k with $1 \leq k \leq n$, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that P_i is D_C-projective if $0 \leq i < k$ and P_j is $\mathcal{P}_C$-projective if $j \geq k$.

(3) For any integer k with $1 \leq k \leq n$, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that P_i is D_C-projective if $0 \leq i < k$ and P_j is $\mathcal{P}_C$-projective if $j \geq k$.

THEOREM 1.2. *Let M be an R-module and n be a non-negative integer. Then the following are equivalent,*

(1) D_C-$pd_R(M) \leq n$;

(2) For some integer k with $0 \leq k \leq n$, there is an exact sequence $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that A_k is D_C-projective and other A_i projective or $\mathcal{P}_C$-projective.

(3) For any integer k with $0 \leq k \leq n$, there is an exact sequence $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that A_k is D_C-projective and other A_i projective or $\mathcal{P}_C$-projective.

Although we do not know whether there is a G_C-projective R-module which is not D_C-projective, we think that this article gives new things. Proposition 2.7, Proposition 2.11, Proposition 2.20 and the above two theorems add a new message to G_C-projective R-modules if G_C-projective R-modules and D_C-projective R-modules happen to coincide.

SETUP AND NOTATION. Throughout this paper, R denotes a commutative ring. C is a fixed semidualizing R-module. ${}_R\mathcal{M}$ denotes the category of R-modules, and $\mathcal{P}(R)$ and $\mathcal{I}(R)$ denote the class of projective and injective modules, respectively.

2. Properties of D_C-projective modules

Now we begin with recall of the definition on semedualizing R-modules.

DEFINITION 2.1. An R-module C is semidualizing if

(a) C admits a degreewise finite projective resolution, that is, there is an exact complex $\cdots \to P_1 \to P_0 \to C \to 0$ with all P_i finitely generated projective R-modules,

(b) the natural homothety map $\chi_C^R : R \to Hom_R(C,\ C)$ is an isomorphism, where χ_C^R satisfies that $\chi_C^R(r)(c) = rc$ for each $r \in R$ and $c \in C$, and

(c) $Ext_R^{n\geq 1}(C,\ C) = 0$.

For any noetherian ring R, a finitely generated R-module C is semidualizing if and only if $\mathbb{R}Hom_R(C,\ C) \cong R$ in $D(R)$, the derived category of the category of R-modules. Clearly, R is a semidualizing R-module.

DEFINITION 2.2. The class of C-projective is defined as

$$\mathcal{P}_C = \{C \otimes_R P \mid P \text{ is projective}\}$$

The $\mathcal{P}_C$-projective dimension of an R-module M is $\mathcal{P}_C$-$pd(M) = \inf\{n \mid 0 \to X_n \to \cdots \to X_0 \to M \to 0$ is exact with all X_i C-projective$\}$. The class of C-flat modules, denoted by $\mathcal{F}_C$ and $\mathcal{F}_C$-flat dimension of M, denoted by $\mathcal{F}_C$-$fd(M)$ are defined similarly.

Definition 2.3. An R-module M is called D_C-projective if there exists a complete PC-resolution of M, which means that

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

is an exact complex such that $M \cong Coker(P_1 \to P_0)$ and each P_i and P^i is projective and such that the complex $Hom_R(\mathbf{P},\ C \otimes_R Q)$ is exact for every flat R-module Q. We use $\mathcal{D}_C\mathcal{P}(R)$ to denote the class of all D_C-projective R-modules. For any R-module M, we say that M has D_C-projective dimension at most n, denoted D_C-$pd_R(M) \leq n$, if M has a D_C-projective resolution of length n, that is, there is an exact complex of the form $0 \to D_n \to \cdots \to D_0 \to M \to 0$, where all D_i are D_C-projective R-modules, and say D_C-$pd_R(M) = n$ if there is not a shorter D_C-projective resolution.

Remark 2.4. It is clear that $\mathcal{D}_C\mathcal{P}(R) \subseteq \mathcal{G}_C\mathcal{P}(R)$. When $C = R$, $\mathcal{D}_C\mathcal{P}(R) = \mathcal{DP}(R)$.

From Definition 2.3 one can obtain the following characterization of D_C-projective R-modules.

Proposition 2.5. *M is D_C-projective if and only if $Ext_R^{n\geq 1}(M,\ C \otimes_R Q) = 0$ and there exists an exact sequence of the form:*

$$\mathbf{X} = 0 \to M \to C \otimes_R P^0 \to C \otimes_R P^1 \to \cdots$$

such that $Hom_R(\mathbf{X},\ C \otimes_R Q)$ is exact for any flat R-module Q.

Recall that White in [18] proved that for any projective P, P and $C \otimes_R P$ are G_C-projective. Moreover, we can show that P and $C \otimes_R P$ are D_C-projective. First we give the following lemma,

Lemma 2.6. *Let P be a projective R-module and $\mathbf{X}$ be a complex. For an R-module A, if the complex $Hom_R(\mathbf{X},\ A)$ is exact, then the complex $Hom_R(P \otimes_R \mathbf{X},\ A)$ is exact. Thus, if $\mathbf{X}$ is a complete PC-resolution of an R-module M, then $P \otimes_R \mathbf{X}$ is a complete PC-resolution of an R-module $P \otimes_R M$. The converses hold in case P is faithfully projective.*

Proof. Since $Hom_R(P,\ -)$ is an exact functor, by the isomorphism of complexes given by Hom-tensor adjointness

$$Hom_R(P \otimes_R \mathbf{X},\ A) \cong Hom_R(P,\ Hom_R(\mathbf{X},\ A)),$$

exactness of the complex $Hom_R(\mathbf{X},\ A)$ implies that the complex $Hom_R(P \otimes_R \mathbf{X},\ A)$ is exact. The remains are trivial. ■

Proposition 2.7. *(1) C and R are D_C-projective;*

(2) For any projective P, P and $C \otimes_R P$ are D_C-projective.

Proof. (1) Since C is semidualizing, there is an exact sequence of the form: $\mathbf{X} = \cdots \to R^{n_1} \to R^{n_0} \to C \to 0$ with all n_i being positive integer numbers. By [18, Lemma 1.11 (b)], $Hom_R(\mathbf{X},\ C \otimes_R Q)$ is exact for any flat R-module Q. On the other hand, there is an exact sequence of the form:

$$\mathbf{Y} = 0 \xrightarrow{0} C \xrightarrow{1} C \xrightarrow{0} C \xrightarrow{1} \cdots.$$

By tensor evaluation homomorphism; see [2, p. 11],

$$Hom_R(\mathbf{Y},\ C \otimes_R Q) \cong Hom_R(\mathbf{Y},\ C) \otimes_R Q \cong \mathbf{Q}$$

is exact, where $\mathbf{Q}$ is the following exact sequence

$$\cdots \xrightarrow{0} Q \xrightarrow{1} Q \xrightarrow{0} Q \xrightarrow{0} 0.$$

Therefore, C is D_C-projective.

It is clear that the complex $Hom_R(\mathbf{X},\ C) = 0 \to R \to C^{n_0} \to C^{n_1} \to \cdots$ is exact. Since R and all C^{n_i} are finitely generated, for any flat R-module F,

$$Hom_R(Hom_R(\mathbf{X},\ C),\ C \otimes_R F) \cong Hom_R(Hom_R(\mathbf{X},\ C),\ C) \otimes_R F \cong \mathbf{X} \otimes_R F$$

is exact. Thus R is D_C-projective.

(2) By Lemma 2.6 and (1), for any projective P, P and $C \otimes_R P$ are D_C-projective. ■

Using a standard argument, we can get the following proposition.

Proposition 2.8. *If* $\mathbf{X}$ *is a complete PC-resolution, and* L *is an* R*-module with* $\mathcal{F}_C$*-*$fd(L) < \infty$*, then the complex* $Hom_R(\mathbf{X},\ L)$ *is exact. Thus if* M *is* D_C*-projective, then* $Ext_R^{\geq 1}(M,\ L) = 0$*.*

In [3, Lemma 2.4], the authors proved that for a D-projective R-module M, either M is projective or $fd_R(M) = \infty$. Now we generalize it as follows:

Proposition 2.9. *If* R*-module* M *is* D_C*-projective, then either* M *is* C*-flat or* $\mathcal{F}_C$*-*$fd_R(M) = \infty$*.*

Proof. Suppose that $\mathcal{F}_C$-$fd_R(M) = n$ with $1 \leq n < \infty$. We show by induction on n that M is C-flat. First assume that $n = 1$, then there is an exact sequence $0 \to X_1 \to X_0 \to M \to 0$ with X_0 and X_1 C-flat. Thus by Proposition 2.8, $Ext_R^1(M,\ X_1) = 0$. So the above short exact sequence is split, and M is a direct summand of X_0. By [13, Proposition 5.5], M is C-flat. Then assume that $n \geq 2$. There is a short exact sequence $0 \to K \to X \to M \to 0$ with X C-flat and $\mathcal{F}_C$-$fd_R(K) \leq n-1$. By induction, we conclude that K is C-flat. Thus $\mathcal{F}_C$-$fd_R(M) \leq 1$. By the above discussion, M is C-flat. ■

It is easy to prove the following two results using standard arguments. We leave the proofs to readers.

Proposition 2.10. *The class of* D_C*-projective* R*-modules is projectively resolving and closed under direct summands.*

Proposition 2.11. *Let M be an R-module with D_C-$pd_R(M) < \infty$ and n be a positive integer. The following are equivalent.*

(1) D_C-$pd_R(M) \leq n$.

(2) $Ext^i_R(M,\ L) = 0$ for all $i > n$ and all R-modules L with $\mathcal{F}_C$-$fd(L) < \infty$.

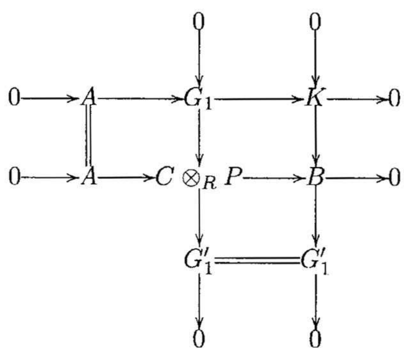
(3) $Ext^i_R(M,\ C \otimes_R F) = 0$ for all $i > n$ and all flat R-modules F.

(4) For any exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ with all G_i D_C-projective, K_n is D_C-projective.

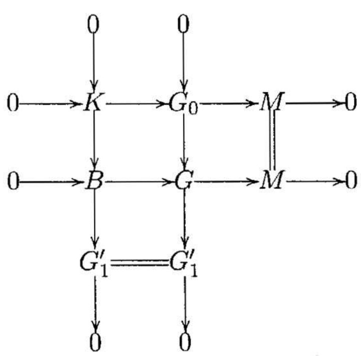
We give the following lemma which plays a crucial role in this paper.

Lemma 2.12. *Let $0 \to A \to G_1 \to G_0 \to M \to 0$ be an exact sequence with G_0 and G_1 D_C-projective. Then there are two exact sequences $0 \to A \to C \otimes_R P \to G \to M \to 0$ with P projective and G D_C-projective and $0 \to A \to H \to Q \to M \to 0$ with Q projective and H D_C-projective.*

Proof. Set $K = Im(G_1 \to G_0)$. Since G_1 is D_C-projective, there is a short exact sequence $0 \to G_1 \to C \otimes_R P \to G'_1 \to 0$ with P projective and G'_1 D_C-projective. Consider the following pushout diagram:



Then consider the following pushout diagram:



By Proposition 2.10, G is D_C-projective, since G_0 and G_1' are D_C-projective. Therefore, we can obtain exact sequence $0 \to A \to C \otimes_R P \to G \to M \to 0$. Similarly, we use pullbacks and can obtain the other exact sequence. ■

THEOREM 2.13. *Let M be an R-module and n be a non-negative integer. Then the following are equivalent,*

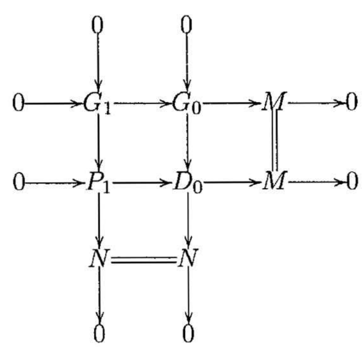
(1) D_C-$pd_R(M) \leq n$;

(2) For some integer k with $1 \leq k \leq n$, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that P_i is D_C-projective if $0 \leq i < k$ and P_j is C-projective if $j \geq k$.

(3) For any integer k with $1 \leq k \leq n$, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ such that P_i is D_C-projective if $0 \leq i < k$ and P_j is C-projective if $j \geq k$.

Proof. (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1): It is clear.

(1) $\Rightarrow$ (3): Let $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ be an exact sequence with all G_i D_C-projective. We prove (3) by induction on n. Let $n = 1$. Since G_1 is D_C-projective, there is a short exact sequence $0 \to G_1 \to P_1 \to N \to 0$ with P_1 C-projective and N D_C-projective. Consider the following pushout diagram:



By Proposition 2.10, D_0 is D_C-projective, since G_0 and N are D_C-projective. Now assume that $n > 1$. Set $A = Ker(G_0 \to M)$, then D_C-$pd_R(A) \leq n - 1$. By the induction hypothesis, for any integer k with $2 \leq k \leq n$, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to A \to 0$ such that P_i is D_C-projective if $1 \leq i < k$ and P_j is C-projective if $j \geq k$. Therefore, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to G_0 \to M \to 0$. Set $B = Ker(P_1 \to G_0)$. For the exact sequence $0 \to B \to P_1 \to G_0 \to M \to 0$, by Lemma 2.16, there is an exact sequence $0 \to B \to P_1' \to G_0' \to M \to 0$ with P_1' C-projective and G_0' D_C-projective. Therefore, we get the wanted exact sequence $0 \to P_n \to \cdots \to P_2 \to P_1' \to G_0' \to M \to 0$. ■

Let $\mathcal{F}$ be a class of R-modules. A morphism $\varphi \colon F \to M$ of $\mathcal{A}$ is called an $\mathcal{F}$-precover of M if $F \in \mathcal{F}$ and $Hom(F', F) \to Hom(F', M) \to 0$ is exact for all $F' \in \mathcal{F}$. φ is called an epic $\mathcal{F}$-precover of M if it is an $\mathcal{F}$-precover and is an epimorphism. If every R-module admits an (epic) $\mathcal{F}$-precover, then we say $\mathcal{F}$ is an (epic) precovering class. M is said to have a special $\mathcal{F}$-precover if there is an exact sequence

$$0 \longrightarrow C \longrightarrow F \longrightarrow M \longrightarrow 0$$

with $F \in \mathcal{F}$ and $Ext^1(\mathcal{F},\ C) = 0$. It is clear that M has an epic $\mathcal{F}$-precover if it has a special $\mathcal{F}$-precover. For more details about precovers, readers can refer to [5, 6, 16].

The authors in [14, Theorem 2.2] proved the following result: If M is an R-module with $D\text{-}pd_R(M) < \infty$, then M admits a special D-projective precover $\varphi : G \twoheadrightarrow M$ where $pd_R(Ker\varphi) = n - 1$ if $n > 0$ and $Ker\varphi = 0$ if $n = 0$. We can use the above theorem to generalize it to the below form,

Corollary 2.14. *If M is an R-module with $D_C\text{-}pd_R(M) = n < \infty$, then M admits a special D_C-projective precover $\varphi\colon G \twoheadrightarrow M$ where $\mathcal{P}_C\text{-}pd_R(Ker\varphi) \leq n - 1$ if $n > 0$ and $Ker\varphi = 0$ if $n = 0$.*

Proof. If $n = 0$, it is trivial. Now assume that $n > 0$. By Theorem 2.13, there is an exact sequence $0 \to P_n \to \cdots \to P_1 \to G \to M \to 0$ such that G is D_C-projective and any P_j is $\mathcal{P}_C$-projective. Then the remainder is trivial. ■

Remark 2.15. In [18, Definition 3.1], the author called a bounded G_C-projective resolution of R-module M a strict G_C-projective resolution if there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

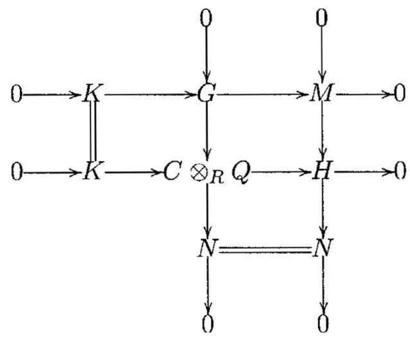
with all G_i C-projective for $i \geq 1$ and G_0 G_C-projective. And it is proved that every R-module M of finite G_C-projective dimension always admits a strict G_C-projective resolution [18, Thereom 3.6]. Using the different method (Theorem 2.13), we can prove that the R-module M of finite D_C-projective dimension has the similar property.

Corollary 2.16. *(1) Let $0 \to G_1 \to G \to M \to 0$ be a short exact sequence with G_1 and G D_C-projective and $Ext^1_R(M,\ F) = 0$ for any C-flat R-module F. Then M is D_C-projective.*

(2) If M is an R-module with $D_C\text{-}pd_R(M) = n$, then there exists an exact sequence $0 \to M \to H \to N \to 0$ with $\mathcal{P}_C\text{-}pd_R(H) \leq n$ and N D_C-projective.

Proof. (1) Since $D_C\text{-}pd_R(M) \leq 1$, by Corollary 2.14, there is an exact sequence $0 \to K \to G \to M \to 0$ where G is D_C-projective and K is C-projective. By the hypothesis $Ext^1_R(M,\ K) = 0$, the exact sequence $0 \to K \to G \to M \to 0$ is split and by Proposition 2.10, M is D_C-projective.

(2) If $n = 0$, by the definition of D_C-projective R-modules, there is an exact sequence $0 \to M \to C \otimes_R P \to K \to 0$ where P is projective and K is D_C-projective. If $n \geq 1$, by Corollary 2.14, there is an exact sequence $0 \to K \to G \to M \to 0$ with $\mathcal{P}_C\text{-}pd_R(K) \leq n - 1$. Since G is D_C-projective, there is $0 \to G \to C \otimes_R Q \to N \to 0$ where Q is projective and N is D_C-projective. Consider the following pushout diagram:



Then $\mathcal{P}_C$-$pd_R(H) \le n$. ■

THEOREM 2.17. *Let M be an R-module and n be a non-negative integer. Then the following are equivalent,*

(1) D_C-$pd_R(M) \le n$;

(2) For some integer k with $0 \le k \le n$, there is an exact sequence $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that A_k is D_C-projective and other A_i projective or C-projective.

(3) For any integer k with $0 \le k \le n$, there is an exact sequence $0 \to A_n \to \cdots \to A_1 \to A_0 \to M \to 0$ such that A_k is D_C-projective and other A_i projective or C-projective.

Proof. (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1): It is clear.

(1) $\Rightarrow$ (3): Let $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ be an exact sequence with all G_i D_C-projective. We prove (3) by induction on n. If $n = 1$, by Lemma 2.12, the assertion is true. Now we assume that $n \ge 2$. Set $K = Ker(G_1 \to G_0)$. For the exact sequence $0 \to K \to G_1 \to G_0 \to M \to 0$, by Lemma 2.12, we get two exact sequences $0 \to K \to G_1' \to P_0 \to M \to 0$ with G_1' D_C-projective and P_0 projective and $0 \to G_n \to \cdots \to G_2 \to G_1' \to P_0 \to M \to 0$. Set $N = Ker(P_0 \to M)$, then D_C-$pd_R(N) \le n-1$. By the induction hypothesis, for any integer k with $1 \le k \le n$, there is an exact sequence $0 \to A_n \to \cdots \to A_1 \to N \to 0$ such that A_k is D_C-projective and other A_i are projective or C-projective. Therefore, we get the wanted exact sequence $0 \to A_n \to \cdots \to A_1 \to P_0 \to M \to 0$. Now we prove the case $k = 0$. Set $A = Ker(G_0 \to M)$, then D_C-$pd_R(A) \le n - 1$. By the induction hypothesis, there is an exact sequence $0 \to B_n \to \cdots \to B_1 \to A \to 0$ such that B_1 is D_C-projective and other B_i projective or C-projective. So we have an exact sequence $0 \to B_n \to \cdots \to B_1 \to G_0 \to M \to 0$. Set $B = Ker(B_1 \to G_0)$. For the exact sequence $0 \to B \to B_1 \to G_0 \to M \to 0$, by Lemma 2.12, we get an exact sequence $0 \to B \to P'' \to G \to M \to 0$ with G D_C-projective and P'' C-projective. Hence the exact sequence $0 \to B_n \to \cdots \to B_2 \to P'' \to G \to M \to 0$ is wanted. ■

Let $\mathcal{F}$ be a class of R-modules. $\mathcal{F}^\perp$ will denote the right orthogonal class of $\mathcal{F}$, that is, $\mathcal{F}^\perp = \{M \mid Ext_R^1(F,\ M) = 0, \forall F \in \mathcal{F}\}$. Analogously, ${}^\perp\mathcal{F} = \{M \mid Ext_R^1(M,\ F) = 0, \forall F \in \mathcal{F}\}$. A cotorsion theory is a pair of classes $(\mathcal{F},\ \mathcal{C})$ of

R-modules such that $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$. A cotorsion theory $(\mathcal{F},\ \mathcal{C})$ is called complete if every R-module has a special $\mathcal{F}$-precover and a special $\mathcal{C}$-preenvelope. It is called hereditary if for any exact sequence $0 \to F' \to F \to F'' \to 0$ with $F,\ F'' \in \mathcal{F}$ implies that $F' \in \mathcal{F}$. For more details about cotorsion theory, readers can refer to [5, 6, 16]. Let $glG_Cpd(R) = sup\{G_C\text{-}pd_R(M) \mid \forall M \in_R \mathcal{M}\}$. We in [12, Theorem 5.1] proved that $(\mathcal{G}_C\mathcal{P}(R),\ \mathcal{G}_C\mathcal{P}(R)^{\perp})$ is a complete hereditary cotorsion theory if $glG_Cpd(R) < \infty$ and [12, Corollary 5.2] $(\mathcal{GP}(R),\ \mathcal{GP}(R)^{\perp})$ is a complete hereditary cotorsion theory if $glGpd(R) < \infty$. Similarly, we prove that $(\mathcal{D}_C\mathcal{P}(R),\ \mathcal{D}_C\mathcal{P}(R)^{\perp})$ is a complete hereditary cotorsion theory if $glD_Cpd(R) < \infty$, where $glD_Cpd(R) = sup\{D_C\text{-}pd_R(M) \mid \forall M \in_R \mathcal{M}\}$.

Theorem 2.18. *Assume that* $glD_Cpd(R) < \infty$. *Then* $(\mathcal{D}_C\mathcal{P}(R),\ \mathcal{D}_C\mathcal{P}(R)^{\perp})$ *is a complete hereditary cotorsion theory.*

Proof. We begin with proving that ${}^{\perp}(\mathcal{D}_C\mathcal{P}(R)^{\perp}) = \mathcal{D}_C\mathcal{P}(R)$. It is clear that ${}^{\perp}(\mathcal{D}_C\mathcal{P}(R)^{\perp}) \supseteq \mathcal{D}_C\mathcal{P}(R)$ because $Ext^1_R(A,\ B) = 0$ for any $A \in \mathcal{D}_C\mathcal{P}(R)$ and $B \in \mathcal{D}_C\mathcal{P}^{\perp}$ by definition. By Corollary 2.14, there is an exact sequence $0 \to K \to G \to M \to 0$ such that G is D_C-projective and $\mathcal{P}_C\text{-}pd(K) < \infty$. By Proposition 2.8, $K \in \mathcal{D}_C\mathcal{P}(R)^{\perp}$. So $Ext^1_R(M,\ K) = 0$, and then $0 \to K \to G \to M \to 0$ is split, i.e., M is a direct summand of G. By Proposition 2.10, M is D_C-projective.

By Proposition 2.10, $\mathcal{D}_C\mathcal{P}(R)$ is projectively resolving, $\mathcal{D}_C\mathcal{P}(R)^{\perp}$ is injectively resolving, so $(\mathcal{D}_C\mathcal{P}(R),\ \mathcal{D}_C\mathcal{P}(R)^{\perp})$ is hereditary. By Corollary 2.14, $(\mathcal{D}_C\mathcal{P}(R),\ \mathcal{D}_C\mathcal{P}(R)^{\perp})$ is complete. ∎

Corollary 2.19. *If* $glDpd(R) = \sup\{Dpd_R(M) \mid \forall M \in_R \mathcal{M}\} < \infty$, $(\mathcal{DP}(R),\ \mathcal{DP}(R)^{\perp})$ *is a complete hereditary cotorsion theory.*

Proposition 2.20. *(1)* $Ext^n_R(G,\ M) = 0$ *for all* $n \geq 1$, $G \in \mathcal{D}_C\mathcal{P}(R)$ *and* $M \in \mathcal{D}_C\mathcal{P}(R)^{\perp}$.

(2) $\mathcal{P}_C = \mathcal{D}_C\mathcal{P}(R) \bigcap \mathcal{D}_C\mathcal{P}(R)^{\perp}$.

(3) If M *be an* R*-module with* $\mathcal{P}_C\text{-}pd_R(M) < \infty$, *then* $\mathcal{P}_C\text{-}pd_R(M) = D_C\text{-}pd_R(M)$.

(4) If M *be an* R*-module with* $D_C\text{-}pd_R(M) < \infty$, *then* $G_C\text{-}pd_R(M) = D_C\text{-}pd_R(M)$.

(5) If M *be an* R*-module with* $pd_R(M) < \infty$, *then* $pd_R(M) = D_C\text{-}pd_R(M)$.

Proof. (1) For any D_C-projective R-module G, there is an exact sequence

$$0 \to G' \to P_{n-1} \to \cdots \to P_1 \to P_0 \to G \to 0$$

where all P_i are projective and G' is D_C-projective. So for any $M \in \mathcal{D}_C\mathcal{P}(R)^{\perp}$, $Ext^n_R(G,\ M) = Ext^1_R(G',\ M) = 0$.

(2) By Propositions 2.7 and 2.8, $\mathcal{P}_C \subseteq \mathcal{D}_C\mathcal{P}(R) \bigcap \mathcal{D}_C\mathcal{P}(R)^{\perp}$. Let $M \in \mathcal{D}_C\mathcal{P}(R) \bigcap \mathcal{D}_C\mathcal{P}^{\perp}$. There is a short exact sequence $0 \to M \to C \otimes_R P \to M' \to 0$ where P is projective and M' is D_C-projective. So $Ext^1_R(M',\ M) = 0$ and

$0 \to M \to C \otimes_R P \to M' \to 0$ is split. Therefore $M \in \mathcal{P}_C$ and $\mathcal{P}_C \supseteq \mathcal{D}_C\mathcal{P}(R) \bigcap \mathcal{D}_C\mathcal{P}(R)^{\perp}$.

(3) It is clear that $\mathcal{P}_C$-$pd_R(M) \geq D_C$-$pd_R(M)$, since every C-projective R-module is D_C-projective. Now we prove that $\mathcal{P}_C$-$pd_R(M) \leq D_C$-$pd_R(M)$. For doing this we assume that D_C-$pd_R(M) = n < \infty$. Since $\mathcal{P}_C$ is precovering [13, Proposition 5.10] and projectively resolving [13, Corollary 6.8], there is an exact sequence

$$0 \to K \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to M \to 0.$$

with K D_C-projective. Since M be an R-module with $\mathcal{P}_C$-$pd_R(M) < \infty$, $\mathcal{P}_C$-$pd_R(K) < \infty$. By (2), K is C-projective.

(4) It is clear that G_C-$pd_R(M) \leq D_C$-$pd_R(M)$, since every D_C-projective R-module is G_C-projective. Now we assume that D_C-$pd_R(M) = n < \infty$. By [18, Proposition 2.12], it is sufficient to find a projective R-module P such that $Ext^n_R(M,\ C \otimes_R P) \neq 0$. By Proposition 2.11, there is a flat R-module F such that $Ext^n_R(M,\ C \otimes_R F) \neq 0$. Since $\mathcal{P}_C$ is precovering [13, Proposition 5.10] and $\mathcal{F}_C$ is projectively resolving [13, Corollary 6.8], there is a short exact sequence $0 \to K \to C \otimes_R P \to C \otimes_R F \to 0$ where K is C-flat. By [15, Theorem 7.3], there is a long exact sequence $\cdots \to Ext^n_R(M,\ C \otimes_R P) \to Ext^n_R(M,\ C \otimes_R F) \to Ext^{n+1}_R(M,\ K) \to \cdots$, where $Ext^{n+1}_R(M,\ K) = 0$. So $Ext^n_R(M,\ C \otimes_R P) \neq 0$.

(5) It is clear that G_C-$pd_R(M) \leq D_C$-$pd_R(M) \leq pd_R(M)$. It is well-known that $pd_R(M) = G_C$-$pd_R(M)$ if $pd_R(M) < \infty$. So $pd_R(M) = D_C$-$pd_R(M)$. ■

We round off this paper with the following questions:

(1) Recall that the author in [14, Theorem 3.1] proved that for any ring R, $r.glGdim(R) = r.glDdim(R)$. So we conjecture that $glG_Cpd(R) = glD_Cpd(R)$, is it true?

(2) Whether is there a G_C-projective R-module which is not D_C-projective?

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Chaoling Huang[a, b], Peihua Zhong[b]

[a]College of Mathematics and Statistics, Hubei Normal University, China

[b]Department of Mathematics, Jiangxi Agricultural University, Nanchang 330045, China

E-mail: `clhuang1978@live.cn, phzhong@163.com`