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TOPOLOGY GENERATED BY CLUSTER SYSTEMS

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Abstract. In this paper, we prove that (X,τ) and the new topology $(X,\tau_{\mathcal{E}})$ have the same semiregularization if $\mathcal{E}$ is a π-network in X with the property $\mathcal{H}$. Also, we discuss the properties of $\mathcal{E}$, $\tau_{\mathcal{E}}$ and study generalized Volterra spaces and discuss their properties. We show that $\tau_{\mathcal{E}}$ coincides with the $\star$-topology for a particular $\mathcal{E}$.

1. Introduction

An ideal $\mathcal{J}$ on a nonempty set X is a collection of subsets of X which satisfies that (i) $A \in \mathcal{J}$ and $B \subset A$ implies $B \in \mathcal{J}$ and (ii) $A \in \mathcal{J}$ and $B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$. Given a topological space (X,τ) with an ideal $\mathcal{J}$ on X and if 2^X is the set of all subsets of X, a set operator $(.)^\star : 2^X \to 2^X$, called a *local function* [7] of A with respect to $\mathcal{J}$ and τ, is defined as follows: for $A \subset X$, $A^\star(\mathcal{J},\tau) = \{x \in X \mid U \cap A \notin \mathcal{J}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts concerning the local functions [6, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\star(A)$ for a topology $\tau^\star(\tau,\mathcal{J})$, called the $\star$-*topology*, finer than τ, is defined by $cl^\star(A) = A \cup A^\star(\mathcal{J},\tau)$ [14]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{J},\tau)$ and $\tau^\star$ or $\tau^\star(\mathcal{J})$ for $\tau^\star(\mathcal{J},\tau)$. An ideal $\mathcal{I}$ is said to be *codense* [6] if $\tau \cap \mathcal{I} = \{\emptyset\}$. By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of A in (X,τ). A subset A of a space is said to be *regular open* (resp. α-*open* [11], *semiopen* [8], *preopen* [9]) if $A = int(cl(A))$ (resp. $A \subset int(cl(int(A)))$, $A \subset cl(int(A))$, $A \subset int(cl(A))$). The family of all preopen (resp. semiopen) sets in (X,τ) is denoted by $PO(X)$ (resp. $SO(X)$). The regular open sets in (X,τ) form a basis for a new topology on X, known as *semiregularization* of τ, denoted τ_s. The topology τ_s is coarser than τ, and τ is said to be *semiregular* if $\tau = \tau_s$. The family of all α-open sets in (X,τ) is denoted by τ^α. τ^α is a topology on X which is finer than τ. The complement of an α-open set is called an α-closed set. The closure and interior of A in (X,τ^α) are

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denoted by $cl_\alpha(A)$ and $int_\alpha(A)$, respectively. If $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X,τ), then $\tau^\star(\mathcal{N}) = \tau^\alpha$ and $cl_\alpha(A) = A \cup A^\star(\mathcal{N})$ [6]. A topological space (X,τ) is said to be *submaximal space* [2] if every dense set is open. A space X is said to be *resolvable* [2] if X is union of two disjoint dense subsets of X.

A nonempty collection $\mathcal{A}$ of nonempty subsets of a set X is called a *grill* [3] if $A \in \mathcal{A}$ and $A \subset B \subset X$ implies $B \in \mathcal{A}$, and $A, B \subset X$ and $A \cup B \in \mathcal{A}$ implies either $A \in \mathcal{A}$ or $B \in \mathcal{A}$. Given a space (X,τ) with a grill $\mathcal{A}$ on X, a set operator $\Phi_\mathcal{A} : 2^X \to 2^X$ [12] with respect to τ and $\mathcal{A}$ is defined as follows: for $A \subset X, \Phi_\mathcal{A}(A) = \{x \in X \mid U \cap A \in \mathcal{A}$ for every $U \in \tau(x)\}$. The operator $\psi : 2^X \to 2^X$ defined by $\psi(A) = A \cup \Phi_\mathcal{A}(A)$ satisfies Kuratowski's closure axioms [12, Theorem 2.4] and $\tau_\mathcal{A} = \{U \subset X \mid \psi(X - U) = X - U\}$ is the unique topology on X induced by $\mathcal{A}$. In [13], Renukadevi proved that $\mathcal{I}$ is a proper ideal on X if and only if $2^X - \mathcal{I}$ is a grill on X and $\mathcal{A}$ is a grill on X if and only if $\mathcal{I} = 2^X - \mathcal{A}$ is an ideal on X. Also, she proved that $A^\star(\mathcal{I}) = \Phi_\mathcal{A}(A)$ for every subset A of X.

Any nonempty system $\mathcal{E} \subset 2^X - \{\emptyset\}$ will be called a *cluster system* in X. If any nonempty open subset of a nonempty open set G contains a set from $\mathcal{E}$, then $\mathcal{E}$ is called a π-*network* [10] in G. For a cluster system $\mathcal{E}$ and a subset A of a space X, $\mathcal{E}(A)$ is the set of all points $x \in X$ such that for any neighborhood U of x, the intersection $U \cap A$ contains a set from $\mathcal{E}$.

In 1993, the class of Volterra spaces was introduced by Gauld and Piotrowski [5]. A topological space (X,τ) is said to be *Volterra* [5] (resp. *weakly Volterra* [5]) if the intersection of any two dense G_δ-sets in X is dense (resp. nonempty). A subset A of X is called *weakly $\mathcal{E}$-Volterrra space* [10] if for any two sets A_1 and A_2 such that $\mathcal{E}(A) \subset \mathcal{E}(A_i)$, i=1,2, $A_1 \cap A_2$ is nonempty. Moreover, if $A \neq \emptyset$ and $cl(A_1 \cap A_2) \supset A$, that is, $A_1 \cap A_2$ is dense in A, then A is called *$\mathcal{E}$-Volterra* [10]. The following lemmas will be useful in the sequel.

Lemma 1.1. [4] *Let (X,τ) be a space. Then the following hold.*

(a) $PO(X,\tau) = PO(X,\tau^\alpha)$.

(b) *If X is submaximal, $PO(X,\tau) = \tau$.*

Lemma 1.2. [4] *For a resolvable space (X,τ), the following are equivalent.*

(a) *$PO(X,\tau)$ is a topology.*

(b) *Every subset of X is preopen.*

(c) *Every open set is closed.*

Lemma 1.3. [6, Lemma 6.3] *Let τ and σ be topologies on X and $\tau \subseteq \sigma$. If $cl_\tau(V) = cl_\sigma(V)$ for every $V \in \sigma$, then $\tau_s = \sigma_s$.*

Lemma 1.4. [1] *If (X,τ) is submaximal, then X remains submaximal when endowed with any finer topology.*

Lemma 1.5. [10, Remark 1 (2)] *Let (X,τ) be a space and G be a nonempty open subset of X. Then $\mathcal{E}$ is a π-network in G if and only if $\mathcal{E}(G) = \mathcal{E}(cl(G)) = cl(G)$.*

Lemma 1.6. [10, Theorem 2] *Let* (X,τ) *be a space with a cluster system* $\mathcal{E}$. *If* $\mathcal{E}$ *is a* π*-network in an open set* X_0, *then* X_0 *is* $\mathcal{E}$*-Volterra if and only if any nonempty open subset of* X_0 *is weakly* $\mathcal{E}$*-Volterra.*

2. Properties of $\mathcal{E}$-operator

In this section, we discuss the properties of $\mathcal{E}(A)$. We have $\mathcal{E}(X) = X$ if and only if $\mathcal{E}$ is a π-network in X. The following Theorem 2.1 gives the properties of $\mathcal{E}$ and Example 2.2 below shows that it can be $\mathcal{E}(X) \neq X$ even if $\mathcal{E}$ is a π-network in a proper open subset of X.

Theorem 2.1. *Let* (X,τ) *be a space with cluster systems* $\mathcal{E}$ *and* $\mathcal{E}_1$ *on* X, *and let* A *and* B *be subsets of* X. *Then the following hold.*

(a) $\mathcal{E}(\emptyset) = \emptyset$.

(b) $\mathcal{E}(\mathcal{E}(A)) \subseteq \mathcal{E}(A)$.

(c) *If* $\mathcal{E} \subseteq \mathcal{E}_1$, *then* $\mathcal{E}(A) \subseteq \mathcal{E}_1(A)$.

(d) $\mathcal{E}(A)$ *is closed,* $\mathcal{E}(A) \subset cl(A)$ *and if* $A \subset B$, *then* $\mathcal{E}(A) \subset \mathcal{E}(B)$ [10, Remark 1(1)].

(e) *If* $U \in \tau$, *then* $U \cap \mathcal{E}(A) = U \cap \mathcal{E}(U \cap A) \subseteq \mathcal{E}(U \cap A)$.

Proof. It is enough to prove (e). $U \cap A \subset A$ implies $\mathcal{E}(U \cap A) \subset \mathcal{E}(A)$ which implies that $U \cap \mathcal{E}(U \cap A) \subset U \cap \mathcal{E}(A)$. If $x \in U \cap \mathcal{E}(A)$, then $x \in U$ and for every $U_x \in \tau(x)$, $U_x \cap A \supset E$ for some $E \in \mathcal{E}$. Take $W = U \cap U_x$. Then $W \in \tau(x)$ with $W \cap A \supset E$ so that $U_x \cap (U \cap A) \supset E$. Therefore, $x \in U \cap \mathcal{E}(U \cap A)$ and so $U \cap \mathcal{E}(A) = U \cap \mathcal{E}(U \cap A)$. ■

Example 2.2. Consider $\mathbb{R}$ with the usual topology τ and $\mathcal{E} = \{G \subset (0,1) \mid G \in \tau - \{\emptyset\}\}$. Clearly, $\mathcal{E}$ is a π-network in $(0,1)$. But $\mathcal{E}(\mathbb{R}) = [0,1] \neq \mathbb{R}$.

Theorem 2.3. *Let* (X,τ) *be a space and* G *be open in* X. *If* $\mathcal{E}$ *is a* π*-network in* G *and* $\mathcal{E}(G) \subset \mathcal{E}(A)$ *for* $A \subset X$, *then* $\mathcal{E}(G) = \mathcal{E}(A \cap G)$.

Proof. Since $A \cap G \subset G$, $\mathcal{E}(A \cap G) \subset \mathcal{E}(G)$ by Theorem 2.1(d). Let $x \in \mathcal{E}(G)$. Since $\mathcal{E}$ is a π-network in G, $\mathcal{E}(G) = cl(G) \subset \mathcal{E}(A)$, by Lemma 1.5. Therefore, for any $U \in \tau(x)$, $U \cap G$ is a nonempty subset of $\mathcal{E}(A)$, hence there is $E \in \mathcal{E}$ such that $E \subset U \cap G \cap A$. So $x \in \mathcal{E}(A \cap G)$. Thus, $\mathcal{E}(A \cap G) \supset \mathcal{E}(G)$ and so $\mathcal{E}(A \cap G) = \mathcal{E}(G)$. ■

Two cluster systems $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be *equivalent* if $\mathcal{E}_1(A) = \mathcal{E}_2(A)$ for every subset A of X. For example, if for any $E_1 \in \mathcal{E}_1$, there is $E_2 \in \mathcal{E}_2$ such that $E_2 \subset E_1$ and vice versa, then the cluster systems $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent. Let $\mathcal{E}_\pi = \{\mathcal{E} \mid \mathcal{E}$ is a π-network in X and every element of $\mathcal{E}$ has nonempty interior$\}$. If $\gamma = \{G \mid G \in \tau - \{\emptyset\}\}$, then $\gamma \in \mathcal{E}_\pi$ is clear. But there are equivalent cluster systems different from γ in $\mathcal{E}_\pi$ as given in the following Example 2.4.

Example 2.4. Consider $\mathbb{R}$ with the usual topology. Let $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ be the cluster systems in $\mathbb{R}$ given by $\mathcal{E}_1 = \{(a,b] \mid a,b \in \mathbb{R}$ and $a < b\}, \mathcal{E}_2 = \{[a,b) \mid a,b \in \mathbb{R}$ and $a < b\}$ and $\mathcal{E}_3 = \{[a,b] \mid a,b \in \mathbb{R}$ and $a < b\}$. Then for $i = 1,2,3$, $\mathcal{E}_i \in \mathcal{E}_\pi$ and $\mathcal{E}_i \neq \gamma$. But for $i \neq j$ and $i,j \in \{1,2,3\}$, each $\mathcal{E}_i$ is equivalent with $\mathcal{E}_j$.

Theorem 2.5. *Let (X,τ) be a space and $A \subset X$. If $\mathcal{E}$ is a π-network in X, then $cl(int(A)) \subset \mathcal{E}(A)$. Equality holds, if every element of $\mathcal{E}$ has nonempty interior.*

Proof. Sice $\mathcal{E}$ is a π-network in X, $cl(int(A)) = \mathcal{E}(int(A)) \subset \mathcal{E}(A)$. Assume that $x \in \mathcal{E}(A)$. Then for every $U_x \in \tau(x)$, there exists $E \in \mathcal{E}$ such that $U_x \cap A \supset E$. Since $E \subset U_x \cap A$, $int(E) \subset U_x \cap int(A)$ and so $U_x \cap int(A) \neq \emptyset$, by hypothesis. Thus, $x \in cl(int(A))$ so that $\mathcal{E}(A) \subset cl(int(A))$. Hence $\mathcal{E}(A) = cl(int(A))$. ∎

The following Example 2.6 shows that the condition "every element of $\mathcal{E}$ has nonempty interior" is necessary for equality in Theorem 2.5.

Example 2.6. Consider $X = [0,\infty)$, $\tau = \{(a,\infty) \mid a \in X\} \cup \{X, \emptyset\}$ and $\mathcal{E} = \{(a,b) \mid a,b \in X\}$. Since every nonempty open subset of X has many element of $\mathcal{E}$, $\mathcal{E}$ is a π-network in X. Also, $int(E) = \emptyset$ for every $E \in \mathcal{E}$. If $A = [1,3)$, then $\mathcal{E}(A) = [0,3)$ and $cl(int(A)) = \emptyset$. Hence $\mathcal{E}(A) \not\subseteq cl(intA)$.

Corollary 2.7. *Let (X,τ) be a space and $A \subset X$. If $\mathcal{E}$ is a π-network in X with $int(E) \neq \emptyset$ for every $E \in \mathcal{E}$, then $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(A)$.*

Proof. By Theorem 2.5, $\mathcal{E}(\mathcal{E}(A)) = \mathcal{E}(cl(int(A))) = cl(int(cl(int(A)))) = cl(int(A)) = \mathcal{E}(A)$. ∎

Corollary 2.8. *Let (X,τ) be a space and $A \subset X$. If $\mathcal{E}$ is a π-network in X, then the following hold.*

(a) *If $\mathcal{E} \subset SO(X)$, then $\mathcal{E}(A) = cl(int(A))$.*

(b) *$A \subset \mathcal{E}(A)$ for every $A \in SO(X)$.*

We say that a cluster system $\mathcal{E}$ on X satisfies the *property $\mathcal{I}$*, whenever $E_1, E_2 \in \mathcal{E}$ implies that $E_1 \cap E_2 \in \mathcal{E}$. A cluster system $\mathcal{E}$ is said to satisfy the *property $\mathcal{H}$* if for every $U_x \in \tau(x)$ and $A, B \subset X$ such that $U_x \cap (A \cup B) \supset E$ implies $U_x \cap A \supset E_1$ or $U_x \cap B \supset E_2$ for some E_1 or E_2 in $\mathcal{E}$. If we consider a cluster system $\mathcal{E}$ with the property $\mathcal{H}$, then a system $\mathcal{E}'$ of all supersets of all sets from $\mathcal{E}$ is equivalent with $\mathcal{E}$ and $2^X - \mathcal{E}'$ is an ideal. The following Example 2.9 shows that a cluster system with $\mathcal{H}$-property need not be a grill.

Example 2.9. (a) Consider $\mathbb{R}$ with the usual topology. If $\mathcal{E} = \{\{r\} : r \in \mathbb{Q}\}$, then $\mathcal{E}$ is a cluster system in $\mathbb{R}$. Also, $\mathcal{E}$ is a π-network in $\mathbb{R}$ satisfying the property $\mathcal{H}$. But $\mathcal{E}$ is not a grill.

(b) In any topological space (X,τ) with a proper ideal $\mathfrak{I}$ on X, $\mathfrak{I} - \{\emptyset\}$ is a cluster system satisfying the property $\mathcal{H}$. But $\mathfrak{I} - \{\emptyset\}$ is not a grill.

Theorem 2.10. *Let (X,τ) be a space and $A_1, A_2 \subset X$. If $\mathcal{E}$ is a cluster system with the property $\mathcal{I}$, then $\mathcal{E}(A_1 \cap A_2) = \mathcal{E}(A_1) \cap \mathcal{E}(A_2)$.*

Proof. Since $A_1 \cap A_2$ is contained in both A_1 and A_2, $\mathcal{E}(A_1 \cap A_2) \subset \mathcal{E}(A_1) \cap \mathcal{E}(A_2)$. Let $x \in \mathcal{E}(A_1) \cap \mathcal{E}(A_2)$. Then for every $U_x \in \tau(x)$, there exist $E_1, E_2 \in \mathcal{E}$ such that $U_x \cap A_1 \supset E_1$ and $U_x \cap A_2 \supset E_2$. Now $U_x \cap A_1 \supset E_1$ and $U_x \cap A_2 \supset E_2$

implies that $(U_x\cap A_1)\cap(U_x\cap A_2)\supset E_1\cap E_2$ which implies that $U_x\cap(A_1\cap A_2)\supset E_3$ where $E_3=E_1\cap E_2\in\mathcal{E}$. Hence $x\in\mathcal{E}(A_1\cap A_2)$. Therefore, $\mathcal{E}(A_1\cap A_2)=\mathcal{E}(A_1)\cap\mathcal{E}(A_2)$. ■

THEOREM 2.11. *Let (X,τ) be a space and A, $B\subset X$. If $\mathcal{E}$ is a cluster system with the property $\mathcal{H}$, then the following hold.*

(a) $\mathcal{E}(A\cup B)=\mathcal{E}(A)\cup\mathcal{E}(B)$.

(b) $\mathcal{E}(A)-\mathcal{E}(B)=\mathcal{E}(A-B)-\mathcal{E}(B)\subset\mathcal{E}(A-B)$.

Proof. (a) By Theorem 2.1(d), $\mathcal{E}(A\cup B)\supset\mathcal{E}(A)\cup\mathcal{E}(B)$. For the reverse inclusion, if $x\in\mathcal{E}(A\cup B)$, then for every $U_x\in\tau(x)$, $U_x\cap(A\cup B)\supset E$ for some $E\in\mathcal{E}$. By hypothesis, there exist $E_1,E_2\in\mathcal{E}$ such that $U_x\cap A\supset E_1$ or $U_x\cap B\supset E_2$ and so $x\in\mathcal{E}(A)$ or $x\in\mathcal{E}(B)$ so that $x\in\mathcal{E}(A)\cup\mathcal{E}(B)$. Hence $\mathcal{E}(A\cup B)=\mathcal{E}(A)\cup\mathcal{E}(B)$.

(b) Clearly, $\mathcal{E}(A-B)-\mathcal{E}(B)\subset\mathcal{E}(A)-\mathcal{E}(B)$. Let $x\in\mathcal{E}(A)-\mathcal{E}(B)$. Then $x\in\mathcal{E}(A)$ implies that for any neighborhood U_x of x, $U_x\cap A$ contains a set from $\mathcal{E}$ and $x\notin\mathcal{E}(B)$ implies that there exists a neighborhood V_x of x such that $V_x\cap B$ does not contain any element of $\mathcal{E}$. If $W_x=U_x\cap V_x$, then $W_x\cap A\supset E$ for some $E\in\mathcal{E}$ and $W_x\cap B\not\supset E$ for every $E\in\mathcal{E}$. By the $\mathcal{H}$ property of $\mathcal{E}$ and $E\subset W_x\cap((A-B)\cup B)$, there exists $E_1\in\mathcal{E}$ such that $E_1\subset W_x\cap(A-B)$. Therefore, $E_1\subset U_x\cap(A-B)$. Hence $x\in\mathcal{E}(A-B)$ and so $x\in\mathcal{E}(A-B)-\mathcal{E}(B)$. Thus, $\mathcal{E}(A-B)-\mathcal{E}(B)\supset\mathcal{E}(A)-\mathcal{E}(B)$. ■

The following Example 2.12 shows that the property $\mathcal{H}$ on $\mathcal{E}$ cannot be dropped in the above Theorem 2.11.

EXAMPLE 2.12. Consider $X=[0,\infty)$, $\tau=\{(a,\infty)\mid a\in X\}\cup\{X,\emptyset\}$ and $\mathcal{E}=\{(n,n+1)\mid n\in\mathbb{W}\}$ where $\mathbb{W}=\mathbb{N}\cup\{0\}$. Clearly, $\mathcal{E}$ does not satisfy the property $\mathcal{H}$.

(a) If $A=(2,3)\cup[3.5,4.5]$ and $B=(2,3)\cup[4.5,5)$, then $A\cup B=(2,3)\cup[3.5,5]$. Also, $\mathcal{E}(A)=[0,2]=\mathcal{E}(B)$ and $\mathcal{E}(A\cup B)=[0,4]$. Therefore, $\mathcal{E}(A)\cup\mathcal{E}(B)\neq\mathcal{E}(A\cup B)$.

(b) If $A=[2,3.5]$ and $B=[1,2.5]$, then $A-B=(2.5,3.5]$. Also, $\mathcal{E}(A)=[0,2]$, $\mathcal{E}(B)=[0,1]$, $\mathcal{E}(A)-\mathcal{E}(B)=(1,2]$ and $\mathcal{E}(A-B)=\emptyset$. Therefore, $\mathcal{E}(A-B)-\mathcal{E}(B)\not\supset\mathcal{E}(A)-\mathcal{E}(B)$.

THEOREM 2.13. *Let (X,τ) be a space and $A\subset X$. If $\mathcal{E}$ is a cluster system with the property that every nonempty subset of element of $\mathcal{E}$ is in $\mathcal{E}$, then $\mathcal{E}(A)=cl(A)$ for $A\in\mathcal{E}$.*

Proof. By Theorem 2.1(d), $\mathcal{E}(A)\subset cl(A)$. Let $x\in cl(A)$. Then for every $U_x\in\tau(x)$, $U_x\cap A\neq\emptyset$. Since $A\in\mathcal{E}$ every nonempty subset of A is also in $\mathcal{E}$ implies that $U_x\cap A\in\mathcal{E}$ and so $x\in\mathcal{E}(A)$. Hence $cl(A)\subset\mathcal{E}(A)$ which completes the proof. ■

The following Example 2.14 shows that the property that every nonempty subset of element of $\mathcal{E}$ is also in $\mathcal{E}$, cannot be dropped in Theorem 2.13.

Example 2.14. Consider R with the usual topology with a cluster system $\mathcal{E} = \{(a,b) \mid a,b \in \mathbb{Z}\}$ where $\mathbb{Z}$ is the set of all integers and $a < b$. If $A = (1,2)$, then $\mathcal{E}(A) = \emptyset$ and $cl(A) = [1,2]$.

3. $\mathcal{E}$-topology and its properties

Throughout this section, we consider the cluster system with the property $\mathcal{H}$. By Theorem 2.1 and Theorem 2.11, we have $cl_{\mathcal{E}} : 2^X \to 2^X$ defined by $cl_{\mathcal{E}}(A) = A \cup \mathcal{E}(A)$ is a Kuratowski closure operator on 2^X. We will denote by $\tau_{\mathcal{E}}$ the topology generated by $cl_{\mathcal{E}}$, called *$\mathcal{E}$-topology*, where τ is the original topology on X, that is, $\tau_{\mathcal{E}} = \{A \subset X \mid cl_{\mathcal{E}}(X - A) = X - A\}$. If $\mathcal{E} = 2^X - \{\emptyset\}$ or $\mathcal{E} = \{\{x\} \mid$ for every $x \in X\}$, then $\mathcal{E}(A) = cl(A)$. Hence in this case, $cl_{\mathcal{E}}(A) = cl(A)$ and $\tau_{\mathcal{E}} = \tau$.

We observe that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are cluster systems on X with the property $\mathcal{H}$, then $\mathcal{E}_1 \vee \mathcal{E}_2 = \{E_1 \cup E_2 \mid E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2\}$, $\mathcal{E}_1 \cup \mathcal{E}_2 = \{E \mid E \in \mathcal{E}_1$ or $E \in \mathcal{E}_2\}$ are also cluster systems on X and $\mathcal{E}_1 \cup \mathcal{E}_2$ satisfies the property $\mathcal{H}$. But $\mathcal{E}_1 \vee \mathcal{E}_2$ need not satisfy the property $\mathcal{H}$ as shown by the following Example 3.1. Corollary 3.3 below follows from Theorem 2.1(c) and Theorem 3.2.

Example 3.1. Consider the topological space (X,τ) where $X = \{a,b\}$ and $\tau = \{\emptyset, X\}$. Let $\mathcal{E}_1 = \{\{a\}\}$ and $\mathcal{E}_2 = \{\{b\}\}$. Then $\mathcal{E}_1 \vee \mathcal{E}_2 = \{\{a,b\}\}$. If $A = \{a\}$, $B = \{b\}$ and $E = \{a,b\}$, then $E \subset X \cap (A \cup B) = X$, but $X \cap A$ and $X \cap B$ do not contain a set from $\mathcal{E}_1 \vee \mathcal{E}_2$, respectively. Thus, $\mathcal{E}_1 \vee \mathcal{E}_2$ does not satisfy the property $\mathcal{H}$.

Theorem 3.2. *Let (X,τ) be a space with two cluster systems $\mathcal{E}_1$ and $\mathcal{E}_2$ in X. Then the following hold.*

(a) $(\mathcal{E}_1 \cup \mathcal{E}_2)(A) = \mathcal{E}_1(A) \cup \mathcal{E}_2(A)$.

(b) $(\mathcal{E}_1 \vee \mathcal{E}_2)(A) = \mathcal{E}_1(A) \cap \mathcal{E}_2(A)$.

Proof. (a) is clear.

(b) $x \in (\mathcal{E}_1 \vee \mathcal{E}_2)(A)$ if and only if for every U_x, $U_x \cap A \supset E_1 \cup E_2$ for some $E_1 \in \mathcal{E}_1$ and $E_2 \in \mathcal{E}_2$ if and only if $U_x \cap A \supset E_1$ and $U_x \cap A \supset E_2$ if and only if $x \in \mathcal{E}_1(A)$ and $x \in \mathcal{E}_2(A)$ if and only if $x \in \mathcal{E}_1(A) \cap \mathcal{E}_2(A)$. ∎

Corollary 3.3. *Let (X,τ) be a space with two cluster systems $\mathcal{E}_1$ and $\mathcal{E}_2$. Then the following hold.*

(a) $\mathcal{E}_1 \subseteq \mathcal{E}_2$ *implies* $\tau_{\mathcal{E}_2} \subseteq \tau_{\mathcal{E}_1}$.

(b) $\tau_{\mathcal{E}_1 \cup \mathcal{E}_2} = \tau_{\mathcal{E}_1} \cap \tau_{\mathcal{E}_2}$.

In general, we do not have any cluster system $\mathcal{E}$ which produces $\tau_{\mathcal{E}} =$ discrete topology. The following Theorem 3.4 shows that in a T_1 space, $\tau_{\mathcal{E}}$ can be discrete.

Theorem 3.4. *Let (X,τ) be a T_1 space. If $\mathcal{E} = \{\{x_0\}\}$ for some $x_0 \in X$, then $\tau_{\mathcal{E}}$ is discrete.*

Proof. Let A be any nonempty subset of X. If $x_0 \notin A$, then for any open set U, $U \cap A$ contains no set from $\mathcal{E} = \{\{x_0\}\}$ and so $\mathcal{E}(A) = \emptyset$. If $x_0 \in A$, then for any $U \in \tau(x_0)$, $U \cap A \supset \{x_0\}$ and so $x_0 \in \mathcal{E}(A)$. If $y \neq x_0$, then there exists a $U \in \tau(y)\,(U = X - \{x_0\})$ such that $U \cap A$ contains no set from $\mathcal{E}$, so $y \notin \mathcal{E}(G)$ and $\mathcal{E}(G) = \{x_0\}$. Hence $\tau_{\mathcal{E}}$ is discrete. ∎

Remark 3.5. If (X, τ) is T_1 and $\{\{x_0\}\} = \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3 = 2^X - \{\emptyset\}$ for some $x_0 \in X$, then by Corollary 3.3(a), we have $\tau = \tau_{\mathcal{E}_3} \subset \tau_{\mathcal{E}_2} \subset \tau_{\mathcal{E}_1} =$ discrete topology.

If $\mathcal{E}$ is a cluster system in X with the property $\mathcal{H}$, then the system $\mathcal{E}'$ of all supersets of all sets from $\mathcal{E}$ is equivalent with $\mathcal{E}$ and $\mathcal{I} = 2^X - \mathcal{E}'$ is an ideal on X. Therefore, by Theorem 2.1 of [13], $\mathcal{E}(A) = A^\star(\mathcal{I})$ and so $\tau_{\mathcal{E}} = \tau^\star(\mathcal{I})$. The following Theorem 3.6 shows that $\mathcal{I}$ is a codense ideal if and only if $\mathcal{E}$ is a π-network.

Theorem 3.6. *Let (X, τ) be a topological space with a cluster system $\mathcal{E}$. If $\mathcal{I} = 2^X - \mathcal{E}'$, then $\mathcal{E}$ is a π-network in X if and only if $\mathcal{I}$ is codense.*

Proof. Suppose that $\mathcal{E}$ is a π-network in X. Let $\emptyset \neq A \in \tau \cap \mathcal{I}$. Then $A \in \tau$ and $A \in \mathcal{I}$. Since $\mathcal{E}$ is a π-network, there exists $E \in \mathcal{E}$ such that $E \subset A$. Since $\mathcal{I}$ is an ideal and $E \subset A$, $E \in \mathcal{I}$ which contradicts the fact that $E \in \mathcal{E}'$. Hence $\mathcal{I}$ is codense. Converse follows from Theorem 2.2 of [13]. ∎

From Theorem 3.6 and the construction of $\mathcal{E}'$, we have the following Theorem 3.7 whose routine proof is omitted. Theorem 3.7(b) shows that (X, τ) and $(X, \tau_{\mathcal{E}})$ have the same semiregularizations if $\mathcal{E}$ is a π-network in X. The proof of (c) and (d) follows from (b).

Theorem 3.7. *Let (X, τ) be a space and $\mathcal{E}$ be a π-network in (X, τ). Then the following hold.*

(a) *$V \subset \mathcal{E}(V)$ for every $V \in \tau_{\mathcal{E}}$.*

(b) *$\tau_s = (\tau_{\mathcal{E}})_s$.*

(c) *Semiregular properties are shared by (X, τ) and $(X, \tau^\star)$.*

(d) *If $(X, \tau_{\mathcal{E}})$ is semiregular, then $\tau = \tau_{\mathcal{E}}$.*

Proof. (a) Observe that a subset A of X is $\tau_{\mathcal{E}}$-closed if and only if $\mathcal{E}(A) \subset A$. Let $V \in \tau_{\mathcal{E}}$. Then $X - V$ is $\tau_{\mathcal{E}}$-closed implies that $\mathcal{E}(X - V) \subset X - V$ which implies $\mathcal{E}(X) - \mathcal{E}(V) \subset X - V$, by Theorem 2.11(b). Since $\mathcal{E}$ is a π-network in X, $\mathcal{E}(X) = X$. Therefore, $X - \mathcal{E}(V) \subset X - V$ so that $V \subset \mathcal{E}(V)$. ∎

Theorem 3.8. *Let (X, τ) be a space and $\mathcal{E}$ be a π-network in X. Then $\tau_{\mathcal{E}} \subset PO(X, \tau)$.*

Proof. Let $A \in \tau_{\mathcal{E}}$. Then $\mathcal{E}(X - A) \subset X - A$. By Theorem 2.5, $cl(int(X - A)) \subset X - A$ which implies $A \subset X - cl(int(X - A))$ so that $A \subset int(cl(A))$. Hence $A \in PO(X, \tau)$. Therefore, $\tau_{\mathcal{E}} \subset PO(X, \tau)$. ∎

Theorem 3.9. *Let (X, τ) be a space and $\mathcal{E} \in \mathcal{E}_\pi$. Then the following hold.*

(a) *$\tau_{\mathcal{E}} = PO(X)$.*

(b) *If X is submaximal, then $\tau = \tau^\alpha = \tau^*(\mathcal{N}) = \tau_{\mathcal{E}} = PO(X)$.*

(c) *If X is resolvable, then $\tau_{\mathcal{E}}$ is discrete.*

Proof. (a) Let $A \in PO(X)$. Then $A \subset int(cl(A))$ implies that $X - int(cl(A)) \subset X - A$ which implies $cl(int(X-A)) \subset X - A$ and so $\mathcal{E}(X-A) \subset X-A$, by Theorem 2.5. Hence $A \in \tau_{\mathcal{E}}$ and so $\tau_{\mathcal{E}} \supset PO(X)$. Thus, $\tau_{\mathcal{E}} = PO(X)$, by Theorem 3.8.

(b) follows from (a) and Lemma 1.1(b).

(c) follows from (a) and Lemma 1.2. ■

From Example 2.6, we assure that the condition "every element of $\mathcal{E}$ has nonempty interior" is necessary for equality in Theorem 3.9(a). Consider $X = [0,\infty)$, $\tau = \{(a,\infty) \mid a \in X\} \cup \{X, \emptyset\}$ and $\mathcal{E} = \{(a,b) \mid a,b \in X\}$. Since every open subset of X contains many elements of $\mathcal{E}$, $\mathcal{E}$ is a π-network in X and so $\tau_{\mathcal{E}} \subset PO(X,\tau)$, by Theorem 3.8. But $int(E) = \emptyset$ for every $E \in \mathcal{E}$. If $A = [1,\infty)$, then $\mathcal{E}(A) = X$. Since $\mathcal{E}(A) \subset cl(A)$, $int(cl(A)) = X$ and so A is a preopen set in (X,τ). Now $X - A = [0,1)$ and $\mathcal{E}(X-A) = [0,1]$. But $cl_{\mathcal{E}}(X-A) = [0,1] \neq X - A$. Therefore, $A \notin \tau_{\mathcal{E}}$. Hence $\tau_{\mathcal{E}} \neq PO(X,\tau)$.

Given a space (X,τ) and a proper ideal $\mathcal{J}$ on X, we can form a cluster system $\mathcal{E}$ which satisfies the property $\mathcal{H}$ such that $\tau^\star = \tau_{\mathcal{E}}$. For $A \subset X$ and $x \in X$, consider $\mathcal{J}(A,x) = \{B \subset U_x \cap A \mid U_x \cap A \in \mathcal{J}\}$ and $\mathcal{J}' = \bigcup_{A,x} \mathcal{J}(A,x)$, $\mathcal{E}^{\mathcal{J}'} = 2^X - \mathcal{J}'$ and also $\mathcal{E}^{\mathcal{J}^c} = 2^X - \mathcal{J}$.

Lemma 3.10. *Let (X,τ) be any topological space with an ideal $\mathcal{J}$ and $A \subset X$. Then the following hold.*

(a) $\mathcal{E}^{\mathcal{J}'}(A) = A^\star$.

(b) $\mathcal{E}^{\mathcal{J}^c}(A) = A^\star$.

Proof. (a) Let $x \in A^\star$. Then for every $U_x \in \tau(x)$, $U_x \cap A \notin \mathcal{J}$. Now $U_x \cap A \notin \mathcal{J}$ implies $U_x \cap A \in \mathcal{E}^{\mathcal{J}'}$ which implies $x \in \mathcal{E}^{\mathcal{J}'}(A)$ so that $A^\star \subset \mathcal{E}^{\mathcal{J}'}(A)$. Again, $x \notin A^\star$ implies that there exists $U_x \in \tau(x)$ such that $U_x \cap A \in \mathcal{J}$ so that every subset of $U_x \cap A$ is not in $\mathcal{E}^{\mathcal{J}'}$ and so $U_x \cap A \not\supset E$ for every $E \in \mathcal{E}$ which implies that $x \notin \mathcal{E}^{\mathcal{J}'}(A)$. Therefore, $A^\star \supset \mathcal{E}^{\mathcal{J}'}(A)$. Hence $A^\star = \mathcal{E}^{\mathcal{J}'}(A)$.

(b) If $x \in A^\star$, then for every $U_x \in \tau(x)$, $U_x \cap A \notin \mathcal{J}$ and so $U_x \cap A \in \mathcal{E}^{\mathcal{J}^c}$ which implies that $x \in \mathcal{E}^{\mathcal{J}^c}(A)$. Therefore, $A^\star \subset \mathcal{E}^{\mathcal{J}^c}(A)$. Let $x \notin A^\star$. Then there exists $U_x \in \tau(x)$ such that $U_x \cap A \in \mathcal{J}$ so that every subset of $U_x \cap A$ is not in $\mathcal{E}^{\mathcal{J}^c}$ and so $U_x \cap A \not\supset E$ for every $E \in \mathcal{E}$ which implies that $x \notin \mathcal{E}^{\mathcal{J}^c}(A)$. Thus, $A^\star \supset \mathcal{E}^{\mathcal{J}^c}(A)$. Hence $A^\star = \mathcal{E}^{\mathcal{J}^c}(A)$. ■

Lemma 3.11. *Let (X,τ) be any topological space with an ideal $\mathcal{J}$ on X. Then the cluster systems $\mathcal{E}^{\mathcal{J}'}$ and $\mathcal{E}^{\mathcal{J}^c}$ satisfy the property $\mathcal{H}$.*

Proof. Suppose that for every $U \in \tau(x)$, $U \cap (A \cup B) \supset E$ for some $E \in \mathcal{E}^{\mathcal{J}'}$. Then $U \cap (A \cup B) \notin \mathcal{J}$ for every $U \in \tau(x)$ and so $x \in (A \cup B)^\star$. Since $(A \cup B)^\star = A^\star \cup B^\star$, $x \in A^\star$ or $x \in B^\star$. By Lemma 3.10, $x \in \mathcal{E}^{\mathcal{J}'}(A)$ or $x \in \mathcal{E}^{\mathcal{J}'}(B)$. Similar proof can be written for $\mathcal{E}^{\mathcal{J}^c}$. Hence the lemma is proved. ■

Theorem 3.12. *Let (X,τ) be any topological space with an ideal $\mathcal{J}$ on X. Then the three topologies $\tau^\star$, $\tau_{\mathcal{E}_{\mathcal{J}'}}$ and $\tau_{\mathcal{E}_{\mathcal{J}^c}}$ are the same. That is, $\tau^\star = \tau_{\mathcal{E}_{\mathcal{J}'}} = \tau_{\mathcal{E}_{\mathcal{J}^c}}$.*

4. Generalized Volterra spaces

In this section, we characterize $\mathcal{E}'$-Volterra spaces by choosing proper cluster system.

Lemma 4.1. [10, Remark 1 (3)] *Let (X,τ) be a space and $\mathcal{E}$ be any cluster system on X. If A is weakly $\mathcal{E}$-Volterra and $\mathcal{E}(A) \subset \mathcal{E}(A_1)$ for any $A_1 \subset X$, then $A_1 \cap A \neq \emptyset$.*

In Example 4.2 of [10], Matejdes proved that a subset A of X is weakly $\mathcal{E}$-Volterra if and only if A is cofinite. Also, he proved that there is no subset which is $\mathcal{E}$-Volterra. Here we show that weakly $\mathcal{E}$-Volterra need not imply $\mathcal{E}$-Volterra. Note that $\mathcal{E}$ is not a π-network.

Example 4.2. Let $X = \{0, \frac{1}{2}, \frac{1}{3}, \dots\}$ with the usual topology and $\mathcal{E} = \{E : E$ is cofinite $\}$. Then every element of $\mathcal{E}$ does not contain finitely many elements of X. Also, $\mathcal{E}$ is not a π-network in X, since every $\{x\}$, $x \neq 0$, is an open set not containing any element of $\mathcal{E}$. To prove $\mathcal{E}(X) = \{0\}$. Since $\{0\}$ is the only limit point of X, for given $\epsilon > 0$, $(0, 0+\epsilon)$ does not contain finitely many elements of X and so $(0, 0+\epsilon) \in \mathcal{E}$. Therefore, $0 \in \mathcal{E}(X)$. Also, every point other than 0 does not belong to $\mathcal{E}$. If $x \neq 0 \in X$, then $\{x\}$ does not contain any element of $\mathcal{E}$, since $\{x\}$ is open in X and every element of $\mathcal{E}$ is countable. Therefore, $\mathcal{E}(X) = \{0\}$. Let A_1, A_2 be two sets such that $\mathcal{E}(X) \subset \mathcal{E}(A_i)$, i=1,2. Since $\{0\} \subset \mathcal{E}(A_i)$, there exists $E \in \mathcal{E}$ such that $U \cap A_i \supset E$ for every open set $U \in \mathcal{N}(0)$. Therefore, $U \cap A_i$ contains every points of X except the finitely many points. Hence $A_1 \cap A_2 \neq \emptyset$ and so X is weakly $\mathcal{E}$-Volterra. Take $A_1 = X - \{\frac{1}{2}\}$ and $A_2 = X - \{\frac{1}{3}\}$. Then $\mathcal{E}(X) \subset \mathcal{E}(A_i), i = 1, 2$. But $A_1 \cap A_2 = X - \{\frac{1}{2}, \frac{1}{3}\}$ is not dense, since $\{\frac{1}{2}\}$ is an open set in X which does not intersect $A_1 \cap A_2$. Hence X is not $\mathcal{E}$-Volterrra.

In view of Matejdes, we introduce $\mathcal{E}'$-Volterra as follows. A subset A is said to be $\mathcal{E}'$*-Volterra* if for any two sets A_1 and A_2 of X such that $\mathcal{E}(A) \subset \mathcal{E}(A_i), i = 1, 2$ $A \subset \mathcal{E}(A_1 \cap A_2)$. Clearly, every $\mathcal{E}'$-Volterra set is both $\mathcal{E}$-Volterra and weakly $\mathcal{E}$-Volterra. The following Example 4.3 shows that a weakly $\mathcal{E}$-Volterra set need not be a $\mathcal{E}'$-Volterra set even though $\mathcal{E}$ is a π-network.

Example 4.3. Consider $X = (0, \infty)$ with the topology $\tau = \{(a, \infty) : a \in X\} \cup \{X, \emptyset\}$ and the cluster system $\mathcal{E} = \{(n, n + 2.5) : n \in \mathbb{N}\}$. Clearly, $\mathcal{E}$ is a π-network on X and hence every open set of X. If $G = (2, \infty)$, then G is $\mathcal{E}$-Volterra and hence weakly $\mathcal{E}$-Volterra. Also, $\mathcal{E}(G) = X$. Let $A = X - \{2i : i$ is an odd natural number$\}$ and $B = X - \{2i : i$ is an even natural number$\}$. Then $\mathcal{E}(A) = X$ and $\mathcal{E}(B) = X$. By construction, $A \cap B \neq \emptyset$. Also, $A \cap B$ is dense in X and hence in G. But $\mathcal{E}(A \cap B) = \emptyset$ and so G is not $\mathcal{E}'$-Volterra.

The proof of the following Theorem 4.4 follows from Lemma 1.6 and the fact that every $\mathcal{E}'$-Volterra space is $\mathcal{E}$-Volterra. The converse of Theorem 4.4 need not be true. In Example 4.3, it is clear that every open subset of X is weakly $\mathcal{E}$-Volterra but X is not $\mathcal{E}'$-Volterra. Theorem 4.5 below shows that the converse of Theorem 4.4 holds if $\mathcal{E}$ satisfies the property $\mathcal{I}$. Since every $\mathcal{E}$-Volterra set is weakly $\mathcal{E}$-Volterra, Theorem 4.6 follows from Theorem 4.5. Also, Example 4.3 shows that the property $\mathcal{I}$ is necessary in Theorem 4.5 and Theorem 4.6.

Theorem 4.4. *Let (X,τ) be a space and $\mathcal{E}$ be a π-network in a nonempty open set X_0 of X. If X_0 is $\mathcal{E}'$-Volterra, then any nonempty open subset of X_0 is weakly $\mathcal{E}$-Volterra.*

Theorem 4.5. *Let (X,τ) be a space and $\mathcal{E}$ be a π-network in a nonempty open set X_0 with the property $\mathcal{I}$. If any nonempty open subset of X_0 is weakly $\mathcal{E}$-Volterra, then X_0 is $\mathcal{E}'$-Volterra.*

Proof. Since X_0 itself an open subset of X_0, X_0 is weakly $\mathcal{E}$-Volterra. Let A_1 and A_2 be two sets such that $\mathcal{E}(X_0) \subset \mathcal{E}(A_i), i = 1,2$. By Theorem 2.10, $\mathcal{E}(A_1 \cap A_2) = \mathcal{E}(A_1) \cap \mathcal{E}(A_2)$. Therefore, $\mathcal{E}(X_0) \subset \mathcal{E}(A_i), i = 1,2$ implies that $\mathcal{E}(X_0) \subset \mathcal{E}(A_1) \cap \mathcal{E}(A_2) = \mathcal{E}(A_1) \cap \mathcal{E}(A_2)$ which implies $X_0 \subset \mathcal{E}(A_1 \cap A_2)$, since $\mathcal{E}$ is a π-network in X_0. Hence X_0 is $\mathcal{E}'$-Volterra. ■

Theorem 4.6. *Let (X,τ) be a space and $\mathcal{E}$ be a π-network in a nonempty open set X_0 with the property $\mathcal{I}$. Then X_0 is $\mathcal{E}$-Volterra if and only if X_0 is $\mathcal{E}'$-Volterra.*

Theorem 4.7. *Let (X,τ) be a submaximal space. Then for every π-network $\mathcal{E}$ in X, every nonempty open subset of X is $\mathcal{E}'$-Volterra and hence $\mathcal{E}$-Volterra. In particular, X is $\mathcal{E}'$-Volterra, hence $\mathcal{E}$-Volterra.*

Theorem 4.8. *Let (X,τ) be a space. If $\mathcal{E} \in \mathcal{E}_\pi$, then every nonempty open subset of X is $\mathcal{E}'$-Volterra, hence $\mathcal{E}$-Volterra. In particular, X is also $\mathcal{E}'$-Volterra.*

Proof. Let G be any nonempty open subset of X and A,B be two subsets of X such that $\mathcal{E}(G) \subset \mathcal{E}(A)$ and $\mathcal{E}(G) \subset \mathcal{E}(B)$. By Theorem 2.5, $G \subset cl(G) \subset cl(int(A))$ and $G \subset cl(G) \subset cl(int(B))$. Let $x \in cl(G)$ and U_x be any open set in G. Now $x \in U_x \subset cl(int(A))$ implies that $U_x \cap int(A) \neq \emptyset$. Therefore, there exists some $y \in U_x \cap int(A)$. Since $cl(G) \subset cl(int(B))$, $y \in U_x \cap int(A) \subset cl(G) \subset cl(int(B))$ implies that $U_x \cap int(A) \cap int(B) \neq \emptyset$ implies that $U_x \cap int(A \cap B) \neq \emptyset$. Therefore, $x \in cl(int(A \cap B))$. Hence $\mathcal{E}(G) \subset \mathcal{E}(A \cap B)$. Therefore, G is $\mathcal{E}'$-Volterra. ■

Here we partially answer the question that if (X,τ) is $\mathcal{E}'$-Volterra whether the new space $(X,\tau_{\mathcal{E}})$ is $\mathcal{E}'$-Volterra. The proof of Theorem 4.9 follows from Theorem 4.7 and Theorem 4.8.

Theorem 4.9. *Let (X,τ) be a submaximal space and $\mathcal{E} \in \mathcal{E}_\pi$. Then every open subset of $(X,\tau_{\mathcal{E}})$ is $\mathcal{E}'$-Volterra, and hence $\mathcal{E}$-Volterra. In particular, $(X,\tau_{\mathcal{E}})$ is $\mathcal{E}'$-Volterra, hence $\mathcal{E}$-Volterra.*

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