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A NOTE ON I-CONVERGENCE AND $I^\star$-CONVERGENCE OF SEQUENCES AND NETS IN TOPOLOGICAL SPACES

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Abstract. In this paper, we use the idea of I-convergence and $I^\star$-convergence of sequences and nets in a topological space to study some important topological properties. Further we derive characterization of compactness in terms of these concepts. We introduce also the idea of I-sequentially compactness and derive a few basic properties in a topological space.

1. Introduction

The concept of convergence of a sequence of real numbers was extended to statistical convergence independently by H. Fast [3] and I.J. Schoenberg [15] as follows:

If K is a subset of the set of all natural numbers $\mathbb{N}$ then natural density of the set K is defined by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exits [4,13] where $|K_n|$ stands for the cardinality of the set $K_n = \{k \in K : k \le n\}$.

A sequence $\{x_n\}$ of real numbers is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$ the set

$$K(\varepsilon) = \{k \in \mathbb{N} : |x_k - \ell| \ge \varepsilon\}$$

has natural density zero [3,15].

This idea of statistical convergence of real sequence was generalized to the idea of I-convergence of real sequences [6,7] using the notion of ideal I of subsets of the set of natural numbers. Several works on I-convergence and on statistical convergence have been done in [1,2,6,7,9,12].

The idea of I-convergence of real sequences coincides with the idea of ordinary convergence if I is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if I is the ideal of subsets of $\mathbb{N}$ of natural density zero. The concept of $I^\star$-convergence is closely related to that of I-convergence and this notion arises

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from an equivalent characterization of statistical convergence of real sequence by T. Šalát [14]. Later B.K. Lahiri and P. Das [10] extended the idea of I-convergence and $I^\star$-convergence to an arbitrary topological space and observed that the basic properties are preserved also in a topological space. They also introduced [11] the idea of I-convergence and $I^\star$-convergence of nets in a topological space and examined how far it affects the basic properties.

In this paper, we have studied further some important properties of I-convergence and $I^\star$-convergence of sequences and nets in a topological space which were not studied before and examined some further consequences in a topological space like characterization of compactness in terms of I-cluster points etc. Also, we have introduced the notion of I-sequential compactness and have found out its relation with the countable compactness in a topological space.

2. *I*-convergence and *I**-convergence of sequences in topological spaces

We recall the following definitions.

Definition 2.1. [8] If X is a non-void set then a family of sets $I \subset 2^X$ is called an *ideal* if

(i) $A, B \in I$ implies $A \cup B \in I$ and

(ii) $A \in I, B \subset A$ imply $B \in I$.

The ideal is called *nontrivial* if $I \neq \{\emptyset\}$ and $X \notin I$.

Definition 2.2. [8] A nonempty family F of subsets of a non-void set X is called a *filter* if

(i) $\emptyset \notin F$

(ii) $A, B \in F$ implies $A \cap B \in F$ and

(iii) $A \in F$, $A \subset B$ imply $B \in F$.

If I is a nontrivial ideal on X then $F = F(I) = \{A \subset X : X \setminus A \in I\}$ is clearly a filter on X and conversely.

A nontrivial ideal I is called *admissible* if it contains all the singleton sets. Several examples of nontrivial admissible ideals may be seen in [6].

Let (X, τ) be a topological space and I be a nontrivial ideal of $\mathbb{N}$, the set of all natural numbers.

Definition 2.3. [10] A sequence $\{x_n\}$ in X is said to be I-convergent to $x_0 \in X$ if for any nonempty open set U containing x_0, $\{n \in \mathbb{N} : x_n \notin U\} \in I$.

In this case, x_0 is called an I-limit of $\{x_n\}$ and written as x_0=I-lim x_n.

Note. If I is an admissible ideal then ordinary convergence implies I-convergence and if I does not contain any infinite set then converse is also true.

The following properties of convergence in a topological space have been verified in [10] to be valid in case of I-convergence.

Theorem 2.1. [10] *If X is a Hausdorff space then an I-convergent sequence has a unique I-limit.*

Theorem 2.2. [10] *A continuous function $f : X \to X$ preserves I-convergence. Again if I is an admissible ideal and X is a first axiom T_1 space then continuity of $f : X \to X$ is necessary to preserve I-convergence.*

Definition 2.4. [10] A sequence $\{x_n\}$ in a topological space (X, τ) is said to be $I^\star$-convergent to $x \in X$ if and only if there exists a set $M \in F(I)(i.e., \mathbb{N} \backslash M \in I)$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $\lim_{k\to\infty} x_{m_k} = x$.

In this case we write $I^\star$-$\lim x_n = x$ and x is called an $I^\star$-limit of $\{x_n\}$.

It has been proved in [10] that if I is an admissible ideal then $I^\star$-$\lim x_n = x$ implies I-$\lim x_n = x$ and so in addition if X is a Hausdorff space then $I^\star$-$\lim x_n$ is unique. Conversely if X has no limit point (i.e, X is a discrete space) then I-$\lim x_n = x$ implies $I^\star$-$\lim x_n = x$ for every admissible ideal I.

Definition 2.5. [10] Let $x = \{x_n\}$ be a sequence of elements of a topological space (X, τ). Then

(i) $y \in X$ is called an I-limit point of x if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M \notin I$ and $\lim_{k\to\infty} x_{m_k}=y$.

(ii) $y \in X$ is called an I-cluster point of x if for every open set U containing y, $\{n \in \mathbb{N} : x_n \in U\} \notin I$.

In [10], it has been proved that if I is an admissible ideal then
(a) $I(L_x) \subset I(C_x)$ and
(b) $I(C_x)$ is a closed set in X
where $I(L_x)$ and $I(C_x)$ denote respectively the set of all I-limit points and set of all I-cluster points of x.

We now prove two important results in a topological space which were not studied in [10]. Let I be a nontrivial ideal of the set $\mathbb{N}$ of natural numbers consisting of all finite subsets of $\mathbb{N}$ and (X, τ) be a topological space.

Theorem 2.3. *Every sequence $\{x_n\}$ has an I-cluster point if and only if every infinite set in X has an ω-accumulation point.*

Proof. Suppose that every sequence in (X, τ) has an I-cluster point and let A be an infinite subset of the space X. Then there is a sequence $\{x_n\}$ (say) of distinct points in A. Let y be an I-cluster point of $\{x_n\}$. Then for any open set V containing y we have $\{n \in \mathbb{N} : x_n \in V\} \notin I$. Hence the set $\{n \in \mathbb{N} : x_n \in V\}$ must be an infinite set. Consequently V contains infinitely many points of the sequence $\{x_n\}$, i.e., V contains infinitely many elements of A. Thus by definition y becomes an ω-accumulation point of A.

Conversely, let every infinite subset of the space X has an ω-accumulation point. Let $\{x_n\}$ be a sequence of points in X. If the range of the sequence is infinite then let y be an ω-accumulation point of $\{x_n\}$. So for each open set V containing y,

$\{n \in \mathbb{N} : x_n \in V\}$ is an infinite set and so $\{n \in \mathbb{N} : x_n \in V\} \notin I$. Hence y becomes an I-cluster point of $\{x_n\}$. Otherwise let for some point y of the space X we have $x_n = y$ for infinitely many positive integers n. So for every open set V containing y we get $\{n \in \mathbb{N} : x_n \in V\}$ is an infinite subset of $\mathbb{N}$ and so $\{n \in \mathbb{N} : x_n \in V\} \notin I$. Thus y becomes an I-cluster point of $\{x_n\}$. ∎

Throughout, I will stand for a nontrivial admissible ideal of $\mathbb{N}$ and (X, τ) stands for a topological space unless otherwise stated. Below we obtain a sufficient condition for a Lindelöf space to be compact.

THEOREM 2.4. *If (X, τ) is a Lindelöf space such that every sequence in X has an I-cluster point then (X, τ) is compact.*

Proof. Let (X, τ) be a Lindelöf space such that every sequence in X has an I-cluster point. We have to show that any open cover of the space X has a finite subcover. Let $\{A_\alpha : \alpha \in \Lambda\}$ be an open cover of the space X, where Λ is an index set. Since (X, τ) is a Lindelöf space so this open cover admits a countable subcover say $\{A_1, A_2, \ldots, A_n, \ldots\}$. Proceeding inductively let $B_1 = A_1$ and for each $m > 1$ let B_m be the first member of the sequence of A's which is not covered by $B_1 \cup B_2 \cup \cdots \cup B_{m-1}$. If this choice becomes impossible at any stage then the sets already selected becomes a required finite subcover. Otherwise it is possible to select a point b_n in B_n for each positive integer n such that $b_n \notin B_r$, for $r < n$. Let x be an I-cluster point of the sequence $\{b_n\}$. Then $x \in B_p$ for some p. Now we have by definition of I-cluster point that the set $M = \{n \in \mathbb{N} : b_n \in B_p\} \notin I$. Hence M must be an infinite subset of $\mathbb{N}$, since I is an admissible ideal of $\mathbb{N}$. So there is some $q > p$ such that $q \in M$ i.e., there exists some $q > p$ such that $b_q \in B_p$ which leads to a contradiction. Thus the result follows. ∎

We now recall the definition of I-convergence of a real sequence which will be needed in the next section.

DEFINITION 2.6. [1] A real sequence $\{x_n\}$ is said to converge to x with respect to an ideal I of the set of natural numbers $\mathbb{N}$ (or I-convergent to x) if for any $\varepsilon > 0$, $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in I$.

In this case we write $I - \lim_{n\to\infty} x_n = x$.

3. *I*-convergence and $I^\star$-convergence of nets in topological spaces

The following two definitions are widely known.

DEFINITION 3.1. [5] Let D be a non-void set and '$\geq$' be a binary relation on D such that '$\geq$' is reflexive, transitive and for any two elements $m, n \in D$ there is an element $p \in D$ such that $p \geq m$ and $p \geq n$. The pair $(D, \geq)$ is called a *directed set.*

DEFINITION 3.2. [5] Let $(D, \geq)$ be a directed set and let X be a nonempty set. A mapping $s : D \to X$ is called a *net* in X, denoted by $\{s_n; n \in D\}$ or simply by $\{s_n\}$ when the set D is clear from the context.

Throughout our discussion (X, τ) will denote a topological space (which will be written sometimes as X) and I will denote a non-trivial ideal of a directed set D. Also the symbol $\mathbb{N}$ is reserved for the set of all natural numbers. For $n \in D$ let $D_n = \{k \in D : k \geq n\}$. Then the collection $F_0 = \{A \subset D : A \supset D_n,$ for some $n \in D\}$ forms a filter in D. Let $I_0 = \{B \subset D : D \setminus B \in F_0\}$. Then I_0 is also a non-trivial ideal in D.

Definition 3.3. [11] A non-trivial ideal I of D will be called *D-admissible* if $D_n \in F(I)$ for all $n \in D$.

We are reproducing below the definition of I-convergence of a net where I is an ideal of D.

Definition 3.4. [11] A net $\{s_n; n \in D\}$ in X is said to be *I-convergent* to $x_0 \in X$ if for any open set U containing x_0, $\{n \in D : s_n \notin U\} \in I$.

Symbolically we write $I\text{-}\lim s_n = x_0$ and we say that x_0 is an I-limit of the net $\{s_n\}$.

Note. If I is D-admissible, then convergence of a net in a topological space implies I-convergence and the converse holds if $I = I_0$. Also if $D = \mathbb{N}$ with the natural ordering then the concepts of D-admissibility and admissibility coincide and in that case I_0 is the ideal of all finite subsets of $\mathbb{N}$.

We recall the following definition of an I-cluster point of a net $\{s_n; n \in D\}$ in a topological space (X, τ).

Definition 3.5. [11] $y \in X$ is called an *I-cluster point* of a net $\{s_n; n \in D\}$ if for every open set U containing y, $\{n \in D : s_n \in U\} \notin I$.

The following result holds in case of I-convergence in a topological space which is true for ordinary convergence of net also.

Theorem 3.1. *For every net $\{s_n; n \in D\}$ in X there is a filter F on X such that x is an I-limit of the net $\{s_n; n \in D\}$ if and only if x is the limit of the filter F and, y is an I-cluster point of the net $\{s_n; n \in D\}$ if and only if y is the cluster point of the filter F.*

Proof. Let $\{s_n; n \in D\}$ be a net in the space X. Let I be a non-trivial ideal of D and $F(I)$ be the associated filter on D. Let us construct for each $M \in F(I)$ the set $A_M = \{s_n : n \in M\}$. Then the family $B = \{A_M : M \in F(I)\}$ forms a filter base on X. Indeed, each A_M is non-empty, since each M is non-empty and if $A_M, A_R \in B$ where $M, R \in F(I)$ then $A_{M \cap R} \subset A_M \cap A_R$ where $M \cap R \in F(I)$, since $F(I)$ is a filter. Thus our conclusion is valid. Let F be the filter generated by this filter base B. Now we show that F has the required property.

Let the net $\{s_n; n \in D\}$ be I-convergent to x. Then for any neighbourhood V of x we have $\{n \in D : s_n \notin V\} \in I$. This implies that $\{n \in D : s_n \in V\} \in F(I)$. We write $M = \{n \in D : s_n \in V\}$. Then by our construction $A_M = \{s_n : n \in M\} \subset V$.

Since $A_M \in F$ we get $V \in F$ and since V is an arbitrary neighbourhood of x, we conclude that $V \in F$ for all neighbourhood V of x. Hence the filter F is convergent to x.

Again let the filter F be convergent to x. Then the neighbourhood filter η_x of the point x is a subfamily of F i.e., $\eta_x \subset F$. Let $V \in \eta_x$ be arbitrary. Then $A_M \subset V$ for some $M \in F(I)$. This implies that $M \subset \{n \in D : s_n \in V\}$ which further implies that $\{n \in D : s_n \in V\} \in F(I)$ i.e., $\{n \in D : s_n \notin V\} \in I$. This shows that the net $\{s_n; n \in D\}$ is also I-convergent to x.

Now we suppose that y is an I-cluster point of the net $\{s_n; n \in D\}$. Then for any neighbourhood V of y we have $\{n \in D : s_n \in V\} \notin I$ i.e., $\{n \in D : s_n \notin V\} \notin F(I)$. Hence we conclude that the set $\{n \in D : s_n \notin V\}$ contains no M for any $M \in F(I)$. So for every $M \in F(I)$ there exists some $m \in M$ such that $m \notin \{n \in D : s_n \notin V\}$ i.e., there exists $m \in M$ for each $M \in F(I)$ such that $s_m \in V$. Thus we get $V \cap A_M \neq \emptyset$ for all $M \in F(I)$ so that y becomes a cluster point of the filter F.

Next let y be a cluster point of the filter F. Then for any neighbourhood V of y we have $V \cap A_M \neq \emptyset$ for all $M \in F(I)$ i.e., $\{n \in D : s_n \in V\} \cap M \neq \emptyset$ for all $M \in F(I)$. We conclude that $\{n \in D : s_n \in V\} \notin I$. For if $\{n \in D : s_n \in V\} \in I$ then this it would imply that $\{n \in D : s_n \notin V\} \in F(I)$. So, if we write $E = \{n \in D : s_n \notin V\}$ then $V \cap A_E = \emptyset$ and this leads to a contradiction. Hence $\{n \in D : s_n \in V\} \notin I$ so that y becomes an I-cluster point of the net $\{s_n; n \in D\}$. ■

We know that a topological space is compact if and only if each family of closed sets which has the finite intersection property [FIP for short] has a non-void intersection. We now prove a very important result regarding compactness of a topological space.

THEOREM 3.2. *In a compact topological space (X, τ) each net $\{s_n; n \in D\}$ has an I-cluster point corresponding to any non-trivial ideal I of D.*

Proof. Let (X, τ) be a compact topological space and $\{s_n; n \in D\}$ be a net in X. Let I be a non-trivial ideal of D and $F(I)$ be the filter on D associated with the ideal I. For each $M \in F(I)$ consider the set $A_M = \{s_n : n \in M\}$. Then the family containing all such A_M has FIP, since $F(I)$ is a filter. Hence the family $B = \{\overline{A_M} : M \in F(I)\}$ is a family of closed sets possessing FIP. Since X is a compact space, $\cap\{\overline{A_M} : M \in F(I)\} \neq \emptyset$. So there is some $x_0 \in X$ such that $x_0 \in \cap\{\overline{A_M} : M \in F(I)\}$. Then for every neighbourhood V of x_0 we have $V \cap A_M \neq \emptyset$. Now we consider the set $K = \{n \in D : s_n \notin V\}$. If $K \in F(I)$ then the corresponding set $A_K = \{s_n : n \in K\}$ does not intersect V i.e., $A_K \cap V = \emptyset$ which contradicts the fact deduced above. Hence, $K \notin F(I)$ which implies that $\{n \in D : s_n \in V\} \notin I$. Thus, x_0 becomes an I-cluster point of the net $\{s_n; n \in D\}$. ■

A sort of converse of the above theorem is given below.

THEOREM 3.3. *A topological space is compact if every net $\{s_n; n \in D\}$ has an I-cluster point corresponding to a D-admissible ideal I.*

The proof is omitted.

Here we show that I-convergence of a net in a product topological space can be described in terms of the projections.

THEOREM 3.4 *Let $\{X_a : a \in \mathcal{A}\}$ be a family of topological spaces where $\mathcal{A}$ is any indexing set. A net $\{s_n; n \in D\}$ in a product space $X = \times\{X_a : a \in \mathcal{A}\}$ is I-convergent to a point x if and only if the net $\{P_a(s_n) : n \in D\}$ is I-convergent to x_a where $P_a : X \to X_a$ is the a-th projection mapping and $P_a(x) = x_a$ and where I is a non-trivial ideal of the domain D of the net.*

Proof. We know that projection map into each co-ordinate space is continuous. Let x be a point of the product space $\times\{X_a : a \in \mathcal{A}\}$ and P_a be the a-th projection map into the factor space X_a for some $a \in \mathcal{A}$. Let $\{s_n; n \in D\}$ be a net in the product space $\times\{X_a : a \in \mathcal{A}\}$ which is I-convergent to the point x in the product space where I is a non-trivial ideal of the domain D of the net. Let V_a be any open set in X_a containing $P_a(x) = x_a$. Then by continuity of P_a there is some open set V containing x such that $P_a(V) \subset V_a$. So the set $\{n \in D : s_n \notin V\} \in I$. Now since $\{n \in D : P_a(s_n) \notin V_a\} \subset \{n \in D : s_n \notin V\}$, we have $\{n \in D : P_a(s_n) \notin V_a\} \in I$. Since V_a is an arbitrary open set containing $P_a(x) = x_a$ we conclude the first part.

For the converse part let $\{s_n; n \in D\}$ be a net in the product space such that $\{P_a(s_n) : n \in D\}$ is I-convergent to $x_a \in X_a$ for each a in $\mathcal{A}$. Let us write $x =< x_a : a \in \mathcal{A} >$. We shall show that $\{s_n; n \in D\}$ is I-convergent to the point x in the product space. Now for each open set V_a in X_a containing x_a we have $\{n \in D : P_a(s_n) \notin V_a\} \in I$ i.e., $\left\{n \in D : s_n \notin P_a^{-1}(V_a)\right\} \in I$. This in turn implies that $\left\{n \in D : s_n \in P_a^{-1}(V_a)\right\} \in F(I)$ where $F(I)$ is the filter on D associated with the ideal I. Hence if Λ be any finite subfamily of the indexing set $\mathcal{A}$ we have $\bigcap_{a\in\Lambda}\left\{n \in D : s_n \in P_a^{-1}(V_a)\right\} \in F(I)$ i.e., $\left\{n \in D : s_n \in \bigcap_{a\in\Lambda} P_a^{-1}(V_a)\right\} \in F(I)$. Again this implies $\left\{n \in D : s_n \notin \bigcap_{a\in\Lambda} P_a^{-1}(V_a)\right\} \in I$. Since the family of such finite intersections is a base for the neighbourhood system of the point x in the product topology so the net $\{s_n; n \in D\}$ is I-convergent to x in the product space. ■

4. Countable compactness and *I*-sequential compactness of a topological space

We now introduce the following definition.

DEFINITION 4.1. A topological space (X, τ) is said to be I-sequentially compact if every sequence in X has an I-cluster point, where I is a non-trivial ideal of the set $\mathbb{N}$ of all positive integers.

The notions of I-sequential compactness and sequential compactness of a topological space are different as shown in the following two examples.

EXAMPLE 4.1. In this example we show that a sequence in a topological space has a cluster point without having an I-cluster point corresponding to a non-trivial ideal I of $\mathbb{N}$, the set of all positive integers.

Let I be a non-trivial ideal of $\mathbb{N}$ generated by all subsets of the set of all even positive integers and all finite subsets of the set of all odd positive integers. Let us consider the topological space $(\mathbb{R}, \tau)$, the set of all real numbers $\mathbb{R}$ endowed with the usual topology τ and a sequence $\{x_n\}$ in $\mathbb{R}$, where

$$x_n = \begin{cases} 0 & \text{if n is even} \\ n+1 & \text{if n is odd.} \end{cases}$$

Then clearly $\{x_n\}$ has a convergent subsequence. But $\{x_n\}$ has no I-cluster point.

EXAMPLE 4.2. This example demonstrates to us that there is a sequence in a topological space which has an I-cluster point corresponding to a non-trivial ideal I of the set $\mathbb{N}$ but has no cluster point.

Let I be a non-trivial ideal of $\mathbb{N}$ containing all subsets of the set of all even positive integers. Let us consider the topological space $(\mathbb{R}, \tau)$, the set of all real numbers $\mathbb{R}$ endowed with the usual topology τ and a sequence $\{x_n\}$ in $\mathbb{R}$ where $x_n = n$, for all $n \in \mathbb{N}$. Now clearly $\{x_n\}$ has no cluster point in $\mathbb{R}$ but every odd positive integer becomes an I-cluster point of the sequence $\{x_n\}$.

We show below that under certain condition there is some relation between countable compactness and I-sequential compactness of a topological space.

Now we recall the following result.

LEMMA. *For a topological space (X, τ) the following are equivalent.*

(a) *(X, τ) is countably compact.*

(b) *For every countable collection of closed subsets of X satisfying the finite intersection property has non-empty intersection.*

(c) *If $F_1 \supset F_2 \supset F_3 \supset \cdots \supset F_n \supset \cdots$ is a descending family of non-empty closed subsets of X then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

Let I be an admissible ideal of the set $\mathbb{N}$.

THEOREM 4.1. *If (X, τ) is I-sequentially compact then (X, τ) becomes a countably compact space.*

Proof. Suppose (X, τ) is an I-sequentially compact space. Let $\{V_n\}_{n=1}^{\infty}$ be a countable open cover of X which has no finite subcover. Then we may pick $x_n \in X - \bigcup_{i=1}^{n} V_i$. Now the sequence $\{x_n\}$ must have an I-cluster point say $x_0 \in X$. Let $x_0 \in V_r$ for some $r \in \mathbb{N}$. Then by definition $\{n \in \mathbb{N} : x_n \in V_r\} \notin I$. Since I is an admissible ideal of $\mathbb{N}$ so the set $A = \{n \in \mathbb{N} : x_n \in V_r\}$ must be an infinite subset of $\mathbb{N}$. Hence there is some $m > r$ such that $x_m \in V_r$. But by our construction $x_m \notin V_r$ and so we arrive at a contradiction. Thus (X, τ) must be countably compact. ∎

THEOREM 4.2. *If (X, τ) is a first countable countably compact space then (X, τ) becomes I-sequentially compact.*

Proof. Suppose (X, τ) is a first countable countably compact space. Let $\{x_n : n \in \mathbb{N}\}$ be a sequence of distinct points of X. Let us take $T_n = \{x_m : m \geq n\}$ for each positive integer n. Then $\{\overline{T_n}\}$ is a descending sequence of non-empty closed sets and hence by above lemma $\bigcap_{n=1}^{\infty}\overline{T_n}\neq\emptyset$. Let $x_0\in\bigcap_{n=1}^{\infty}\overline{T_n}$. Since (X,τ) is a first countable space, suppose that $\{B_n(x_0)\}_{n=1}^{\infty}$ is a countable local base at the point $x_0\in X$ such that $B_n\supset B_{n+1}$ for all $n\in\mathbb{N}$. Now $B_m(x_0)\cap T_m\neq\emptyset$. So there exists some $k_m\geq m$ such that $x_{k_m}\in B_m(x_0)$. Since $B_1(x_0)\cap T_1\neq\emptyset$, we choose a positive integer k_1 such that $x_{k_1}\in B_1(x_0)$. Again since $B_2(x_0)\cap T_{k_1}\neq\emptyset$, choose a positive integer $k_2>k_1$ such that $x_{k_2}\in B_2(x_0)$. Suppose $k_1<k_2<\cdots<k_n$ have been chosen such that $x_{k_i}\in B_i(x_0)$ for $i=1,2,\ldots,n$. Again since $B_{n+1}(x_0)\cap T_{k_{n+1}}\neq\emptyset$, there is some $k_{n+1}>k_n$ such that $x_{k_{n+1}}\in B_{n+1}(x_0)$. Thus we get a subsequence $\{x_{k_n}\}_{n=1}^{\infty}$ of the sequence $\{x_n\}$ such that $x_{k_r}\in B_r(x_0),\forall r\in\mathbb{N}$. We show that this subsequence converges to x_0. Let $x_0\in V$ where V is an open subset of X. Then there exists some positive integer m such that $B_m(x_0)\subset V$. Then for all $n>m$ we have $x_{k_n}\in B_n(x_0)\subset B_m(x_0)\subset V$. Since I is an admissible ideal of $\mathbb{N}$, the sequence $\{x_{k_n}\}$ is I-convergent to x_0. This implies that for every open set U containing x_0 we have $\{n\in\mathbb{N}:x_{k_n}\notin U\}\in I$. Since I is a non-trivial ideal, $\{n\in\mathbb{N}:x_{k_n}\in U\}\notin I$ i.e., x_0 becomes an I-cluster point of the sequence $\{x_{k_n}\}$. Now since $\{n\in\mathbb{N}:x_n\in U\}\supset\{n\in\mathbb{N}:x_{k_n}\in U\}$ so we obtain $\{n\in\mathbb{N}:x_n\in U\}\notin I$, which in turn implies that x_0 becomes an I-cluster point of the sequence $\{x_n\}$. Thus (X,τ) is an I-sequentially compact space. ■

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