

## EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF NONLOCAL ELLIPTIC SYSTEMS WITH MULTIPLE PARAMETERS

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**Abstract.** In this paper, we study the existence of positive solutions to the following nonlocal elliptic systems

$$\begin{cases} -M_1 \left( \int_{\Omega} |\nabla u|^p dx \right) \Delta_p u = \alpha_1 a(x) f_1(v) + \beta_1 b(x) g_1(u), & x \in \Omega, \\ -M_2 \left( \int_{\Omega} |\nabla v|^q dx \right) \Delta_q v = \alpha_2 c(x) f_2(u) + \beta_2 d(x) g_2(v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with smooth boundary  $\partial\Omega$ ,  $1 < p, q < N$ ,  $M_i : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , are continuous and nondecreasing functions,  $a, b, c, d \in C(\overline{\Omega})$ , and  $\alpha_i, \beta_i$ ,  $i = 1, 2$ , are positive parameters.

### 1. Introduction

In this paper, we study the existence of positive solutions to the following nonlocal elliptic systems

$$\begin{cases} -M_1 \left( \int_{\Omega} |\nabla u|^p dx \right) \Delta_p u = \alpha_1 a(x) f_1(v) + \beta_1 b(x) g_1(u), & x \in \Omega, \\ -M_2 \left( \int_{\Omega} |\nabla v|^q dx \right) \Delta_q v = \alpha_2 c(x) f_2(u) + \beta_2 d(x) g_2(v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  with smooth boundary  $\partial\Omega$ ,  $1 < p, q < N$ ,  $M_i : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ ,  $i = 1, 2$ , are continuous and nondecreasing functions, where  $\mathbb{R}_0^+ = [0, +\infty)$ ,  $a, b, c, d \in C(\overline{\Omega})$ , and  $\alpha_i, \beta_i$ ,  $i = 1, 2$ , are positive parameters.

We assume throughout this paper the following hypotheses

(H1)  $a, b, c, d \in C(\overline{\Omega})$  and  $a(x) \geq a_0 > 0$ ,  $b(x) \geq b_0 > 0$ ,  $c(x) \geq c_0 > 0$ ,  $d(x) \geq d_0 > 0$  for all  $x \in \Omega$ ;

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(H2)  $M_i : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ ,  $i = 1, 2$ , are two continuous and increasing functions and  $0 < m_i \leq M_i(t) \leq m_{i,\infty}$  for all  $t \in \mathbb{R}_0^+$ .

Since the first two equations in (1.1) contain an integral over  $\Omega$ , it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where  $u$  describes a process which depends on the average of itself, such as the population density, see [8]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.2}$$

presented by Kirchhoff in 1883, see [16]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings:  $L$  is the length of the string,  $h$  is the area of the cross-section,  $E$  is the Young modulus of the material,  $\rho$  is the mass density, and  $P_0$  is the initial tension.

In the recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [2, 9, 11, 15, 17, 19, 20] in which the authors have used variational method and topological method to get the existence of solutions. In [1, 4, 10, 14], the authors studied the existence of solutions for Kirchhoff type problems by using sub and supersolutions method. Motivated by the papers mentioned above and the ideas in [3–5, 7, 10, 13, 18], in this note, we study the existence of positive solutions for Kirchhoff type system (1.1). Our result improves the previous one introduced by J. Ali et al. [3] in which  $M_1(t) = M_2(t) \equiv 1$  and  $h(x) = k(x) \equiv 1$  in  $\Omega$ . We emphasize that it is really necessary to impose the boundedness of the Kirchhoff functions  $M_i$ ,  $i = 1, 2$ . The difference between this work and the previous one [10] is that system (1.1) involves multiple parameters.

We make the following assumptions on the functions  $f_i, g_i$ ,  $i = 1, 2$ :

(H3)  $f_i, g_i \in C^1(0, \infty) \cap C[0, \infty)$ ,  $i = 1, 2$ , are monotone functions such that  $\lim_{t \rightarrow \infty} f_i(t) = \lim_{t \rightarrow \infty} g_i(t) = \infty$ ;

(H4)  $\lim_{t \rightarrow \infty} f_1 \left( L[f_2(t)]^{\frac{1}{q-1}} \right) / t^{p-1} = 0$  for every  $L > 0$ ;

(H5)  $\lim_{t \rightarrow \infty} g_1(t) / t^{p-1} = \lim_{t \rightarrow \infty} g_2(t) / t^{q-1} = 0$ .

Our main result in this paper is given by the following theorem.

**THEOREM 1.1.** *Assume that the conditions (H1)-(H5) are satisfied. Then system (1.1) has a positive weak solution provided  $a_0\alpha_1 + b_0\beta_1$  and  $c_0\alpha_2 + d_0\beta_2$  are large.*

## 2. Preliminaries

In this paper, we denote by  $W_0^{1,r}(\Omega)$  ( $1 \leq r < \infty$ ) the completion of  $C_0^\infty(\Omega)$ , with respect to the norm

$$\|u\|_r = \left( \int_\Omega |\nabla u|^r dx \right)^{\frac{1}{r}}.$$

Let us consider the following eigenvalue problem for the  $r$ -Laplace operator  $-\Delta_r u$ , see [7, 12, 13]:

$$\begin{cases} -\Delta_r u = \lambda |u|^{r-2} u & \text{in } \Omega, \\ u = 0 & \text{on } x \in \partial\Omega. \end{cases} \tag{2.1}$$

Let  $\phi_{1,r} \in C^1(\bar{\Omega})$  be the eigenfunction corresponding to the first eigenvalue  $\lambda_{1,r}$  of (2.1) such that  $\phi_{1,r} > 0$  in  $\Omega$  and  $\|\phi_{1,r}\|_\infty = 1$ . It can be shown that  $\frac{\partial \phi_{1,r}}{\partial \eta} < 0$  on  $\partial\Omega$  and hence, depending on  $\Omega$ , there exist positive constants  $m, \eta, \sigma$  such that

$$\begin{cases} |\nabla \phi_{1,r}|^r - \lambda_{1,r} \phi_{1,r}^r \geq m & \text{on } \bar{\Omega}_\eta, \\ \phi_{1,r} \geq \sigma & \text{on } x \in \Omega \setminus \bar{\Omega}_\eta, \end{cases} \tag{2.2}$$

where  $\bar{\Omega}_\eta := \{x \in \Omega : d(x, \partial\Omega) \leq \eta\}$ , see [6].

We also consider the unique solution  $e_r \in W_0^{1,r}(\Omega)$  of the boundary value problem

$$\begin{cases} -\Delta_r e_r = 1 & \text{in } \Omega, \\ e_r = 0 & \text{on } x \in \partial\Omega \end{cases} \tag{2.3}$$

to discuss our result. It is known that  $e_r > 0$  in  $\Omega$  and  $\frac{\partial e_r}{\partial \eta} < 0$  on  $\partial\Omega$ .

We will prove our results by using the method of sub- and supersolutions, we refer the readers to recent papers [1, 4, 10, 14] on the topic. A pair of functions  $(\psi_1, \psi_2)$  is said to be a subsolution of problem (1.1) if it is in  $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$  such that

$$\begin{aligned} M_1 \left( \int_\Omega |\nabla \psi_1|^p dx \right) \int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla w dx \\ \leq \alpha_1 \int_\Omega a(x) f_1(\psi_2) w dx + \beta_1 \int_\Omega b(x) g_1(\psi_1) w dx, \quad \forall w \in W, \end{aligned}$$

and

$$\begin{aligned} M_2 \left( \int_\Omega |\nabla \psi_2|^q dx \right) \int_\Omega |\nabla \psi_2|^{q-2} \nabla \psi_2 \cdot \nabla w dx \\ \leq \alpha_1 \int_\Omega c(x) f_2(\psi_1) w dx + \beta_1 \int_\Omega d(x) g_2(\psi_2) w dx, \quad \forall w \in W, \end{aligned}$$

where  $W := \{w \in C_0^\infty(\Omega) : w \geq 0 \text{ in } \Omega\}$ . A pair of functions  $(z_1, z_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$  is said to be a supersolution if

$$\begin{aligned} M_1 \left( \int_\Omega |\nabla z_1|^p dx \right) \int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla w dx \\ \geq \alpha_1 \int_\Omega a(x) f_1(z_2) w dx + \beta_1 \int_\Omega b(x) g_1(z_1) w dx, \quad \forall w \in W, \end{aligned}$$

and

$$\begin{aligned} M_2 \left( \int_\Omega |\nabla z_2|^q dx \right) \int_\Omega |\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla w dx \\ \geq \alpha_1 \int_\Omega c(x) f_2(z_1) w dx + \beta_1 \int_\Omega d(x) g_2(z_2) w dx, \quad \forall w \in W. \end{aligned}$$

The following result plays an important role in our arguments, we refer the interested readers to [1, 10, 14] for details.

LEMMA 2.1. *Assume that  $M : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$  is continuous and increasing, and there exists  $m_0 > 0$  such that  $M(t) \geq m_0$  for all  $t \in \mathbb{R}_0^+$ . If the functions  $u, v \in W_0^{1,r}(\Omega)$  satisfy*

$$M\left(\int_{\Omega} |\nabla u|^r dx\right) \int_{\Omega} |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi dx \leq M\left(\int_{\Omega} |\nabla v|^r dx\right) \int_{\Omega} |\nabla v|^{r-2} \nabla v \cdot \nabla \varphi dx$$

for all  $\varphi \in W_0^{1,r}(\Omega)$ ,  $\varphi \geq 0$ , then  $u \leq v$  in  $\Omega$ .

From Lemma 2.1 we can establish the basic principle of the sub- and super-solutions method for nonlocal systems. Indeed, we consider the following nonlocal system

$$\begin{cases} -M_1\left(\int_{\Omega} |\nabla u|^p dx\right) \Delta_p u = h(x, u, v) \text{ in } \Omega, \\ -M_2\left(\int_{\Omega} |\nabla v|^q dx\right) \Delta_q v = k(x, u, v) \text{ in } \Omega, \\ u = v = 0 \text{ on } x \in \partial\Omega, \end{cases} \quad (2.4)$$

where  $\Omega$  is a bounded smooth domain of  $\mathbb{R}^N$  and  $h, k : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the following conditions:

- (HK1)  $h(x, s, t)$  and  $k(x, s, t)$  are Carathéodory functions and they are bounded if  $s, t$  belong to bounded sets.
- (KH2) There exists a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  being continuous, nondecreasing, with  $g(0) = 0$ ,  $0 \leq g(s) \leq C(1 + |s|^{\min\{p,q\}-1})$  for some  $C > 0$ , and applications  $s \mapsto h(x, s, t) + g(s)$  and  $t \mapsto k(x, s, t) + g(t)$  are nondecreasing, for a.e.  $x \in \Omega$ .

If  $u, v \in L^\infty(\Omega)$ , with  $u(x) \leq v(x)$  for a.e.  $x \in \Omega$ , we denote by  $[u, v]$  the set  $\{w \in L^\infty(\Omega) : u(x) \leq w(x) \leq v(x) \text{ for a.e. } x \in \Omega\}$ . Using Lemma 2.1 and the method as in the proof of Theorem 2.4 of [18] (see also Section 4 of [5]), we can establish a version of the abstract lower and upper-solution method for our class of the operators as follows.

PROPOSITION 2.2. *Let  $M_1, M_2 : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$  be two functions satisfying the condition (H2). Assume that the functions  $h, k$  satisfy the conditions (HK1) and (HK2). Assume that  $(\underline{u}, \underline{v}), (\bar{u}, \bar{v})$ , are respectively, a weak subsolution and a weak supersolution of system (2.4) with  $\underline{u}(x) \leq \bar{u}(x)$  and  $\underline{v}(x) \leq \bar{v}(x)$  for a.e.  $x \in \Omega$ . Then there exists a minimal  $(u_*, v_*)$  (and, respectively, a maximal  $(u^*, v^*)$ ) weak solution for system (2.4) in the set  $[\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$ . In particular, every weak solution  $(u, v) \in [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}]$  of system (2.4) satisfies  $u_*(x) \leq u(x) \leq u^*(x)$  and  $v_*(x) \leq v(x) \leq v^*(x)$  for a.e.  $x \in \Omega$ .*

### 3. Proof of the main results

Let  $\lambda_{1,r}, \phi_{1,r}$  ( $r = p, q$ ), and  $\delta, m, \sigma, \Omega_\eta$  be as described in Section 2. Let  $k_0 > 0$  such that  $f_i(t) \geq -k_0$  and  $g_i(t) \geq -k_0$  for all  $t \geq 0, i = 1, 2$ . We now construct our positive subsolution.

We shall verify that  $(\psi_1, \psi_2)$  is a subsolution of (1.1) for  $a_0\alpha_1 + b_0\beta_1$  and  $c_0\alpha_2 + d_0\beta_2$  large, where

$$\begin{aligned}\psi_1 &= \left[ \frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1} \right]^{\frac{1}{p-1}} \left( \frac{p-1}{p} \right) \phi_{1,p}^{\frac{p}{p-1}}, \\ \psi_2 &= \left[ \frac{k_0(c_0\alpha_2 + d_0\beta_2)}{mm_2} \right]^{\frac{1}{q-1}} \left( \frac{q-1}{q} \right) \phi_{1,q}^{\frac{q}{q-1}}.\end{aligned}$$

Let the test function  $w \in W := \{w \in C_0^\infty(\Omega) : w \geq 0 \text{ in } \Omega\}$ . We have

$$\begin{aligned}& \int_{\Omega} |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla w \, dx \\ &= \frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1} \int_{\Omega} |\nabla \phi_{1,p}|^{p-2} \phi_{1,p} \nabla \phi_{1,p} \cdot \nabla w \, dx \\ &= \frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1} \left[ \int_{\Omega} |\nabla \phi_{1,p}|^{p-2} \nabla \phi_{1,p} \cdot \nabla (\phi_{1,p} w) \, dx - \int_{\Omega} |\nabla \phi_{1,p}|^p w \, dx \right] \\ &= \frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1} \int_{\Omega} [\lambda_{1,p} \phi_{1,p}^p - |\nabla \phi_{1,p}|^p] w \, dx.\end{aligned}\tag{3.1}$$

Similarly, we have

$$\int_{\Omega} |\nabla \psi_2|^{q-2} \nabla \psi_2 \cdot \nabla w \, dx = \frac{k_0(c_0\alpha_2 + d_0\beta_2)}{mm_2} \int_{\Omega} [\lambda_{1,q} \phi_{1,q}^q - |\nabla \phi_{1,q}|^q] w \, dx\tag{3.2}$$

for all function  $w \in W := \{w \in C_0^\infty(\Omega) : w \geq 0 \text{ in } \Omega\}$ .

Now, by (2.2), we have in  $\bar{\Omega}_\eta$ ,  $\lambda_{1,p} \phi_{1,p}^p - |\nabla \phi_{1,p}|^p \leq -m$  and  $\lambda_{1,q} \phi_{1,q}^q - |\nabla \phi_{1,q}|^q \leq -m$ . It follows that in  $\bar{\Omega}_\eta$ ,

$$\begin{aligned}& \frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1} M_1 \left( \int_{\Omega} |\nabla \psi_1|^p \, dx \right) [\lambda_{1,p} \phi_{1,p}^p - |\nabla \phi_{1,p}|^p] \\ & \leq -k_0(a_0\alpha_1 + b_0\beta_1) \\ & \leq \alpha_1 a(x) f_1(\psi_2) + \beta_1 b(x) g_1(\psi_1)\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}& \frac{k_0(c_0\alpha_2 + d_0\beta_2)}{mm_2} M_2 \left( \int_{\Omega} |\nabla \psi_2|^q \, dx \right) [\lambda_{1,q} \phi_{1,q}^q - |\nabla \phi_{1,q}|^q] \\ & \leq -k_0(c_0\alpha_2 + d_0\beta_2) \\ & \leq \alpha_2 c(x) f_2(\psi_1) + \beta_2 d(x) g_2(\psi_2).\end{aligned}\tag{3.4}$$

Next, in  $\Omega \setminus \overline{\Omega}_\eta$ , we have  $\phi_{1,p} \geq \sigma > 0$  and  $\phi_{1,q} \geq \sigma > 0$ . By the hypotheses  $(\mathcal{H}1)$ - $(\mathcal{H}3)$ , for  $a_0\alpha_1 + b_0\beta_1$  and  $c_0\alpha_2 + d_0\beta_2$  large we deduce that

$$\begin{aligned} & \frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1} M_1 \left( \int_{\Omega} |\nabla \psi_1|^p dx \right) [\lambda_{1,p} \phi_{1,p}^p - |\nabla \phi_{1,p}|^p] \\ & \leq \frac{k_0(a_0\alpha_1 + b_0\beta_1)}{mm_1} m_{1,\infty} \lambda_{1,p} \\ & \leq a_0\alpha_1 f_1(\psi_2) + b_0\beta_1 g_1(\psi_1) \\ & \leq \alpha_1 a(x) f_1(\psi_2) + \beta_1 b(x) g_1(\psi_1) \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} & \frac{k_0(c_0\alpha_2 + d_0\beta_2)}{mm_2} M_2 \left( \int_{\Omega} |\nabla \psi_2|^q dx \right) [\lambda_{1,q} \phi_{1,q}^q - |\nabla \phi_{1,q}|^q] \\ & \leq \frac{k_0(c_0\alpha_2 + d_0\beta_2)}{mm_2} m_{2,\infty} \lambda_{1,q} \\ & \leq c_0\alpha_2 f_2(\psi_1) + d_0\beta_2 g_2(\psi_2) \\ & \leq \alpha_2 c(x) f_2(\psi_1) + \beta_2 d(x) g_2(\psi_2) \end{aligned} \quad (3.6)$$

for all  $x \in \Omega$ . From (3.1)–(3.6), it follows that  $(\psi_1, \psi_2)$  is a subsolution of system (1.1).

Next, we construct a supersolution  $(z_1, z_2)$  of system (1.1). Let

$$z_1 = C e_p, \quad z_2 = \left( \frac{\|c\|_{\infty} \alpha_2 + \|d\|_{\infty} \beta_2}{m_2} \right)^{\frac{1}{q-1}} (f_2(C \|e_p\|_{\infty}))^{\frac{1}{q-1}} e_q,$$

where  $e_p, e_q$  are given by (2.3) and  $C > 0$  is large and to be chosen later. We shall verify that  $(z_1, z_2)$  is a supersolution of system (1.1). To this end, let  $w \in W := \{w \in C_0^\infty(\Omega) : w \geq 0 \text{ in } \Omega\}$ . Then we obtain from (2.3) and the condition  $(\mathcal{H}2)$  that

$$\begin{aligned} & M_1 \left( \int_{\Omega} |\nabla z_1|^p dx \right) \int_{\Omega} |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla w dx \\ & = C^{p-1} M_1 \left( \int_{\Omega} |\nabla z_1|^p dx \right) \int_{\Omega} w dx \\ & \geq m_1 C^{p-1} \int_{\Omega} w dx. \end{aligned}$$

By  $(\mathcal{H}4)$  and  $(\mathcal{H}5)$ , we can choose  $C$  large enough so that

$$\begin{aligned} m_1 C^{p-1} & \geq \alpha_1 \|a\|_{\infty} f_1 \left[ \left( \frac{\|c\|_{\infty} \alpha_2 + \|d\|_{\infty} \beta_2}{m_2} \right)^{\frac{1}{q-1}} (f_2(C \|e_p\|_{\infty}))^{\frac{1}{q-1}} \|e_q\|_{\infty} \right] \\ & \quad + \beta_1 \|b\|_{\infty} g_1(C \|e_p\|_{\infty}) \\ & \geq \alpha_1 a(x) f_1(z_2) + \beta_1 b(x) g_1(z_1) \end{aligned}$$

for all  $x \in \Omega$ . Hence,

$$\begin{aligned} M_1 & \left( \int_{\Omega} |\nabla z_1|^p dx \right) \int_{\Omega} |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla w dx \\ & \geq \alpha_1 \int_{\Omega} a(x) f_1(z_2) w dx + \beta_1 \int_{\Omega} b(x) g_1(z_1) w dx. \end{aligned}$$

Also,

$$\begin{aligned} M_2 & \left( \int_{\Omega} |\nabla z_2|^q dx \right) \int_{\Omega} |\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla w dx \\ & \geq (\|c\|_{\infty} \alpha_2 + \|d\|_{\infty} \beta_2) \int_{\Omega} f_2(C \|e_p\|_{\infty}) w dx \\ & \geq \alpha_2 \int_{\Omega} c(x) f_2(z_1) w dx + \beta_2 \int_{\Omega} d(x) f_2(C \|e_p\|_{\infty}) w dx. \end{aligned} \quad (3.7)$$

Again by (H3) and (H5), for  $C$  large enough we have

$$\begin{aligned} f_2(C \|e_p\|_{\infty}) & \geq g_2 \left[ \left( \frac{\|c\|_{\infty} \alpha_2 + \|d\|_{\infty} \beta_2}{m_2} \right)^{\frac{1}{q-1}} (f_2(C \|e_p\|_{\infty}))^{\frac{1}{q-1}} \|e_q\|_{\infty} \right] \\ & \geq g_2(z_2). \end{aligned} \quad (3.8)$$

From (3.7) and (3.8) we have

$$\begin{aligned} M_2 & \left( \int_{\Omega} |\nabla z_2|^q dx \right) \int_{\Omega} |\nabla z_2|^{q-2} \nabla z_2 \cdot \nabla w dx \\ & \geq \alpha_2 \int_{\Omega} c(x) f_2(z_1) w dx + \beta_2 \int_{\Omega} d(x) g_2(z_2) w dx \end{aligned}$$

and thus  $(z_1, z_2)$  is a supersolution of system (1.1). Obviously, we have  $\psi_i(x) \leq z_i(x)$  in  $\Omega$  with large  $C$  for  $i = 1, 2$ . Thus, by the comparison principle, there exists a solution  $(u, v)$  of (1.1) with  $\psi_1 \leq u \leq z_1$  and  $\psi_2 \leq v \leq z_2$ . This completes the proof of Theorem 1.1.

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