МАТЕМАТИЧКИ ВЕСНИК
67, 4 (2015), 269–276
December 2015

оригинални научни рад
research paper

# PARTIAL ISOMETRIES AND NORM EQUALITIES FOR OPERATORS

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**Abstract.** Let $H$ be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on $H$. In this paper we shall show that if $A \in B(H)$ is a nonzero closed range operator, then the injective norm $\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda$ attains its minimal value 2 if and only if $A/\|A\|$ is a partial isometry. Also we shall give some characterizations of partial isometries and normal partial isometries in terms of norm equalities for operators. These characterizations extend previous ones obtained by A. Seddik in [On the injective norm and characterization of some subclasses of normal operators by inequalities or equalities, J. Math. Anal. Appl. 351 (2009), 277–284], and by M. Khosravi in [A characterization of the class of partial isometries, Linear Algebra Appl. 437 (2012)].

## 1. Introduction and preliminary results

Let $H$ be a complex Hilbert space and $B(H)$ be the space of all bounded linear operators on $H$. We denote by $\mathfrak{F}_1(H)$ the class of all rank one operators in $B(H)$. For an operator $A \in B(H)$, we write $A^*$ for its adjoint, $|A| = (A^*A)^{\frac{1}{2}}$ for its modulus, $R(A)$ for its range, and $N(A)$ for its kernel. An operator $A \in B(H)$ is said to be an orthogonal projection if $A^2 = A = A^*$, normal if $AA^* = A^*A$. The Moore-Penrose inverse of $A \in B(H)$, denoted by $A^+$, is the unique solution to the equations

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad AA^+ = \left(AA^+\right)^*, \quad A^+A = \left(A^+A\right)^*.$$

Notice that $A^+$ exists if and only if $R(A)$ is closed [4]. In this case $AA^+$ and $A^+A$ are the orthogonal projections onto $R(A)$ and $R(A^*)$, respectively and $R(A^+) = R(A^*)$. An operator $A \in B(H)$ is EP if and only if $AA^+ = A^+A$; EP stands for "equal projection", as in this case $R(A^*) = R(A)$. It is well known that if $A$ is a normal operator with closed range then $A$ is EP. The converse is not true even in a finite dimensional space.

Recall that an operator $A \in B(H)$ is said to be a partial isometry provided that $\|Ax\| = \|x\|$ for every $x \in N(A)^\perp$, that is, $A^*$ is the Moore-Penrose inverse of

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*2010 Mathematics Subject Classification*: 47A30, 47A05, 47B15.

*Keywords and phrases*: Closed range operator; Moore-Penrose inverse; injective norm; partial isometry; normal operator; EP operator; operator equality.

$A$ (i.e. $AA^*A = A$); see [8], [9] and Chapter 13 of [3] for details. Then it is easy to see that if $A$ is a partial isometry, then $A$ is normal if and only if $A$ is EP.

Let $A_1, \ldots, A_n$, $B_2, \ldots, B_n \in B(H)$. Then the injective norm of $\sum_{i=1}^{n} A_i \otimes B_i$ in the tensor product space $B(H) \otimes B(H)$ is denied as

$$\| \sum_{i=1}^{n} A_i \otimes B_i \|_\lambda = \sup | \sum_{i=1}^{n} f(A_i) g(B_i)|,$$

where the supremum is taken over all bounded functionals $f, g$ on $B(H)$ with $\|f\| = \|g\| = 1$.

It was proved in [7] that

$$\| \sum_{i=1}^{n} A_i \otimes B_i \|_\lambda = \sup \{ \| \sum_{i=1}^{n} A_i X B_i \| : X \in B(H), \|X\| = 1 = \operatorname{rank} X \}.$$

In [12], A. Seddik proved that $\|A^* \otimes A^{-1} + A^{-1} \otimes A^*\|_\lambda \geq 2$ holds for every invertible operator $A \in B\,(H)$, and this last inequality becomes an equality if and only if $A$ is a unitary operator multiplied by a nonzero scalar. More generally, M. Khosravi [6], proved that if $A$ is a nonzero closed range operator with $R(A) = R(A^*)$ (i.e, $A$ is EP), then $\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda = 2$ if and only if $A$ is a nonzero real scalar of a normal partial isometry. In this paper, we shall show that if $A \in B\,(H)$ is with closed range, then $\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda \geq 2$, and if $\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda$ gets its minimal value 2, then $\frac{A}{\|A\|}$ is a partial isometry. Also, we shall give new characterizations of partial isometries, normal partial isometries and closed range normal operators in $B(H)$ by equalities or inequalities. These characterizations extend previous ones obtained in [6], [12] and [13].

For this purpose, we need to start with the following known results.

For $T \in B\,(H)$, the reduced minimum modulus is defined by

$$\gamma(T) = inf\{\|Tx\| : \operatorname{dist}(x, N(T)) = 1\} \quad (\gamma(T) = +\infty \;\; if \;\; T = 0).$$

It is well known [2], that $\gamma(T) > 0$ if and only if $R(T)$ is closed and in this case $\gamma(T) = \frac{1}{\|T^+\|}$.

In [10], Mbekhta has given the following characterization of partial isometries in Hilbert spaces.

Lemma 1.1. [11] *Let $T$ be a bounded operator on $H$. Then $T$ is a nonzero partial isometry if and only if $\gamma(T) = \|T\| = 1$.*

Lemma 1.2. *Let $A$ and $B$ be two operators in $B\,(H)$ and let the two-sided multiplication $M_{A,B} : B(H) \to B(H)$ be defined by $M_{A,B}(X) = AXB$. Then $\|M_{A,B}\| = \sup_{\|X\|=1=\operatorname{rank} X} \|AXB\| = \|A\|\|B\|$.*

## 2. Main results

In the following proposition, we give some preliminary characterizations of partial isometries.

PROPOSITION 2.1. *Let $A \in B(H)$ be a nonzero operator with closed range. Then the following statements are equivalent:*

(i) $\dfrac{A}{\|A\|}$ *is a partial isometry,*

(ii) $\|A\|\|A^+\| = 1$,

(iii) $\forall X \in B(H)$, $\|AXA^+\| = \|A^+AXA^+A\|$,

(iv) $\forall X \in B(H)$, $\|A^+XA\| = \|AA^+XAA^+\|$.

*Proof.* Since $\gamma(\dfrac{A}{\|A\|}) = \dfrac{1}{\|A\|\|A^+\|}$, then the equivalence of (i) and (ii) follows immediately from Lemma 1.1.

(ii) $\Rightarrow$ (iii). If $\|A\|\|A^+\| = 1$, then for every $X \in B(H)$, we get

$$\|AXA^+\| = \|AA^+AXA^+AA^+\| \leq \|A^+AXA^+A\| \leq \|AXA^+\|.$$

Thus, (iii) holds.

(iii) $\Rightarrow$ (iv). If we replace $X$ by $A^+XA$ in $(iii)$, we obtain the following equality

$$\forall X \in B(H), \ \ \|AA^+XAA^+\| = \|A^+AA^+XAA^+A\| = \|A^+XA\|.$$

Hence, (iv) is satisfied.

(iv) $\Rightarrow$ (ii). From (iv), it follows that

$$\sup_{\|X\|=1} \|A^+XA\| = \sup_{\|X\|=1} \|AA^+XAA^+\|.$$

Since $AA^+$ is an orthogonal projection, then by Lemma 1.2, we obtain $\|A\|\|A^+\| = 1$. ■

The following lemma was given in [6]. Here, we shall prove it by an easier and direct proof.

LEMMA 2.2. *Let $A \in B(H)$ be a nonzero operator with closed range. If $A$ is an EP operator, then*

$$\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda = \|A\|\|A^+\| + \frac{1}{\|A\|\|A^+\|}.$$

*Proof.* Since $A$ is EP, we conclude that the operator $A$ has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : R(A) \oplus^\perp N(A) \longrightarrow R(A) \oplus^\perp N(A),$$

where $A_1$ is invertible on $R(A)$. Then

$$A^+ = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : R(A) \oplus^\perp N(A) \longrightarrow R(A) \oplus^\perp N(A).$$

Hence,

$$\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda = \|A_1^* \otimes A_1^{-1} + A_1^{-1} \otimes A_1^*\|_\lambda$$

From [13, Theorem 4], it follows that

$$\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda = \|A_1\|\|A_1^{-1}\| + \frac{1}{\|A_1\|\|A_1^{-1}\|}.$$

Finally we obtain

$$\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda = \|A\|\|A^+\| + \frac{1}{\|A\|\|A^+\|}. \quad \blacksquare$$

Recently, in [11], we have proved that if $A \in B(H)$ is with closed range, then $\|A^*XA^+ + A^+XA^*\| \geq 2\|AA^+XA^+A\|$. From Lemma 1.2, it follows that $\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda \geq 2$ holds for every nonzero closed range operator $A$ in $B(H)$. In the following theorem, we shall characterize the class of operators $A \in B(H)$ for which the injective norm $\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda$ gets its minimal value 2.

THEOREM 2.3. *Let $A \in B(H)$ be a nonzero operator with closed range. Then the following statements are equivalent:*

(i) $\dfrac{A}{\|A\|}$ *is a partial isometry,*

(ii) $\forall X \in B(H)$, $\|A^*XA^+ + A^+XA^*\| = 2\|AA^+XA^+A\|$,

(iii) $\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda = 2$.

*Proof.* (i) $\Rightarrow$ (ii). Assume (i) holds. Put $B = \frac{A}{\|A\|}$. Since $B$ is a partial isometry, then $\|B\| = 1$ and $B^+ = B^*$. Therefore, for every $X \in B(H)$, we obtain

$$\begin{aligned}\|A^*XA^+ + A^+XA^*\| &= 2\|B^*XB^*\| = 2\|B^*BB^*XB^*BB^*\| \\ &\leq 2\|BB^*XB^*B\| \leq 2\|B^*XB^*\|\end{aligned}$$

Thus, $\|A^*XA^+ + A^+XA^*\| = 2\|BB^*XB^*B\| = 2\|AA^+XA^+A\|$.

(ii) $\Rightarrow$ (iii). From (ii) and Lemma 1.2, it follows that

$$\begin{aligned}\|A^* \otimes A^+ + A^+ \otimes A^*\|_\lambda &= \sup_{\|X\|=1=\operatorname{rank} X} \|A^*XA^+ + A^+XA^*\| \\ &= \sup_{\|X\|=1=\operatorname{rank} X} 2\|AA^+XA^+A\| = 2.\end{aligned}$$

(iii) $\Rightarrow$ (i). Assume that (iii) holds. Then for every operator $X \in \mathfrak{F}_1(H)$, we obtain

$$\|A^*XA^+ + A^+XA^*\| \leq 2\|X\|.$$

By replacing $X$ by $AA^+XA^+A$ in this last inequality and by using $A^*AA^+ = A^*$ and $A^+AA^* = A^*$, we obtain

$$\forall X \in \mathfrak{F}_1(H), \ \ \|A^*XA^+ + A^+XA^*\| \leq 2\|AA^+XA^+A\|.$$

Now, Let $A = U|A|$ be the polar decomposition of $A$, where $U$ is an isometry on $R(A^*)$. Since $A^* = |A|U^*$, $A^+ = |A|^+U^*$ and using this last inequality, we obtain also

$$\forall X \in \mathfrak{F}_1(H), \ \|U(|A|^+X^*U|A|^+|A|X^*U|A|^+)\| \leq 2\|U(|A|^+|A|X^*U|A||A|^+)\|,$$

and since $U$ is an isometry on $R(A^*)$, then for every operator $X \in \mathfrak{F}_1(H)$, we have

$$\||A|^+X^*U|A|^+|A|X^*U|A|^+\| \leq 2\||A|^+|A|X^*U|A||A|^+\|.$$

By replacing again $X$ by $UX^*$ in this last inequality, and using the fact that $U^*U|A| = |A|$ and $U^*U|A|^+ = |A|^+$, we find that

$$\||A|^+X|A| + |A|X|A|^+\| \le 2\||A|^+|A|X|A||A|^+\|,$$

holds for every operator $X \in \mathfrak{F}_1(H)$, which implies that

$$\sup_{\|X\|=1=\operatorname{rank} X} \||A|^+X|A| + |A|X|A|^+\| \le 2 \sup_{\|X\|=1=\operatorname{rank} X} \||A|^+|A|X|A||A|^+\|.$$

So, $\||A| \otimes |A|^+ + |A|^+ \otimes |A|\|_\lambda \le 2$.

Since $|A|$ is an EP operator, then by Lemma 2.2, we get

$$\||A|\|\||A|^+\| + \frac{1}{\||A|\|\||A|^+\|} \le 2$$

Hence $\||A|\|\||A|^+\| = 1$, and since $\|A\|\|A^+\| = \||A|\|\||A|^+\|$, so (i) follows immediately from Proposition 2.1. ∎

As an immediate consequence of Proposition 1.1 in [11] and Theorem 2.3, we have the following result.

Corollary 2.4. *Let $A \in B(H)$ be a nonzero operator with closed range. Then the following statements are equivalent:*

(i) $\dfrac{A}{\|A\|}$ *is a partial isometry,*

(ii) $\forall X \in B(H)$, $\|A^*XA^+\| + \|A^+XA^*\| \le 2\|AA^+XA^+A\|$.

In the following theorem, we give some characterizations of normal partial isometries in terms of norm operator equalities.

Theorem 2.5. *Let $A \in B(H)$ be a nonzero operator with closed range. Then the following statements are equivalent:*

(i) $\dfrac{A}{\|A\|}$ *is a normal partial isometry,*

(ii) $\forall X \in B(H)$, $\|AXA^+\| = \|A^+AXAA^+\|$,

(iii) $\forall X \in B(H)$, $\|AXA^+\| + \|A^+XA\| = 2\|AA^+XA^+A\|$.

*Proof.* (i) ⇔ (ii). Assume that (i) holds. It follows from Proposition 2.1 that

$$\forall X \in B(H),\ \|AXA^+\| = \|A^+AXA^+A\|.$$

Since $AA^+ = A^+A$ because $A$ is normal, then we deduce that the statement (ii) is satisfied.

Conversely, from (ii), we obtain

$$\|A\|\|A^+\| = \sup_{\|X\|=1} \|AXA^+\| = \sup_{\|X\|=1} \|AA^+XAA^+\| = \|AA^+\|\|A^+A\| = 1.$$

Then using Proposition 2.1, we get that $\frac{A}{\|A\|}$ that is a partial isometry and $\|A^+AXA^+A\| = \|A^+AXAA^+\|$ holds for every $X \in B(H)$. Let $x, y$ be two vectors in $H$ such that $Ax \neq 0$. By taking $X = x \otimes y$ in this last equality, we obtain $\|A^+Ay\| = \|AA^+y\|$, for every $y \in B(H)$. Hence $AA^+ = A^+A$ Since $AA^+$ and $A^+A$ are positive operators, so $A$ is EP. Thus $A$ is normal.

(i) $\Rightarrow$ (iii). The implication follows from Proposition 2.1 and the fact that $AA^+ = A^+A$.

(iii) $\Rightarrow$ (i). First, we prove that $A$ is EP.

Applying (iii) for $X = I - AA^+$, we get that $\|AA^+ - A^2(A^+)^2\| = 0$, and that

$$A^2(A^+)^2 = AA^+. \tag{$*$}$$

Applying again (iii) for $X = I - A^+A$, we get that $\|A^+A - (A^+)^2A^2\| = 0$ and therefore

$$(A^+)^2A^2 = A^+A. \tag{$**$}$$

From $(*)$ and $(**)$, it follows that

$$A^2(A^+)^2A^2 = AA^+A^2 = A^2 \quad \text{and} \quad (A^+)^2A^2(A^+)^2 = A^+A(A^+)^2 = (A^+)^2,$$

and since $A^2(A^+)^2$ and $(A^+)^2A^2$ are orthogonal projections, we conclude that $(A^2)^+ = (A^+)^2$. Then by [1, Theorem 2.2(c)], we obtain

$$R(A^*A^2) \subset R(A) \quad \text{and} \quad R(A(A^*)^2) \subset R(A^*).$$

Since $(A^2)^+ = (A^+)^2$ and from $(*)$ and $(**)$, it follows that $R(A^2) = R(A)$ and $N((A)^2) = N(A)$. This last relation is equivalent to $R((A^*)^2) = R(A^*)$. Notice the following

$$R(A^*A^2) = A^*R(A^2) = A^*R(A) = R(A^*A) = R(A^*)$$

and

$$R(A(A^*)^2) = AR((A^*)^2) = AR(A^*) = R(AA^*) = R(A)$$

Therefore, $R(A^*) \subset R(A)$ and $R(A) \subset R(A^*)$, that is $R(A) = R(A^*)$. Hence, $A$ is EP.

Now, we prove that $\frac{A}{\|A\|}$ is a partial isometry. Since $A$ is EP, then $A$ has the following matrix representation in accordance with this decomposition:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : R(A) \oplus^{\perp} N(A) \longrightarrow R(A) \oplus^{\perp} N(A),$$

where $A_1$ is invertible. Then it follows from (iii) that

$$\forall X_1 \in B(R(A)), \quad \|A_1X_1A_1^{-1}\| + \|A_1^{-1}X_1A_1\| = 2\|X_1\|.$$

Using [12, Theorem 6], we obtain that $A_1$ is a unitary operator in $B(R(A))$ multiplied by a nonzero scalar. Hence $\frac{A}{\|A\|}$ is a normal partial isometry. ∎

The following theorem is a generalization of Proposition 2 in [13].

Theorem 2.6. *Let $A \in B(H)$ have a closed range. Then the following statements are equivalent:*

(i) *$A$ is normal,*

(ii) *$\forall X \in \mathfrak{L}(H)$, $\|AXA^+\| + \|A^+XA\| \leq \|A^*XA^+\| + \|A^+XA^*\|$,*

(iii) *$\forall X \in \mathfrak{F}_1(H)$, $\|AXA^+\| + \|A^+XA\| \leq \|A^*XA^+\| + \|A^+XA^*\|$.*

*Proof.* (i) $\Rightarrow$ (ii). Since $A$ is normal, then for every operator $X \in B(H)$, we obtain

$$\left\|AXA^+\right\| = \left\|A^*XA^+\right\| \text{ and } \left\|A^+XA\right\| = \left\|A^+XA^*\right\|.$$

So (ii) holds.

(ii) $\Rightarrow$ (iii). This implication is trivial.

(iii) $\Rightarrow$ (i). By choosing $X = x \otimes y$, for $x, y \in H$, then using (iii) we get

$$\|Ax \otimes (A^*)^+y\| + \|A^+x \otimes A^*y\| \leq \|A^*x \otimes (A^*)^+y\| + \|A^+x \otimes Ay\|.$$

Hence,

$$\|Ax\|\|(A^*)^+y\| + \|A^+x\|\|A^*y\| \leq \|A^*x\|\|(A^*)^+y\| + \|A^+x\|\|Ay\|.$$

Since $N(A^+) = N(A^*)$ and $N(A) = N((A^*)^+)$, then by taking $y \in N(A)$ and $A^*x \neq 0$ in this last inequality, we obtain $A^*y = 0$. Hence $N(A) \subset N(A^*)$. The same argument shows that $N(A^*) \subset N(A)$. Consequently, $N(A) = N(A^*)$, and so $A$ is an EP operator. Then $A$ has a matrix representation on $H = R(A) \oplus^\perp N(A)$ of the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $A_1$ is invertible on $R(A)$.

By taking $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$ on $H = R(A) \oplus^\perp N(A)$ and using (iii), we get

$$\forall X_1 \in B(R(A)), \quad \left\|A_1X_1A_1^{-1}\right\| + \left\|A_1^{-1}X_1A_1\right\| \leq \left\|A_1^*X_1A_1^{-1}\right\| + \left\|A_1^{-1}X_1A_1^*\right\|.$$

Then, by [13, Proposition 2], we obtain that $A$ is normal. ∎

Acknowledgement. The author would like to thank the referee for his/her useful comments and suggestions on the presentation of this paper.

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(received 17.09.2014; in revised form 03.02.2015; available online 01.04.2015)

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