МАТЕМАТИЧКИ ВЕСНИК
67, 2 (2015), 102–114
June 2015

оригинални научни рад
research paper

THE RAINBOW DOMINATION SUBDIVISION NUMBERS OF GRAPHS

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Abstract. A 2-*rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\cup_{u\in N(v)} f(u) = \{1,2\}$ is fulfilled. The *weight* of a 2RDF f is the value $\omega(f) = \Sigma_{v\in V}|f(v)|$. The 2-*rainbow domination number* of a graph G, denoted by $\gamma_{r2}(G)$, is the minimum weight of a 2RDF of G. The 2-*rainbow domination subdivision number* $\mathrm{sd}_{\gamma_{r2}}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the 2-rainbow domination number. In this paper, we initiate the study of 2-rainbow domination subdivision number in graphs.

1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *leaf* of a graph G is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves.

For a positive integer k, a k-*rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\dots,k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1,2,\dots,k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v\in V} |f(v)|$. The k-*rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A $\gamma_{rk}(G)$-*function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [2] and has been studied by several authors [3–5,11,13]. For a more thorough treatment of domination parameters and for terminology not presented here see [10].

2010 Mathematics Subject Classification: 05C69
Keywords and phrases: domination number; 2-rainbow domination number; 2-rainbow domination subdivision number.

The 2-*rainbow domination subdivision number* $\mathrm{sd}_{\gamma_{r2}}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the 2-rainbow domination number of G. (An edge $uv \in E(G)$ is *subdivided* if the edge uv is deleted, but a new vertex x is added, along with two new edges ux and vx. The vertex x is called a *subdivision vertex*). Observation 3 below shows that the 2-rainbow domination number of graphs cannot decrease when an edge of graph is subdivided. Since the 2-rainbow domination number of the graph K_2 does not change when its only edge is subdivided, in the study of 2-rainbow domination subdivision number we must assume that the graph has maximum degree at least two.

The purpose of this paper is to initiate the study of the 2-rainbow domination subdivision number $\mathrm{sd}_{\gamma_{r2}}(G)$. Although it may not be immediately obvious that the 2-rainbow domination subdivision number is defined for all graphs with maximum degree at least two, we will show this shortly.

We make use of the following results in this paper.

Theorem A. [6] *Let G be a graph of order $n \geq k+1$. Then $\gamma_{rk}(G) = k$ if and only if there exists a vertex set A with $|A| \leq k$ such that every vertex of $V(G) - A$ is adjacent to every vertex of A.*

Theorem B. [3] *For $n \geq 1$,*

$$\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Corollary 1. *For $n \geq 3$,*

$$\mathrm{sd}_{\gamma_{r2}}(P_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Theorem C. [3] *For $n \geq 3$,*

$$\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

Corollary 2. *For $n \geq 3$,*

$$\mathrm{sd}_{\gamma_{r2}}(C_n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 4 \\ 3 & \text{if } n \equiv 2 \pmod 4 \\ 2 & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

Observation 3. *Let G be a connected graph of order $n \geq 3$ and $e = uv \in E(G)$. If G' is the graph obtained from G by subdividing the edge e, then $\gamma_{r2}(G') \geq \gamma_{r2}(G)$.*

Proof. Let G' be the graph obtained from G by subdividing the edge e with subdivision vertex x, and let f be a $\gamma_{r2}(G')$-function. It is easy to see that the function $g : V(G) \to \mathcal{P}(\{1,2\})$ defined by $g(u) = f(u) \cup f(x)$ and $g(w) = f(w)$ for $w \in V(G) - \{u\}$ is a 2-rainbow dominating function of G which implies that $\gamma_{r2}(G') = \omega(f) \geq \gamma_{r2}(G)$. This completes the proof. ∎

Observation 4. *Let v be a vertex of G with $\deg(v) \geq 2$ and let $N(v) = \{v_1, \ldots, v_k\}$. Assume $1 \leq r \leq k$ and G' is obtained from G by subdividing the edge vv_i with vertex x_i for $i = 1, \ldots, r$. If G' has a $\gamma_{r2}(G')$-function f such that $|f(v)| + \sum_{i=1}^{r} |f(x_i)| \geq 3$, then $\gamma_{r2}(G') \geq \gamma_{r2}(G)$.*

Proof. Define $g : V \to \mathcal{P}(\{1,2\})$ by $g(v) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(G) - \{v\}$. Obviously, g is a 2RDF of G of weight less than $\gamma_{r2}(G')$ that implies $\gamma_{r2}(G') \geq \gamma_{r2}(G)$. ∎

The proof of the following observation is straightforward and therefore omitted.

Observation 5. *For $n \geq 3$, $\mathrm{sd}_{\gamma_{r2}}(K_n) = 2$.*

Observation 6. *If $G = K_{m,n}$ is the complete bipartite graph and $m \geq n \geq 4$, then $\mathrm{sd}_{\gamma_{r2}}(K_{m,n}) = 3$.*

Proof. Note that $\gamma_{r2}(K_{m,n}) = 4$ when $m \geq n \geq 4$. Let $X = \{u_1, u_2, \ldots, u_n\}$ and $Y = \{v_1, v_2, \ldots, v_m\}$ be the partite sets of $K_{m,n}$. First we show that $\mathrm{sd}_{\gamma_{r2}}(K_{m,n}) \geq 3$. Let $e_1 = u_i v_j, e_2 = u_{i'} v_{j'}$ be two arbitrary edges of $K_{m,n}$ and let G be obtained from $K_{m,n}$ by subdividing the edges e_1, e_2. If $u_i = u_{i'}$ (the case $v_j = v_{j'}$ is similar), then the function $g : V \to \mathcal{P}(\{1,2\})$ defined by $g(u_i) = \{1,2\}, g(v_j) = \{1\}$ and $g(v_{j'}) = \{2\}$ is a 2-rainbow dominating function of G of weight 4. Let $u_i \neq u_{i'}$ and $v_j \neq v_{j'}$. Then the function $g : V \to \mathcal{P}(\{1,2\})$ defined by $g(u_i) = g(v_{j'}) = \{1\}$ and $g(u_{i'}) = g(v_j) = \{2\}$ is a 2-rainbow dominating function of G of weight 4. It follows that $\mathrm{sd}_{\gamma_{r2}}(K_{m,n}) > 2$.

To prove $\mathrm{sd}_{\gamma_{r2}}(K_{m,n}) \leq 3$, assume that G' is the graph obtained from $K_{m,n}$ by subdividing the edges u_1v_1, u_1v_2, u_1v_3 with vertices x_1, x_2, x_3, respectively. Let f be a $\gamma_{r2}(G')$-function. By Observation 4, we may assume $|f(u_1)| + \sum_{i=1}^{3} |f(x_i)| \leq 2$. We consider three cases.

Case 1. $f(u_1) = \{1,2\}$.

Then $f(x_i) = \emptyset$ for $i = 1,2,3$. If $|f(u_i)| \geq 1$ for each $i \geq 2$ or $|f(v_i)| \geq 1$ for $i = 1,2,3$, then $\gamma_{r2}(G') = \omega(f) \geq 5$ and we are done. Suppose $|f(u_i)| = 0$ and $|f(v_j)| = 0$ for some i, j, say $i = 2$ and $j = 1$. To dominate u_2 and v_1, we must have $\sum_{i=1}^{m} |f(v_i)| \geq 2$ and $\sum_{i=2}^{n} |f(u_i)| \geq 2$, respectively. It follows that $\gamma_{r2}(G') = \omega(f) \geq 5$ as desired.

Case 2. $f(u_1) = \emptyset$.

Then to dominate x_i, we must have $f(x_i) \neq \emptyset$ or $f(v_i) = \{1,2\}$ for $i = 1,2,3$. If $\sum_{i=1}^{3}(|f(x_i)| + |f(v_i)|) \geq 4$ then to dominate v_4, we must have $|f(v_4)| + \sum_{i=2}^{n} |f(u_i)| \geq 1$ implying that $\gamma_{r2}(G') = \omega(f) \geq 5$ as desired. Let $\sum_{i=1}^{3}(|f(x_i)| + |f(v_i)|) \leq 3$. It follows that $|f(x_i)| = 1$ and $|f(v_i)| = 0$ for $i = 1,2,3$. If $|f(u_i)| \geq 1$

for each $i \geq 2$, then $\gamma_{r2}(G') = \omega(f) \geq 5$. Suppose $|f(u_i)| = 0$ for some $i \geq 2$, say $i = 2$. To dominate u_2, we must have $\sum_{i=4}^{m} |f(v_i)| \geq 2$ which implies $\gamma_{r2}(G') = \omega(f) \geq 5$.

Case 3. $|f(u_1)| = 1$.

We may assume, without loss of generality, that $f(u_1) = \{1\}$. To dominate x_i, we must have $|f(x_i)| + |f(v_i)| \geq 1$ for $i = 1, 2, 3$. Now to dominate v_4, we must have $|f(v_4)| + \sum_{i=2}^{n} |f(u_i)| \geq 1$. It follows that $\gamma_{r2}(G') = \omega(f) \geq 5$ and hence $\mathrm{sd}_{\gamma_{r2}}(K_{m,n}) \leq 3$. This completes the proof. ■

2. Sufficient conditions for small rainbow domination subdivision number

LEMMA 7. *Let G be a connected graph. If there is a path $v_3v_2v_1$ in G with* $\deg(v_2) = 2$ *and* $\deg(v_1) = 1$*, then G has a $\gamma_{r2}(G)$-function f such that $f(v_1) = \{1\}$, and $2 \in f(v_3)$.*

Proof. Assume that g is a $\gamma_{r2}(G)$-function. To dominate v_1, we must have $|g(v_1)| + |g(v_2)| \geq 1$. Consider two cases.

Case 1. $|g(v_1)| + |g(v_2)| \geq 2$.

Define $f : V(G) \to \mathcal{P}(\{1, 2\})$ by $f(v_1) = \{1\}, f(v_2) = \emptyset, f(v_3) = g(v_3) \cup \{2\}$ and $f(x) = g(x)$ for $x \in V(G) - \{v_1, v_2, v_3\}$. It is easy to see that f is a $\gamma_{r2}(G)$-function with the desired property.

Case 2. $|g(v_1)| + |g(v_2)| = 1$.

Then $|g(v_1)| = 1$ and $|g(v_2)| = 0$. We may assume, without loss of generality, that $g(v_1) = \{1\}$. To dominate v_2, we must have $2 \in g(v_3)$ and hence g has the desired property and the proof is completed. ■

PROPOSITION 8. *If G contains a strong support vertex, then* $\mathrm{sd}_{\gamma_{r2}}(G) = 1$.

Proof. Let w be a strong support vertex of G and let $u, v \in V$ be two leaves adjacent to w. Suppose G' is the graph obtained from G by subdividing the edge uw with vertex x. By Lemma 7, G has a $\gamma_{r2}(G')$-function f such that $f(u) = f(v) = \{1\}$ and $2 \in f(w)$. Define $g : V \to \mathcal{P}(\{1, 2\})$ by $g(u) = g(v) = \emptyset, g(w) = \{1, 2\}$ and $g(z) = f(z)$ for each $z \in V \setminus \{u, v, w\}$. Clearly, g is a 2RDF of G with $w(g) < w(f)$ and hence $\mathrm{sd}_{r2}(G) = 1$. ■

THEOREM 9. *For any tree T of order $n \geq 3$,* $\mathrm{sd}_{\gamma_{r2}}(T) \leq 2$.

Proof. If $\mathrm{diam}(T) = 2$ then T is a star and by Proposition 8, $\mathrm{sd}_{r2}(T) = 1$. Let $\mathrm{diam}(T) \geq 3$ and let $v_1v_2 \ldots v_d$ be a diametral path in T. By Proposition 8, we may assume $\deg(v_2) = 2$. Let T' be obtained from T by subdividing the edge v_iv_{i+1} with subdivision vertex x_i for $i = 1, 2$. Suppose f is a $\gamma_{r2}(T')$-function. By Lemma 7, we may assume $f(v_1) = \{1\}$ and $2 \in f(v_2)$. Define $g : V(T) \to \mathcal{P}(\{1, 2\})$ by $g(v_1) = f(v_2), g(v_2) = f(x_2)$ and $g(z) = f(z)$ for each $z \in V(T) \setminus \{v_1, v_2\}$. It is easy to see that g is a 2RDF of T of weight less than $\gamma_{r2}(T')$. Thus $\mathrm{sd}_{\gamma_{r2}}(T) \leq 2$ and the proof is completed. ■

Proposition 10. *If G is a connected graph of order $n \geq 3$ with $\gamma_{r2}(G) = 2$, then* $\mathrm{sd}_{\gamma_{r2}}(G) \leq 2$. *This bound is sharp for complete graphs.*

Proof. Let w be a vertex of degree at least two and let u_1, u_2 be its neighbors. Assume G' is the graph obtained from G by subdividing the edge wu_i with subdivision vertex x_i for $i = 1, 2$. We claim that $\gamma_{r2}(G') \geq 3$. Suppose to the contrary that $\gamma_{r2}(G') = 2$. By Theorem A, there exists a vertex set A with $|A| \leq 2$ such that every vertex of $V(G') - A$ is adjacent to every vertex of A. Since $|V(G') - A| \geq 3$, we have $x_1, x_2 \notin A$. Then $|A| = 2$, for otherwise G' must have a vertex of degree $|V(G')| - 1$, a contradiction. Then all neighbors of x_1, x_2 must belong to A which leads to a contradiction, because there are three neighbors. This completes the proof. ■

Proposition 11. *If G is a connected graph of order $n \geq 3$ with $\gamma_{r2}(G) = 3$, then* $\mathrm{sd}_{\gamma_{r2}}(G) \leq 3$.

Proof. If $\Delta(G) = 2$, then G is a path or cycle and the result follows from Corollaries 1 and 2. Let $\Delta(G) \geq 3$. Let $v \in V(G)$ be a vertex of maximum degree and $v_1, v_2, v_3 \in N(v)$. Assume G' is the graph obtained from G by subdividing the edge vv_i with vertex x_i for $i = 1, 2, 3$. We show that $\gamma_{r2}(G') > \gamma_{r2}(G)$ which implies $\mathrm{sd}_{\gamma_{r2}}(G) \leq 3$. Let f be a $\gamma_{r2}(G')$-function. By Observation 4, we may assume $|f(v)| + \sum_{i=1}^{3} |f(x_i)| \leq 2$. We consider three cases.

Case 1. $f(v) = \{1, 2\}$.

Then $f(x_i) = \emptyset$ for $i = 1, 2, 3$. If $|f(v_i)| \geq 1$ for each $1 \leq i \leq 3$, then $\gamma_{r2}(G') = \omega(f) \geq 5$ and we are done. Suppose $|f(v_i)| = 0$ for some $1 \leq i \leq 3$, say $i = 1$. To dominate v_1, we must have $\sum_{x \in V(G) - \{v, v_1\}} |f(x)| \geq 2$. It follows that $\gamma_{r2}(G') = \omega(f) \geq 4$.

Case 2. $f(v) = \emptyset$.

Then to dominate x_i, we must have $f(x_i) \neq \emptyset$ or $f(v_i) = \{1, 2\}$ for $i = 1, 2, 3$. If $\sum_{i=1}^{3}(|f(x_i)| + |f(v_i)|) \geq 4$, we are done. Let $\sum_{i=1}^{3}(|f(x_i)| + |f(v_i)|) \leq 3$. Then $|f(x_i)| = 1$ and $|f(v_i)| = 0$ for $i = 1, 2, 3$. To dominate v_1, we have $\sum_{x \in V(G) - \{v\}} |f(x)| \geq 2$. Hence $\gamma_{r2}(G') = \omega(f) \geq 5$.

Case 3. $|f(v)| = 1$.

To dominate x_i, we must have $|f(x_i)| + |f(v_i)| \geq 1$ for $i = 1, 2, 3$. Hence $\gamma_{r2}(G') = \omega(f) \geq 4$.

Thus $\gamma_{r2}(G') > \gamma_{r2}(G)$ and the proof is completed. ■

3. Bounds on the rainbow domination subdivision number

In this section we present some upper bounds on $\mathrm{sd}_{\gamma_{r2}}(G)$ in terms of the vertex degree and the minimum degree of G. We start with the following lemma.

Lemma 12. *Let G be a connected graph of order $n \geq 3$, $v \in V(G)$, $N(v) = \{v_1, \dots, v_k\}$ and $F \subseteq E(G) - \{vv_i \mid 1 \leq i \leq k\}$ (possibly empty set). Let G' be the* *graph obtained from G by subdividing the edge vv_i with vertex x_i for $1 \leq i \leq k$ and the edges in F. If G' has a $\gamma_{r2}(G')$-function, f, such that $f(v) \neq \emptyset$ and $f(x_i) \neq \emptyset$ for some i, then $\gamma_{r2}(G') \geq \gamma_{r2}(G) + 1$.*

Proof. Let f be a $\gamma_{r2}(G')$-function such that $f(v) \neq \emptyset$ and $f(x_i) \neq \emptyset$ for some i. Then $|f(v)| + \sum_{i=1}^{k} |f(x_i)| \geq 2$. Suppose G_1 is the graph obtained from G by subdividing the edges in F. Define $g : V(G_1) \rightarrow \mathcal{P}(\{1,2\})$ by $g(v) = \bigcup_{i=1}^{k} f(x_i)$ and $g(z) = f(z)$ for each $z \in V(G_1) - \{v\}$. Obviously g is a 2RDF of G_1 of weight less than $\omega(f)$. Hence $\gamma_{r2}(G') \geq \gamma_{r2}(G_1) + 1$. It follows from Observation 3 that $\gamma_{r2}(G') \geq \gamma_{r2}(G) + 1$ and the proof is completed. ■

Lemma 13. *Let G be a connected graph of order $n \geq 3$ and let G have a vertex $v \in V(G)$ which is contained in a triangle vuw such that $N(u) \subseteq N[v]$. Then* $\mathrm{sd}_{\gamma_{r2}}(G) \leq \deg(v)$.

Proof. Let $N(v) = \{u = v_1, w = v_2, \ldots, v_k\}$ and let G' be the graph obtained from G by subdividing the edge vv_i with vertex x_i for $1 \leq i \leq k$. Let f be a $\gamma_{r2}(G')$-function. By Observation 4 and Lemma 12, we may assume that $|f(v)| + \sum_{i=1}^{k} |f(x_i)| \leq 2$ and $|f(v)| = 0$ or $\sum_{i=1}^{k} |f(x_i)| = 0$. If $f(v) = \{1,2\}$, then the function $g : V(G) \rightarrow \mathcal{P}(\{1,2\})$ defined by $g(v) = \{1\}$ and $g(z) = f(z)$ for each $z \in V(G) - \{v\}$, is a 2RDF of G of weight less than $\gamma_{r2}(G')$. If $|f(v)| = 1$, then the function $h : V(G) \rightarrow \mathcal{P}(\{1,2\})$ by $h(v_1) = \emptyset$ and $h(z) = f(z)$ for each $z \in V(G) - \{v_1\}$, is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Hence let $f(v) = \emptyset$. Then $\sum_{i=1}^{k} |f(x_i)| = 2$. We consider the following cases.

Case 1. $f(x_1) = \{1,2\}$.

Then $\sum_{i=2}^{k} |f(x_i)| = 0$. Now to dominate x_2, we must have $f(v_2) = \{1,2\}$. So the function f, restricted to G is a 2RDF of G of weight less than $\gamma_{r2}(G')$.

Case 2. $|f(x_1)| = 1$.

Then $\sum_{i=2}^{k} |f(x_i)| = 1$. This implies that $f(v_i) = \{1,2\}$ for each $2 \leq i \leq k$, except one of them. If $f(v_1) \neq \emptyset$ or v_1 has a neighbor with label $\{1,2\}$, then define $g : V(G) \rightarrow \mathcal{P}(\{1,2\})$ by $g(v) = \bigcup_{i=2}^{k} f(x_i)$ and $g(z) = f(z)$ for each $z \in V(G) - \{v\}$. Clearly g is a 2RDF of G of weight less than $\gamma_{r2}(G')$ as desired. Hence let $f(v_1) = \emptyset$ and let v_1 does not have a neighbor with label $\{1,2\}$. Since $v_1v_2 \in E(G)$, we deduce that $|f(v_2)| \leq 1$ and $f(v_i) = \{1,2\}$ for each $i \geq 3$. We may assume, without loss of generality, that $f(x_1) = \{1\}$. Since $N(v_1) \subseteq N[v]$ and since v_1 does not have a neighbor with label $\{1,2\}$, to rainbowly dominate v_1 we must have $2 \in f(v_2)$. Define $g : V(G) \rightarrow \mathcal{P}(\{1,2\})$ by $g(v) = \{1\}$ and $g(z) = f(z)$ for each $z \in V(G) - \{v\}$. Clearly g is a 2RDF of G of weight less than $\omega(f)$.

Case 3. $f(x_1) = \emptyset$.

To dominate x_1, we have $f(v_1) = \{1,2\}$. Then the function $g : V(G) \rightarrow \mathcal{P}(\{1,2\})$ defined by $g(v_1) = \emptyset, g(v) = \bigcup_{i=2}^{k} f(x_i)$ and $g(z) = f(z)$ for each $z \in V(G) - \{v, v_1\}$, is a 2RDF of G of weight less than $\gamma_{r2}(G')$. This yields $\mathrm{sd}_{\gamma_{r2}}(G) \leq \deg(v)$. ■

THEOREM 14. *For any connected graph G with adjacent vertices u and v, each of degree at least two,*

$$\mathrm{sd}_{\gamma_{r2}}(G) \le \deg(u) + \deg(v) - |N(u) \cap N(v)| - 1.$$

Proof. If $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$, then the result is immediate by Lemma 13. Let $N(u) - N[v] \neq \emptyset$, $N(v) = \{v_1, v_2 \ldots, v_k\}$ where $u = v_k$ and $N(u) - N[v] = \{u_1, u_2, \ldots, u_t\}$. Let G' be the graph obtained from G by subdividing the edge vv_i with subdivision vertex x_i for $i = 1, 2, \ldots, k$, and the edge uu_j with subdivision vertex y_j for $j = 1, 2, \ldots, t$. Let f be a $\gamma_{r2}(G')$-function. By Lemma 12, we may assume $|f(v)| = 0$ or $\sum_{i=1}^{k} |f(x_i)| = 0$. On the other hand, by Lemma 13, we can assume $|f(v)| + \sum_{i=1}^{k} |f(x_i)| \le 2$. If $f(v) = \{1, 2\}$, then define $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(v) = \{1\}, g(u_j) = f(u_j) \cup f(y_j)$ for each j, and $g(z) = f(z)$ for $z \in V \setminus \{v, u_1, \ldots, u_t\}$. It is easy to see that g is a 2RDF of G of weight less than $\gamma_{r2}(G')$. We now consider two cases.

Case 1. $|f(v)| = 1$.

Assume, without loss of generality, that $f(v) = \{1\}$. Then $f(x_i) = \emptyset$ for each i because $\sum_{i=1}^{k} |f(x_i)| = 0$. It follows that $2 \in f(v_i)$ for each i. Then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(v) = \emptyset, g(u) = \{1\}, g(u_j) = f(u_j) \cup f(y_j)$ for each j, and $g(z) = f(z)$ for each $z \in V \setminus \{u, v, u_1, \ldots, u_t\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$.

Case 2. $f(v) = \emptyset$.

Then $\bigcup_{i=1}^{k} f(x_i) = \{1, 2\}$ and $\sum_{i=1}^{k} |f(x_i)| = 2$. If $f(x_i) = \{1, 2\}$ for some i, then the function $g : V(G) \to \mathcal{P}(\{1, 2\})$ by $g(v_i) = f(v_i) \cup \{1\}, g(u_j) = f(u_j) \cup f(y_j)$ for each j and $g(z) = f(z)$ otherwise, is a 2RDF of G of weight less than $\gamma_{r2}(G')$.

Thus we assume $f(x_i) = \{1\}, f(x_j) = \{2\}$ for some $i \neq j$. If $f(v_i) \neq \emptyset$, then the function $f_1 : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $f_1(v) = f(x_j), f_1(u_j) = f(u_j) \cup f(y_j)$ for each j and $f_1(z) = f(z)$ otherwise, is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Thus we may assume that $f(v_i) = \emptyset$. Similarly, we assume that $f(v_j) = \emptyset$.

If $k \notin \{i, j\}$ then $f(x_k) = \emptyset$ and to rainbowly dominate x_k, we must have $f(u) = \{1, 2\}$. It is easy to see that the function $f_2 : V(G) \to \mathcal{P}(\{1, 2\})$ by $f_2(v) = \{1, 2\}, f_2(u) = \emptyset$, $f_2(u_j) = f(u_j) \cup f(y_j)$ for each j and $f_2(z) = f(z)$ otherwise, is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $k \in \{i, j\}$. Assume, without loss of generality, that $i = k$. Then $f(u) = \emptyset$. If $1 \in \bigcup_{z \in (N_G(u) \cup N_{G'}(u)) - \{x_k\}} f(z)$, then the function $f_3 : V(G) \to \mathcal{P}(\{1, 2\})$ by $f_3(v) = \{2\}$, $f_3(u_j) = f(u_j) \cup f(y_j)$ for each j and $f_3(z) = f(z)$ otherwise, is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $1 \notin \bigcup_{z \in (N_G(u) \cup N_{G'}(u)) - \{x_k\}} f(z)$. Then $f(y_j) = \{2\}$ for each j and the function $f_4 : V(G) \to \mathcal{P}(\{1, 2\})$ by $f_4(v) = f_4(u) = \{2\}$ and $f_4(z) = f(z)$ for each $z \in V \setminus \{u, v\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Thus $\mathrm{sd}_{\gamma_{r2}}(G) \le k + t = \deg(u) + \deg(v) - |N(u) \cap N(v)| - 1$ and the proof is completed. ■

COROLLARY 15. *For any connected graph G of order $n \ge 3$,*

$$\mathrm{sd}_{\gamma_{r2}}(G) \le n - \gamma_{r2}(G) + 3.$$

Proof. If G is a star, then $\gamma_{r2}(G) = 2$ and hence $\mathrm{sd}_{\gamma_{r2}}(G) = 1 < n - \gamma_{r2}(G) + 3$. Let G have two adjacent vertices u, v, each of degree at least two. Define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(v) = f(u) = \{1,2\}, f(z) = \emptyset$ for $z \in (N(u) \cup N(v)) \setminus \{u,v\}$ and $f(z) = \{1\}$ for each $z \in V \setminus (N[u] \cup N[v])$. It is clear that f is a 2RDF of G and hence $\gamma_{r2}(G) \leq n - |N(u) \cup N(v)| + 4$. Thus $|N(u) \cup N(v)| - 1 \leq n - \gamma_{r2}(G) + 3$. It follows from Theorem 14 that $\mathrm{sd}_{\gamma_{r2}}(G) \leq \deg(u) + \deg(v) - |N(u) \cap N(v)| - 1 = |N(u) \cup N(v)| - 1 \leq n - \gamma_{r2}(G) + 3$. ■

The next result is an immediate consequence of Propositions 10, 11 and Corollary 15.

THEOREM 16. *For any connected connected graph G of order $n \geq 3$,*

$$\mathrm{sd}_{\gamma_{r2}}(G) \leq n - 1.$$

THEOREM 17. *Let G be a connected graph of order $n \geq 3$ with $\delta(G) = 1$. If $v \in V(G)$ is a support vertex v, then $\mathrm{sd}_{\gamma_{r2}}(G) \leq \deg(v)$.*

Proof. Let $u \in V(G)$ be a leaf adjacent to v, where $\deg(v) \geq 2$ and let $N(v) = \{v_1 = u, v_2, \ldots, v_k\}$. Let G_1 be the graph obtained from G by subdividing the edges $vv_1, vv_2, \ldots, vv_k$ with vertices $x_1, x_2, \ldots, x_k$, respectively. Let f be a $\gamma_{r2}(G_1)$-function. Considering the path vx_1u in G_1, we may assume that $f(u) = \{1\}$ and $2 \in f(v)$ by Lemma 7. By Observation 4 and Lemma 12, we can assume $|f(v)| + \sum_{i=1}^{k} |f(x_i)| \leq 2$ and $|f(v)| = 0$ or $\sum_{i=1}^{k} |f(x_i)| = 0$. If $f(v) = \{1,2\}$, then clearly the function $g : V(G) \to \mathcal{P}(\{1,2\})$ defined by $g(u) = \emptyset$ and $g(z) = f(z)$ for each $z \in V \setminus \{u\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. Let $f(v) = \{2\}$. Then $f(x_i) = \emptyset$ for each i. Then $1 \in f(v_i)$ for each i and the function $g : V(G) \to \mathcal{P}(\{1,2\})$ defined by $g(u) = \{2\}, g(v) = \emptyset$ and $g(z) = f(z)$ for each $z \in V \setminus \{u,v\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G_1)$. This completes the proof. ■

THEOREM 18. *Let G be a connected graph of order $n \geq 3$ with $\delta(G) = 1$. Then $\mathrm{sd}_{\gamma_{r2}}(G) \leq \gamma_{r2}(G)$.*

Proof. Let $u \in V(G)$ be a leaf adjacent to v, where $\deg(v) \geq 2$ and let $N(v) = \{v_1 = u, v_2, \ldots, v_k\}$. By Theorem 17, we may assume $\deg(v) > \gamma_{r2}(G)$. Let $\gamma_{r2}(G) = \ell$ and let G' be obtained from G by subdividing the edges $vv_1, vv_2, \ldots, vv_\ell$ with vertices $x_1, x_2, \ldots, x_\ell$, respectively. Let f be a $\gamma_{r2}(G')$-function. We may assume that $f(u) = \{1\}$ and $2 \in f(v)$ by Lemma 7. If $f(v) = \{1,2\}$ or $f(x_i) \neq \emptyset$ for some i, then clearly the function defined in the proof of Theorem 17 is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $f(v) = \{2\}$ and $f(x_i) = \emptyset$ for each i. It follows that $|f(v_i)| \geq 1$ for each $1 \leq i \leq \ell$. Thus $\gamma_{r2}(G') = \omega(f) \geq |f(v)| + \sum_{i=1}^{\ell} |f(v_i)| > \gamma_{r2}(G)$. Hence $\mathrm{sd}_{\gamma_{r2}}(G) \leq \gamma_{r2}(G)$ and the proof is completed. ■

THEOREM 19. *Let G be a connected graph of order $n \geq 3$ with $\delta(G) = 1$. Then $\mathrm{sd}_{\gamma_{r2}}(G) \leq \alpha'(G) + 1$, where $\alpha'(G)$ is the matching number of G.*

Proof. If $\mathrm{sd}_{\gamma_{r2}}(G) \leq 2$, then the result is immediate. Suppose $\mathrm{sd}_{\gamma_{r2}}(G) \geq 3$. Let $u \in V(G)$ be a leaf adjacent to v, where $\deg(v) \geq 2$ and let $N(v) = \{v_1 = u, v_2, \ldots, v_k\}$. By Theorem 17, we may assume $\deg(v) > \alpha'(G) + 1$. Note that subdividing all edges incident to v, increases the rainbow domination number of G. Let S be the smallest subset of edges incident to v such that subdividing the edges of S increases the rainbow domination number of G. We may assume that $S = \{vv_1, vv_2, \ldots, vv_r\}$ where $3 \leq r \leq k$. Let G' be the graph obtained from G by subdividing the edges $vv_1, vv_2, \ldots, vv_{r-1}$ with vertices $x_1, x_2, \ldots, x_{r-1}$ respectively. Then $\gamma_{r2}(G) = \gamma_{r2}(G')$. Let f be a $\gamma_{r2}(G')$-function. By Lemma 7, we may assume that $f(u) = \{1\}, 2 \in f(v)$. If $|f(v)| = 2$ or $\bigcup_{i=1}^{r-1} f(x_i) \neq \emptyset$, then an argument similar to that described in the proof of Theorem 17 leads to a contradiction. Let $f(v) = \{2\}$ and $\bigcup_{i=1}^{r-1} f(x_i) = \emptyset$. It follows from $\bigcup_{i=1}^{r-1} f(x_i) = \emptyset$ that $1 \in f(v_i)$ for each $2 \leq i \leq r-1$.

If there exists some v_i $(2 \leq i \leq r-1)$ such that $f(w) \neq \emptyset$ or $\bigcup_{x \in N(w)-\{v_i\}} f(x) = \{1,2\}$ for each $w \in N(v_i) - \{v\}$, then the function $g : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(u) = f(v_i) = \emptyset, f(v) = \{1,2\}$ and $g(z) = f(z)$ for each $z \in V \setminus \{u, v, v_i\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G)$, a contradiction. Thus for each $2 \leq i \leq r-1$, v_i has a $\{v_2, \ldots, v_{r-1}\}$-private neighbor, say w_i. Now the set $\{uv, v_2w_2, \ldots, v_{r-1}w_{r-1}\}$ is a matching of G implying that $\mathrm{sd}_{\gamma_{r2}}(G) \leq r+1 \leq \alpha'(G) + 1$ and the proof is completed. ∎

A vertex v of a graph G is a simplicial vertex if the induced subgraph $G[N(v)]$, is a clique.

Theorem 20. *If a graph G contains a simplicial vertex u of degree at least two, then* $\mathrm{sd}_{\gamma_{r2}}(G) \leq \deg(u) + 1$.

Proof. Let u be a simplicial vertex in G with $N(u) = \{u_1, u_2, \ldots, u_k\}$, and let G' be the graph obtained from G by subdividing the edge u_1u_2 with subdivision vertex w and the edge uu_i with subdivision vertex x_i for each $1 \leq i \leq k$. We show that $\gamma_{r2}(G') > \gamma_{r2}(G)$. Suppose f is a $\gamma_{r2}(G')$-function. By Lemma 12, we may assume $|f(u)| = 0$ or $\sum_{i=1}^{k} |f(x_i)| = 0$. If $f(u) = \{1,2\}$, then the function $g : V(G) \to \mathcal{P}(\{1,2\})$ by $g(u) = \{1\}, g(u_1) = f(u_1) \cup f(x_1) \cup f(w), g(u_j) = f(u_j) \cup f(x_j)$ for each $2 \leq j \leq k$, and $g(z) = f(z)$ for each $z \in V \setminus \{u, u_1, \ldots, u_k\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $|f(u)| \leq 1$. If $|f(u)| + |f(w)| + \sum_{i=1}^{k} |f(x_i)| \geq 3$, then the function $g : V \to \mathcal{P}(\{1,2\})$ by $g(u) = \{1,2\}$, and $g(z) = f(z)$ for each $z \in V \setminus \{u\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $|f(u)| + |f(w)| + \sum_{i=1}^{k} |f(x_i)| \leq 2$. If $f(u) = \emptyset$, then we must have $\bigcup_{i=1}^{k} f(x_i) = \{1,2\}$ and so $\sum_{i=1}^{k} |f(x_i)| = 2$ and $f(w) = \emptyset$. Then $f(u_1) \cup f(u_2) = \{1,2\}$ and the function f, restricted to G is a 2RDF of G of weight less than $\gamma_{r2}(G')$.

Now let $|f(u)| = 1$. Then $\sum_{i=1}^{k} |f(x_i)| = 0$ and hence $f(x_i) = \emptyset$ for each i. Assume, without loss of generality, that $f(u) = \{1\}$. It follows that $2 \in f(u_i)$ for each i. If $f(w) = \emptyset$, then $f(u_1) \cup f(u_2) = \{1,2\}$ and the function $g : V \to \mathcal{P}(\{1,2\})$ defined by $g(u) = \emptyset$, and $g(z) = f(z)$ for each $z \in V \setminus \{u\}$ is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Let $|f(w)| = 1$. Then the function f, restricted to G

is a 2RDF of G of weight less than $\gamma_{r2}(G')$. Thus $\gamma_{r2}(G') \geq \gamma_{r2}(G)$ and hence $\mathrm{sd}_{\gamma_{r2}}(G) \leq \deg(u) + 1$. This completes the proof. ∎

4. Graphs with large rainbow domination subdivision number

In the previous two sections, we essentially encountered graphs with small rainbow domination subdivision number. Our aim in this section is to show that the rainbow domination subdivision number of a graph can be arbitrarily large. First we define total domination subdivision number and Roman domination subdivision number.

A subset S of vertices of G is a *total dominating set* if $N(S) = V$. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. The *total domination subdivision number* $\mathrm{sd}_{\gamma_t}(G)$ of a graph G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the total domination number. The total domination subdivision number was introduced by Haynes et al. [8].

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined in [12, 14] as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *weight* of an RDF is the value $w(f) = \sum_{v \in V} f(v)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. By definitions, we have

$$\gamma_{r2}(G) \leq \gamma_R(G) \tag{1}$$

for every graph G. The *Roman domination subdivision number* $\mathrm{sd}_{\gamma_R}(G)$ of a graph G is the minimum number of edges that must be subdivided in order to increase the Roman domination number of G. The Roman domination subdivision number was introduced by Atapour et al. [1].

The following graph was introduced by Haynes et al. [9] to prove a similar result on $\mathrm{sd}_{\gamma_t}(G)$ and we keep the main idea of their proof. Let $k \geq 3$ be an integer. Let $X = \{1, 2, \ldots, 3(k-1)\}$ and let $\mathcal{Y} = \{Y \subset X \mid |Y| = k\}$. Thus, $\mathcal{Y}$ consists of all k-subsets of X, and so $|\mathcal{Y}| = \binom{3(k-1)}{k}$. Let $\mathcal{G}$ be the graph with vertex set $X \cup \mathcal{Y}$ and with edge set constructed as follows: add an edge joining every two distinct vertices of X and for each $x \in X$ and $Y \in \mathcal{Y}$, add an edge joining x and Y if and only if $x \in Y$. Then, $\mathcal{G}$ is a (connected graph of order $n = \binom{3(k-1)}{k} + 3(k-1)$. The set X induces a clique in $\mathcal{G}$, while the set $\mathcal{Y}$ is an independent set and each vertex of $\mathcal{Y}$ has degree k in $\mathcal{G}$. Therefore $\delta(\mathcal{G}) = k$.

LEMMA 21. *For any integer $k \geq 3$, $\gamma_{r2}(\mathcal{G}) = 4k - 5$.*

Proof. Atapour et al. [1] proved that $\gamma_R(\mathcal{G}) = 4k - 5$. It follows from (1) that

$$\gamma_{r2}(\mathcal{G}) \leq 4k - 5. \tag{2}$$

Now let f be a $\gamma_{r2}(\mathcal{G})$-function such that $|Z = \{v \in V : |f(v)| = 1\}|$ is minimum. We proceed further with a series of claims that we may assume satisfied by the f.

Claim 1. For each $Y \in \mathcal{Y}$, $|f(Y)| \leq 1$.

Suppose to the contrary that $f(Y) = \{1, 2\}$ for some $Y \in \mathcal{Y}$. Let $y \in Y$ and define $g : V(\mathcal{G}) \longrightarrow \mathcal{P}(\{1, 2\})$ by $g(Y) = \emptyset, g(y) = \{1, 2\}$ and $g(v) = f(v)$ for each $v \in V(G) \setminus \{y, Y\}$. Since $\mathcal{Y}$ is independent and $G[X]$ is a complete graph, we deduce that g is a $\gamma_{r2}(\mathcal{G})$-function. By repeating this process we may assume $|f(Y)| \leq 1$ for each $Y \in \mathcal{Y}$.

Claim 2. $|Z \cap \mathcal{Y}| \leq 2$.

Suppose to the contrary that $|Z \cap \mathcal{Y}| \geq 3$. Let $|f(Y_1)| = |f(Y_2)| = |f(Y_3)| = 1$. If Y_1, Y_2, Y_3 are mutually disjoint, then we must have $|X| \geq 3k$ which is a contradiction. Assume, without loss of generality, that $x \in Y_1 \cap Y_2$. Then the function $g : V(\mathcal{G}) \longrightarrow \mathcal{P}(\{1, 2\})$ by $g(Y_1) = g(Y_2) = \emptyset, g(x) = \{1, 2\}$ and $g(v) = f(v)$ for each $v \in V(G) \setminus \{x, Y_1, Y_2\}$. Obviously, g is a $\gamma_{r2}(\mathcal{G})$-function which contradicts the choice of f. Henceforth, we may assume $|Z \cap \mathcal{Y}| \leq 2$ and the claim is proved.

Claim 3. $|Z \cap \mathcal{Y}| \leq 1$.

Assume to the contrary that $|Z \cap \mathcal{Y}| = 2$. Let $Y_1, Y_2 \in Z \cap \mathcal{Y}$. If $Y_1 \cap Y_2 \neq \emptyset$, then as in the proof of Claim 2, we can obtain a contradiction. Thus $Y_1 \cap Y_2 = \emptyset$. If $|f(x)| = 1$ for some $x \in Y_1$, then the function $g : V(\mathcal{G}) \longrightarrow \mathcal{P}(\{1, 2\})$ by $g(Y_1) = \emptyset, g(x) = \{1, 2\}$ and $g(v) = f(v)$ for each $v \in V(G) \setminus \{x, Y_1\}$, is a $\gamma_{r2}(\mathcal{G})$-function which contradicts the choice of f. Thus $f(x) = \emptyset$ for each $x \in Y_1$. Similarly, $f(x) = \emptyset$ for each $x \in Y_2$. Now let $x_1 \in Y_1$ and $x_2 \in Y_2$. Then the set $(Y_1 - \{x_1\}) \cup \{x_2\}$ is not rainbowly dominated by f, a contradiction.

Claim 4. $|Z \cap \mathcal{Y}| = 1$.

Assume to the contrary that $|Z \cap \mathcal{Y}| = 0$. Let X_i $(i = 1, 2)$ be the set of vertices of X such that $f(x) = \{i\}$ and let X_3 be the set of vertices of X assigned $\emptyset$ by f. Then obviously $|X_1| + |X_3| \leq k - 1$ and $|X_2| + |X_3| \leq k - 1$. It is easy to see that the below integer linear programming

$$\begin{aligned}
\text{Max} \quad & |X_1| + |X_2| + |X_3| \\
s.t. \quad & |X_1| + |X_3| \leq k - 1 \\
& |X_2| + |X_3| \leq k - 1 \\
& |X_i| \in Z_{\geq 0} \ (i = 1, 2, 3)
\end{aligned}$$

has the unique solution $|X_1| = |X_2| = k - 1$ and $|X_3| = 0$. It follows that $\gamma_{r2}(\mathcal{G}) = \omega(f) \geq 4(k - 1)$ which contradicts (2).

Let $Y \in Z \cap \mathcal{Y}$. As in the proof of Claim 3, we must have $f(x) = \emptyset$ for each $x \in Y$. If there is a vertex $w \notin Y$ such that $|f(w)| \leq 1$, then the set $(Y - \{x\}) \cup \{w\}$ is not rainbowly dominated by f for each $x \in Y$, a contradiction. Hence $f(w) = \{1, 2\}$ for each $w \in X \setminus Y$ implying that $\gamma_{r2}(\mathcal{G}) = \omega(f) = 2(2k - 3) + 1 \geq 4k - 5$. Thus $\gamma_{r2}(\mathcal{G}) = 4k - 5$ and the proof is complete. ■

THEOREM 22. *For any integer $k \geq 4$, there exists a simple connected graph G with $sd_{\gamma_{r2}}(G) = k$.*

Proof. Let $G = \mathcal{G}$. First we show that $\mathrm{sd}_{\gamma_{r2}}(G) \le k$. Let $Y = \{x_1, \ldots, x_k\} \in \mathcal{Y}$ and let G' be the graph obtained from G by subdividing the edge Yx_i with subdivision vertex z_i, for each $1 \le i \le k$. Let f be a $\gamma_{r2}(G')$-function such that $|Z = \{v \in V(G') : |f(v)| = 1\}|$ is minimum. If $f(Y) = \{1, 2\}$, then the function $g : V(G) \longrightarrow \mathcal{P}(\{1, 2\})$ defined by $g(Y) = \{1\}, g(x_i) = f(x_i) \cup f(z_i)$ for $1 \le i \le k$ and $g(v) = f(v)$ for $v \in V(G) \setminus \{Y, x_1, \ldots, x_k\}$, is a $\gamma_{r2}(G)$-function of weight less than $\gamma_{r2}(G')$.

Let $f(Y) = \emptyset$. Then obviously $\sum_{i=1}^{k} |f(z_i)| \ge 2$. If $\sum_{i=1}^{k} |f(z_i)| \ge 3$, then the function $g : V(G) \longrightarrow \mathcal{P}(\{1, 2\})$ defined by $g(Y) = \{1, 2\}$ and $g(v) = f(v)$ for each $v \in V(G) \setminus \{Y\}$, is a $\gamma_{r2}(G)$-function of weight less than $\gamma_{r2}(G')$. Suppose $\sum_{i=1}^{k} |f(z_i)| = 2$. Then $f(z_i) = \emptyset$ for some i, implying that $f(x_i) = \{1, 2\}$. Now the function f, restricted to G is a $\gamma_{r2}(G)$-function of weight less than $\gamma_{r2}(G')$.

Finally let $|f(Y)| = 1$. Without loss of generality, assume that $f(Y) = \{1\}$. By Lemma 12 we may assume $f(z_i) = \emptyset$ for each i. Then obviously $2 \in f(x_i)$ for each i. If $\bigcup_{i=1}^{k} f(x_i) = \{1, 2\}$, then by defining $g(Y) = \emptyset$ and $g(v) = f(v)$ for each $v \in V(G) - \{Y\}$ we obtain a 2RDF of G with weight less than $\gamma_{r2}(G')$. Hence $f(x_i) = \{2\}$ for each i. As in Claim 2 in the proof of Lemma 21, we must have $|Z \cap \mathcal{Y}| \le 2$. We claim that $|Z \cap \mathcal{Y}| = 1$. Assume to the contrary that $|Z \cap \mathcal{Y}| = 2$. Let $Y, Y' \in Z \cap \mathcal{Y}$. As in the proof of Claim 3 in Lemma 21, we may assume that $f(x) = \emptyset$ for each $x \in Y'$. Then obviously the set $(Y - \{x_k\}) \cup \{x\}$ is not rainbowly dominated by f, a contradiction. Thus $|Z \cap \mathcal{Y}| = 1$. If $f(x) = \emptyset$ for some $x \in V(G) \setminus Y$, then the set $(Y - \{x_k\}) \cup \{x\}$ is not rainbowly dominated by f, a contradiction. Clearly the number of vertices in X assigned $\{1\}$ under f is at most $k - 1$. Thus $\gamma_{r2}(G') = \omega(f) \ge 2(k - 2) + 2k = 4k - 4 > \gamma_{r2}(G)$. Thus

$$\mathrm{sd}_{\gamma_{r2}}(G) \le k. \tag{3}$$

Now we show that $\mathrm{sd}_{\gamma_{r2}}(G) \ge k$. Atapour et al. [1] proved that $\gamma_R(G) = 4k-5$ and $\mathrm{sd}_{\gamma_R}(G) = k$. Hence $\gamma_R(G) = \gamma_{r2}(G)$ by Lemma 21. Let $F = \{e_1, \ldots, e_{k-1}\}$ be an arbitrary subset of $k-1$ edges of G. Assume H is obtained from G by subdividing each edge in F. It follows from Lemma 21 and the fact $\gamma_{r2}(H) \le \gamma_R(H)$ that

$$\gamma_{r2}(H) \le \gamma_R(H) = \gamma_R(G) = \gamma_{r2}(G) \le \gamma_{r2}(H).$$

Hence $\gamma_{r2}(H) = \gamma_{r2}(G)$ and

$$\mathrm{sd}_{\gamma_{r2}}(G) \ge k. \tag{4}$$

By (3) and (4) we have $\mathrm{sd}_{\gamma_{r2}}(G) = k$ and the proof is complete. ■

As noticed in [9] for $\mathrm{sd}_{\gamma_t}(G)$, the following corollary is an immediate consequence of Theorem 22.

Corollary 23. *There exist simple connected graphs G of arbitrary large order n satisfying $sd_{\gamma_{r2}}(G) \ge \frac{1}{3}\log_2 n + 1$.*

We conclude this paper with an open problem.

Problem 1. Prove or disprove: For any connected graph G of order $n \ge 3$,

$$\mathrm{sd}_{\gamma_{r2}}(G) \le \alpha'(G) + 1.$$

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(received 14.10.2013; in revised form 01.02.2014; available online 03.03.2014)

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