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SOME SPECTRAL PROPERTIES OF GENERALIZED DERIVATIONS

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Abstract. Given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ and Banach space operators $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$, the generalized derivation $\delta_{A,B} \in L(L(\mathcal{Y},\mathcal{X}))$ is defined by $\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB$. This paper is concerned with the problem of transferring the left polaroid property, from operators A and B^* to the generalized derivation $\delta_{A,B}$. As a consequence, we give necessary and sufficient conditions for $\delta_{A,B}$ to satisfy generalized a-Browder's theorem and generalized a-Weyl's theorem. As an application, we extend some recent results concerning Weyl-type theorems.

1. Introduction

Given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ and Banach space operators $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$, let $L_A \in L(L(\mathcal{X}))$ and $R_B \in L(L(\mathcal{Y}))$ be the left and the right multiplication operators, respectively, and denote by $\delta_{A,B} \in L(L(\mathcal{Y},\mathcal{X}))$ the generalized derivation $\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB$. The problem of transferring spectral properties from A and B to L_A, R_B, L_AR_B and $\delta_{A,B}$ was studied by numerous mathematicians, see [6–8,10,11,15,19,22,23] and the references therein. The main objective of this paper is to study the problem of transferring the left polaroid property and its strong version, finitely left polaroid property, from A and B^* to $\delta_{A,B}$. After Section 2 where several basic definitions and facts will be recalled, we will prove that if A is a left polaroid and satisfies property $(\mathcal{P}_l)$ and B is a right polaroid and satisfy property $(\mathcal{P}_r)$, then $\delta_{A,B}$ is a left polaroid. Also, we prove that if A is a finitely left polaroid and B is a finitely right polaroid, then $\delta_{A,B}$ is a finitely left polaroid. In Section 4, we give necessary and sufficient conditions for $\delta_{A,B}$ to satisfy generalized a-Weyl's theorem. In the last section we apply results obtained previously. If $\mathcal{X} = H$ and $\mathcal{Y} = K$ are Hilbert spaces, we prove that if $A \in L(H)$ and $B \in L(K)$ are completely totally hereditarily normaloid operators, then $f(\delta_{A,B})$ satisfies generalized a-Weyl's theorem, for every analytic function f defined on a neighborhood of $\sigma(\delta_{A,B})$ which is non constant on each of the components of its domain. This generalizes results obtained in [8,10,11,14,22,23].

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2. Notation and terminology

Unless otherwise stated, from now on $\mathcal{X}$ (similarly, $\mathcal{Y}$) shall denote a complex Banach space and $L(\mathcal{X})$ (similarly, $L(\mathcal{Y})$) the algebra of all bounded linear maps defined on and with values in $\mathcal{X}$ (resp. $\mathcal{Y}$). Given $T \in L(\mathcal{X})$, $N(T)$ and $R(T)$ will stand for the null space and the range of T, resp. Recall that $T \in L(\mathcal{X})$ is said to be bounded below, if $N(T) = \{0\}$ and $R(T)$ is closed. Denote the approximate point spectrum of T by

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}.$$

Let

$$\sigma_s(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}$$

denote the surjective spectrum of T. In addition, $\mathcal{X}^*$ will denote the dual space of $\mathcal{X}$, and if $T \in \mathcal{X}$, then $T^* \in L(\mathcal{X}^*)$ will stand for the adjoint map of T. Clearly, $\sigma_a(T^*) = \sigma_s(T)$ and $\sigma_a(T) \cup \sigma_s(T) = \sigma(T)$, the spectrum of T. Recall that the ascent $asc(T)$ of an operator T is defined by $asc(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and the descent $dsc(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf \emptyset = \infty$. It is well known that if $asc(T)$ and $dsc(T)$ are both finite, then they are equal.

A complex number $\lambda \in \sigma_a(T)$ (resp. $\lambda \in \sigma_s(T)$) is a left pole (resp. a right pole) of order d of $T \in L(\mathcal{X})$ if $asc(T - \lambda I) = d < \infty$ and $R((T - \lambda I)^{d+1})$ is closed (resp. $dsc(T - \lambda I) = d < \infty$ and $R((T - \lambda I)^d)$ is closed). We say that T is left polar (resp. right polar) of order d at a point $\lambda \in \sigma_a(T)$ (resp. $\lambda \in \sigma_s(T)$) if λ is a left pole of T (resp. right pole of T) of order d. Now, T is a left polaroid (resp. right polaroid) if T is left polar (resp. right polar) at every $\lambda \in iso\sigma_a(T)$ (resp. $\lambda \in iso\sigma_s(T)$), where $iso\mathcal{K}$ is the set of all isolated points of $\mathcal{K}$ for $\mathcal{K} \subseteq \mathbb{C}$. According to [7], a left polar operator $T \in L(\mathcal{X})$ of order $d(\lambda)$ at $\lambda \in \sigma_a(T)$, satisfies property $(\mathcal{P}_l)$ if the closed subspace $N((T - \lambda)^{d(\lambda)}) + R(T - \lambda)$ is complemented in $\mathcal{X}$ for every $\lambda \in iso\sigma_a(T)$. Dually, a right polar operator $T \in L(\mathcal{X})$ of order $d(\lambda)$ at $\lambda \in \sigma_s(T)$, satisfies property $(\mathcal{P}_r)$ if the closed subspace $N(T - \lambda) \cap R((T - \lambda)^{d(\lambda)})$ is complemented in $\mathcal{X}$ for every $\lambda \in iso\sigma_s(T)$. If $\mathcal{X} = H$ is a Hilbert space, then every left polar (resp. right polar) operator $T \in L(H)$ of order $d(\lambda)$ at $\lambda \in iso\sigma_a(T)$ (resp. $\lambda \in iso\sigma_s(T)$) satisfies property $(\mathcal{P}_l)$ (resp. $(\mathcal{P}_r)$). On the other hand, it is known that $T \in L(\mathcal{X})$ is a right polaroid if and only if T^* is a left polaroid and T is a polaroid if it is both left and right polaroid, whenever $iso\sigma(T) = iso\sigma_a(T) \cup iso\sigma_s(T)$.

Recall that $T \in L(\mathcal{X})$ is said to be a Fredholm operator, if both $\alpha(T) = dimN(T)$ and $\beta(T) = dim\mathcal{X}/R(T)$ are finite dimensional, in which case its index is given by $ind(T) = \alpha(T) - \beta(T)$. If $R(T)$ is closed and $\alpha(T)$ is finite (resp. $\beta(T)$ is finite), then $T \in L(\mathcal{X})$ is said to be an upper semi-Fredholm (resp. a lower semi-Fredholm) while if $\alpha(T)$ and $\beta(T)$ are both finite and equal, so the index is zero and T is said to be a Weyl operator. These classes of opertaors generate the Fredholm spectrum, the upper semi-Fredholm spectrum, the lower semi-Fredholm spectrum and the Weyl spectrum of $T \in L(\mathcal{X})$ which will be denoted by $\sigma_e(T)$, $\sigma_{SF_+}(T)$, $\sigma_{SF_-}(T)$ and $\sigma_W(T)$, respectively. The Weyl essential approximate point spectrum and the Browder essential approximate point spectrum of $T \in L(\mathcal{X})$ are the sets

$$\sigma_{aw}(T) = \{\lambda \in \sigma_a(T) : \lambda \in \sigma_{SF_+}(T) \text{ or } 0 < ind(T - \lambda I)\}$$

and

$$\sigma_{ab}(T) = \{\lambda \in \sigma_a(T) : \lambda \in \sigma_{aw}(T) \ \text{ or } \ asc(T - \lambda I) = \infty\}.$$

It is clear that

$$\sigma_{SF_+}(T) \subseteq \sigma_{aw}(T) \subseteq \sigma_{ab}(T) \subseteq \sigma_a(T).$$

For $T \in L(\mathcal{X})$ and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. If for some integer n the range space $R(T^n)$ is closed and the induced operator $T_n \in L(R(T^n))$ is Fredholm, then T will be said to be B-Fredholm. In a similar way, if T_n is an upper semi-Fredholm (resp. lower semi-Fredholm) operator, then T is called upper semi B-Fredholm (resp. lower semi B-Fredholm). In this case the index of T is defined as the index of semi-Fredholm operator T_n, see [9]. $T \in L(\mathcal{X})$ is called semi B-Fredholm if T is upper semi B-Fredholm or lower semi B-Fredholm. Let

$$\Phi_{SBF}(\mathcal{X}) = \{\, T \in L(\mathcal{X}) : T \text{ is semi B-Fredholm} \,\},$$
$$\Phi_{SBF_+^-}(\mathcal{X}) = \{\, T \in \Phi_{SBF}(\mathcal{X}) : T \text{ is upper semi B-Fredholm with } ind(T) \leq 0 \,\},$$
$$\Phi_{SBF_-^+}(\mathcal{X}) = \{\, T \in \Phi_{SBF}(\mathcal{X}) : T \text{ is lower semi B-Fredholm with } ind(T) \geq 0 \,\}.$$

Then the upper semi B-Weyl and lower semi B-Weyl spectrum of T are the sets

$$\sigma_{UBW}(T) = \{\lambda \in \sigma_a(T) : T - \lambda I \notin \Phi_{SBF_+^-}(\mathcal{X})\}$$

and

$$\sigma_{LBW}(T) = \{\lambda \in \sigma_a(T) : T - \lambda I \notin \Phi_{SBF_-^+}(\mathcal{X})\},$$

respectively. $T \in L(\mathcal{X})$ will be said to be B-Weyl, if T is both upper and lower semi B-Weyl (equivalently, T is B-Fredholm operator of index zero). The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \ \text{ is not B-Weyl operator}\}.$$

Let $\Pi^l(T)$ denote the set of left pole of $T \in L(\mathcal{X})$.

$$\Pi^l(T) = \{\lambda \in \sigma_a(T) : asc(T - \lambda I) = d < \infty \ \text{ and } \ R((T - \lambda I)^{d+1}) \text{ is closed}\}.$$

A strong version of the left polaroid property says that $T \in L(\mathcal{X})$ is a finitely left polaroid (resp. a finitely right polaroid) if and only if every $\lambda \in iso\sigma_a(T)$ (resp. $\lambda \in iso\sigma_s(T)$) is a left pole of T and $\alpha(T - \lambda I) < \infty$ (resp. a right pole of T and $\beta(T - \lambda I) < \infty$). Let $\Pi_0^l(T)$ (resp. $\Pi_0^r(T)$) denote the set of finite left poles (resp. the set of finite right poles) of T. Then $T \in L(\mathcal{X})$ is a finitely left polaroid (resp. a finitely right polaroid) if and only if $iso\sigma_a(T) = \Pi_0^l(T)$ (resp. $iso\sigma_a(T) = \Pi_0^r(T)$).

For $T \in L(\mathcal{X})$ define

$$\Delta(T) = \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow R(T^n) \cap N(T) \subseteq R(T^m) \cap N(T)\}.$$

The degree of stable iteration is defined as $dis(T) = \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $dis(T) = \infty$ if $\Delta(T) = \emptyset$. $T \in L(\mathcal{X})$ is said to be quasi-Fredholm of degree d, if there exists $d \in \mathbb{N}$ such that $dis(T) = d$, $R(T^n)$ is a closed subspace of $\mathcal{X}$ for each $n \geq d$ and $R(T) + N(T^n)$ is a closed subspace of $\mathcal{X}$. An operator $T \in L(\mathcal{X})$ is said to be semi-regular, if $R(T)$ is closed and $N(T^n) \subseteq R(T^m)$ for all $m, n \in \mathbb{N}$.

An important property in local spectral theory is the single valued extension property. An operator $T \in L(\mathcal{X})$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc $\mathbb{D}$ centered at λ_0, the only analytic function $f : \mathbb{D} \to \mathcal{X}$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$ is the function $f \equiv 0$. An operator $T \in L(\mathcal{X})$ is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$.

Furthermore, for $T \in L(\mathcal{X})$ the quasi-nilpotent part of T is defined by

$$H_0(T) = \{x \in \mathcal{X} : \lim_{n \to \infty} \|T^n(\mathcal{X})\|^{\frac{1}{n}} = 0\}.$$

It can be easily seen that $N(T^n) \subset H_0(T)$ for every $n \in \mathbb{N}$. The analytic core of an operator $T \in L(\mathcal{X})$ is the subspace $K(T)$ defined as the set of all $x \in \mathcal{X}$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in \mathcal{X}$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$, the spaces $K(T)$ are hyperinvariant under T and satisfy $K(T) \subset R(T^n)$, for every $n \in \mathbb{N}$ and $T(K(T)) = K(T)$, see [1] for information on $H_0(T)$ and $K(T)$.

3. Left polaroid generalized derivation

We begin this section by recalling some results concerning spectra of generalized derivations.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and consider $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. Let $\delta_{A,B} \in L(L(\mathcal{Y}, \mathcal{X}))$ be the generalized derivation induced by A and B, i.e.,

$$\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB \text{ where } X \in L(\mathcal{Y}, \mathcal{X}).$$

According to [20, Theorem 3.5.1], we have that

$$\sigma_a(\delta_{A,B}) = \sigma_a(A) - \sigma_s(B).$$

and it is not difficult to conclude that

$$iso\sigma_a(\delta_{A,B}) = (iso\sigma_a(A) - iso\sigma_a(B^*)) \setminus acc\sigma_a(\delta_{A,B}).$$

The following results concerning upper semi Fredholm spectrum and Browder essential approximate point spectrum of generalized derivation were proved in [8,24]. They will be used in the sequel.

Lemma 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and consider $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. Then the following statements hold.*

i) $\sigma_{SF_+}(\delta_{A,B}) = (\sigma_{SF_+}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \sigma_{SF_-}(B))$.

ii) $\sigma_{ab}(\delta_{A,B}) = (\sigma_{ab}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \sigma_{ab}(B^*))$.

The following lemma concerning the Weyl essential approximate point spectrum of a generalized derivation will also be used in the sequel.

LEMMA 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and consider $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. Then*

$$\sigma_{aw}(\delta_{A,B}) \subseteq (\sigma_{aw}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \sigma_{aw}(B^*)).$$

Proof. Let $\lambda \notin (\sigma_{aw}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \sigma_{aw}(B^*))$. If $\mu_i \in \sigma_a(A)$ and $\nu_i \in \sigma_s(B)$ are such that $\lambda = \mu_i - \nu_i$. Then $\mu_i \notin \sigma_{SF_+}(A)$ and $\nu_i \notin \sigma_{SF_-}(B)$, hence from statement i) of Lemma 3.1 $\lambda \notin \sigma_{SF_+}(\delta_{A,B})$. Now, we will prove that

$$ind(\delta_{A,B} - \lambda I) \leq 0.$$

Suppose to the contrary that $ind(\delta_{A,B} - \lambda I) > 0$. Then $\lambda \notin \sigma_e(\delta_{A,B})$. It follows from [17, Corollary 3.4] that

$$\lambda = \mu_i - \nu_i \quad (1 \leq i \leq n),$$

where $\mu_i \in iso\sigma(A)$ for $1 \leq i \leq m$ and $\nu_i \in iso\sigma(B)$, for $m+1 \leq i \leq n$. We have that $ind(\delta_{A,B} - \lambda I)$ is equal to

$$\sum_{j=m+1}^{n} dimH_0(B - \nu_j)ind(A - \mu_j) - \sum_{k=1}^{m} dimH_0(A - \mu_k)ind(B - \nu_k).$$

Since $\mu_i \in iso\sigma(A)$, for $1 \leq i \leq m$ and $\nu_i \in iso\sigma(B)$, for $m+1 \leq i \leq n$, it follows that $dimH_0(A - \mu_j)$ is finite, for $1 \leq j \leq m$ and $dimH_0(B - \nu_k)$ is finite, for $m+1 \leq k \leq n$ and we have also $ind(A - \mu_i) \leq 0$ and $ind(B - \nu_j) \geq 0$. Thus $ind(\delta_{A,B} - \lambda I) \leq 0$. This a contradiction. Hence $\lambda \notin \sigma_{aw}(\delta_{A,B})$. ∎

According to [7], a left polaroid operator (resp. a right polaroid operator) satisfies property $(\mathcal{P}_l)$, (resp. $(\mathcal{P}_r)$), if it is left polar at every $\lambda \in iso\sigma_a(T)$ (resp. right polar at every $\lambda \in iso\sigma_s(T)$ which satisfies property $(\mathcal{P}_l)$, (resp. property $(\mathcal{P}_r)$. The following lemma is the dual version of [7, Lemma 3.1].

LEMMA 3.3. *Let $\mathcal{X}$ be a Banach space. If $T \in L(\mathcal{X})$ is a right polaroid and satisfies property $(\mathcal{P}_r)$, then for every $\lambda \in iso\sigma_s(T)$ there exist T-invariant closed subspaces N_1 and N_2 such that $\mathcal{X} = N_1 \oplus N_2$, $(T - \lambda)|_{N_1}$ is nilpotent of order $d(\lambda)$ and $(T - \lambda I)|_{N_2}$ is surjective, where $d(\lambda)$ is the order of the right pole at λ. Moreover, $K(T - \lambda I) = R((T - \lambda I)^{d(\lambda)})$.*

Proof. From the hypothesis, $T - \lambda$ is quasi-Fredholm of degree $d(\lambda)$ and closed subspace $N((T - \lambda I)^{d(\lambda)}) + R(T - \lambda)$ is complemented in $\mathcal{X}$. Since $T \in L(\mathcal{X})$ is right polaroid and satisfies property $(\mathcal{P}_r)$, then $N(T - \lambda) \cap R((T - \lambda)^{d(\lambda)})$ is complemented in $\mathcal{X}$. From [25, Theorem 5], there exist T-invariant closed subspaces N_1 and N_2 such that $\mathcal{X} = N_1 \oplus N_2$, $(T - \lambda)|_{N_1}$ is nilpotent of order $d(\lambda)$ and $(T - \lambda I)|_{N_2}$ is semi-regular. Since $dsc(T - \lambda I) = d(\lambda)$, the semi-regular operator $(T - \lambda I)|_{N_2}$ is surjective. Since $K(T - \lambda I) = K((T - \lambda I)|N_1) \oplus K((T - \lambda I)|N_2) = 0 \oplus N_2 = N_2$, we can conclude from [2, Theorem 2.7] that $K(T - \lambda I) = R((T - \lambda I)^{d(\lambda)}$. ∎

Next follows the main result of this section.

Theorem 3.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and let $A \in L(\mathcal{X})$ be a left polaroid and $B \in L(\mathcal{Y})$ be a right polaroid. If A satisfies property $(\mathcal{P}_l)$ and B satisfies property $(\mathcal{P}_r)$, then $\delta_{A,B}$ is a left polaroid.*

Proof. Let $\lambda \in iso\sigma_a(\delta_{A,B})$. Then there exist $\mu \in \sigma_a(A)$ and $\nu \in \sigma_s(B)$ such that $\lambda = \mu - \nu$, and it follows that $\mu \in iso\sigma_a(A)$ and $\nu \in iso\sigma_s(B) = iso\sigma_a(B^*)$. Since A is a left polaroid, then there exist A-invariant closed subspaces M_1 and M_2 such that $\mathcal{X} = M_1 \oplus M_2$, $(A - \mu I)|_{M_1} = A_1 - \mu I|_{M_1}$ is nilpotent of order d_1 where $d_1 = d(\mu)$ is the order of left pole of A at μ and that $(A - \mu I)|_{M_2} = A_2 - \mu I|_{M_2}$ is bounded below. Also, since B is a right polaroid, then there exists B-invariant closed subspaces N_1 and N_2 such that $\mathcal{Y} = N_1 \oplus N_2$, $(B - \nu)|_{N_1} = B_1 - \nu I|_{N_1}$ is nilpotent of order d_2 where $d_2 = d(\nu)$ is the order of right pole of B at ν and $(B-\nu I)|_{N_2} = B_2-\nu I|_{N_2}$ is surjective. Let $d = d_1+d_2$ and $X \in L(N_1\oplus N_2, M_1\oplus M_2)$ have the representation $X = [X_{kl}]_{k,l=1}^2$. We will prove that $asc(\delta_{A,B} - \lambda I)$ is finite.

Let $(\delta_{A,B} - \lambda I)^{d+1}(X) = 0$ imply that $X_{12} = X_{21} = X_{22} = 0$. Since $(\delta_{A_1,B_1} - \lambda I)$ is d-nilpotent it follows that $(\delta_{A,B} - \lambda I)^d(X) = 0$. Hence $asc(\delta_{A,B} - \lambda I) \leq d < \infty$.

Now, we prove that $(\delta_{A,B} - \lambda I)^{d+1}(L(\mathcal{Y}, \mathcal{X}))$ is closed. First, we will prove that $0 \notin \sigma_a(\delta_{A_2-\mu I|_{M_2}, B_2-\nu I|_{N_2}})$. For this, it suffices to prove that $\sigma_a(A_2-\mu I|_{M_2})\cap \sigma_s(B_2 - \nu I|_{N_2}) = \emptyset$. Suppose that there exists a complex number α such that $\alpha \in \sigma_a(A_2-\mu I|_{M_2})\cap\sigma_s(B_2-\nu I|_{N_2})$. Then $\alpha \in \sigma_a(A_2-\mu I|_{M_2})$ and $\alpha \in \sigma_s(B_2-\nu I|_{N_2})$, from [1, Theorem 2.48], $0 \in \sigma_a(A_2 - (\mu + \alpha)I|_{M_2})$ and $0 \in \sigma_s(B_2 - (\nu + \alpha)I|_{N_2})$. Since $(\mu + \alpha)$ is isolated in the approximate point spectrum of A and $(\nu + \alpha)$ is isolated in the surjective spectrum of B, then by the hypothesis A is a left polaroid which satisfies property $(\mathcal{P}_l)$ and B is a right polaroid which satisfies property $(\mathcal{P}_r)$. We conclude that

$$(A - (\mu + \alpha)I)|_{M_2} = A_2 - (\mu + \alpha)I|_{M_2}$$

is bounded below and

$$(B - (\nu + \alpha)I)|_{N_2} = B_2 - (\nu + \alpha)I|_{N_2}$$

is surjective. That is

$$0 \notin \sigma_a(A_2 - (\mu + \alpha)I|_{M_2}) \text{ and } 0 \notin \sigma_s(B_2 - (\nu + \alpha)I|_{N_2}).$$

This is a contradiction, hence $0 \notin \sigma_a(\delta_{A_2-\mu I|_{M_2}, B_2-\nu I|_{N_2}})$. Since $0 \notin \sigma_a(\delta_{A_2,B_2} - \lambda I)$, then from [3, Lemma 1.1] $(\delta_{A_2,B_2} - \lambda I)^{d+1}(L(N_2, M_2))$ is closed. We have that $\delta_{A_1,B_1} - \lambda I$ is nilpotent of order d, and then by [26, Theorem 2.7] it follows that

$$(\delta_{A_1,B_1} - \lambda I)^{d+1}(L(N_1, M_1)) \text{ is closed.}$$

From the fact that $0 \notin \sigma_a(\delta_{A_i,B_j} - \lambda I)$ and [3, Lemma 1.1]AA, it follows that

$$(\delta_{A_i,B_j} - \lambda I)^{d+1}(L(N_j, M_i)) \text{ is closed for } 1 \leq i, j \leq 2 \text{ and } i \neq j.$$

Consequently, $(\delta_{A,B} - \lambda I)^{d+1}(L(\mathcal{X}, \mathcal{Y}))$ is closed. Hence λ is a left pole of $\delta_{A,B}$ which means that $\delta_{A,B}$ is a left polaroid. ■

In the case of Hilbert spaces, we have the following corollary.

Corollary 3.5. *Let H and K be Hilbert spaces and let $A \in L(H)$ and $B \in L(K)$. If A and B^* are left polaroids, then $\delta_{A,B}$ is left polaroid.*

Remark. From [18, Theorem 3.8] we have that if $T \in L(\mathcal{X})$, such that $\alpha(T) < \infty$ and $asc(T) < \infty$, then $R(T^n)$ is closed for some integer $n > 1$, if and only if $R(T)$ is closed. Hence T is a finitely left polaroid if and only if $\alpha(T-\lambda I) < \infty$, $asc(T - \lambda I) < \infty$ and $R(T - \lambda I)$ is closed for every $\lambda \in iso\sigma_a(T)$.

In the following theorem, we characterize finitely left polaroid generalized derivation.

Theorem 3.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. If A and B^* are finitely left polaroid operators, then $\delta_{A,B}$ is a finitely left polaroid.*

Proof. Let $\lambda \in iso\sigma_a(\delta_{A,B})$. Then there exist $\mu \in \sigma_a(A)$ and $\nu \in \sigma_s(B)$ such that $\lambda = \mu - \nu$, hence we have $\mu \in iso\sigma_a(A)$ and $\nu \in iso\sigma_s(B) = iso\sigma_a(B^*)$. Suppose that A and B^* are finitely left polaroids. Then from [27, Corollary 2.2] we have that $\mu \notin \sigma_{ab}(A)$ and $\nu \notin \sigma_{ab}(B^*)$. Applying statement ii) of Lemma 3.1, we get $\lambda \notin \sigma_{ab}(\delta_{A,B})$, hence by [27, Corollary 2.2] $\delta_{A,B}$ is a finitely left polaroid. ∎

4. Consequences to Weyl's type theorem

For $T \in L(\mathcal{X})$, let $E^a(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T - \lambda I)\}$ and $E_0^a(T) = \{\lambda \in E^a(T) : \alpha(T - \lambda I) < \infty\}$. Recall that T is said to satisfy a-Browder's theorem (resp. generalized a-Browder's theorem) if $\sigma_a(T) \setminus \sigma_{aw}(T) = \Pi_0^l(T)$ (resp. $\sigma_a(T) \setminus \sigma_{UBW}(T) = \Pi^l(T)$). From [4, Theorem 2.2] we have that T satisfies a-Browder's theorem if and only if T satisfies generalized a-Browder's theorem. T is said to satisfy a-Weyl's theorem (resp. generalized a-Weyl's theorem) if $\sigma_a(T) \setminus \sigma_{aw}(T) = E_0^a(T)$ (resp. $\sigma_a(T) \setminus \sigma_{UBW}(T) = E^a(T)$).

For $T \in L(\mathcal{X})$, let $E(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda I)\}$ and $E_0(T) = \{\lambda \in E(T) : \alpha(T - \lambda I) < \infty\}$. Recall that T is said to satisfy Weyl's theorem (resp. generalized Weyl's theorem) if $\sigma(T) \setminus \sigma_W(T) = E_0(T)$ (resp. $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$). We know that if T satisfies generalized a-Weyl's theorem then T satisfies a-Weyl's theorem and this implies that T satisfies Weyl's theorem. Next, generalized a-Weyl's theorem for $\delta_{A,B}$ will be studied.

Theorem 4.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. Suppose that A and B^* satisfy a-Browder's theorem. If A is a left polaroid and satisfies property $(\mathcal{P}_l)$ and B is a right polaroid and satisfies $(\mathcal{P}_r)$, then the following assertions are equivalent.*

i) *$\delta_{A,B}$ satisfies generalized a-Weyl's theorem.*

ii) $\sigma_{aw}(\delta_{A,B}) = (\sigma_{aw}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \sigma_{aw}(B^*))$.

Proof. If A and B^* satisfy a-Browder theorem, then they satisfy generalized a-Browder theorem. By [8, Theorem 4.2] it follows that $\delta_{A,B}$ satisfies generalized a-Browder's theorem if and only if $\sigma_{aw}(\delta_{A,B}) = (\sigma_{aw}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \sigma_{aw}(B^*))$. That is $\sigma_a(\delta_{A,B}) \setminus \sigma_{UBW}(\delta_{A,B}) = \Pi^l(\delta_{A,B})$. Since A is a left polaroid and B is a right polaroid, then from Theorem 3.4 $\delta_{A,B}$ is a left polaroid, consequently $\Pi^l(\delta_{A,B}) = E^a(\delta_{A,B})$. Thus $\delta_{A,B}$ satisfies generalized a-Weyl's theorem. The reverse implication is obvious from the fact that $\delta_{A,B}$ satisfies generalized a-Weyl's theorem implies $\delta_{A,B}$ satisfies generalized a-Browder's theorem ∎

In the case of Hilbert spaces operators, we have the following corollaries.

Corollary 4.2. *Let H and K be two Hilbert spaces and let $A \in L(H)$ and $B \in L(K)$. Suppose that A and B^* satisfy a-Browder's theorem. If A is a left polaroid and B is a right polaroid, then the following assertions are equivalent.*

i) $\delta_{A,B}$ *satisfies generalized a-Weyl's theorem.*

ii) $\sigma_{aw}(\delta_{A,B}) = (\sigma_{aw}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \overline{\sigma_{aw}(B^*)})$.

Corollary 4.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. Suppose that A and B^* satisfy a-Browder's theorem. If A is a left polaroid and satisfies property $(\mathcal{P}_l)$ and B is a right polaroid and satisfies property $(\mathcal{P}_r)$, then the following assertions are equivalent.*

i) $\delta_{A,B}$ *has SVEP at* $\lambda \notin \sigma_{UBW}(\delta_{A,B})$.

ii) $\delta_{A,B}$ *satisfies a-Browder's theorem.*

iii) $\delta_{A,B}$ *satisfies a-Weyl's theorem.*

iv) $\delta_{A,B}$ *satisfies generalized a-Weyl's theorem.*

v) $\sigma_{aw}(\delta_{A,B}) = (\sigma_{aw}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \sigma_{aw}(B^*))$.

Proof. $(i) \Leftrightarrow (ii)$ follows from [5, Theorem 2.1], $(iii) \Leftrightarrow (iv)$ follows from [3, Theorem 3.7] and $(iv) \Leftrightarrow (v)$ follows from Theorem 4.1. ∎

In the following result, we give sufficient conditions for $\delta_{A,B}$ to satisfy a-Browder's theorem.

Theorem 4.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and let $A \in L(\mathcal{X})$ and $B \in L(\mathcal{Y})$. If A has SVEP at $\mu \in \sigma_a(A) \setminus \sigma_{SF_+}(A)$ and B has SVEP at $\nu \in \sigma_a(B^*) \setminus \sigma_{SF_-}(B)$, then $\delta_{A,B}$ satisfies a-Browder's theorem.*

Proof. Let $\lambda \in \sigma_a(\delta_{A,B}) \setminus \sigma_{aw}(\delta_{A,B})$. Then $\lambda \in \sigma_a(\delta_{A,B}) \setminus \sigma_{SF_+}(\delta_{A,B})$, and from statement i) of Lemma 3.1 there exist $\mu \in \sigma_a(A) \setminus \sigma_{SF_+}(A)$ and $\nu \in \sigma_s(B) \setminus \sigma_{SF_-}(B)$ such that $\lambda = \mu - \nu$. Since A has SVEP at $\mu \notin \sigma_{SF_+}(A)$ and B has SVEP at $\nu \notin \sigma_{SF_-}(B)$, it follows from [27, Corollary 2.2] that $\mu \notin \sigma_{ab}(A)$ and $\nu \notin \sigma_{ab}(B^*)$; applying statement ii) of Lemma 3.1 we get $\lambda \notin \sigma_{ab}(\delta_{A,B})$. Hence $\lambda \in \Pi_0^l(\delta_{A,B})$. Let $\lambda \in \Pi_0^l(\delta_{A,B})$; according to [27, Corollary 2.2], we have $\lambda \in \sigma_a(\delta_{A,B}) \setminus \sigma_{ab}(\delta_{A,B})$. Since $\sigma_{aw}(T) \subseteq \sigma_{ab}(T)$, then $\lambda \in \sigma_a(\delta_{A,B}) \setminus \sigma_{aw}(\delta_{A,B})$. Hence $\delta_{A,B}$ satisfy a-Browder's theorem. ∎

5. Application

A Banach space operator $T \in L(\mathcal{X})$ is said to be hereditary normaloid, $T \in \mathcal{HN}$, if every part of T (i.e., the restriction of T to each of its invariant subspaces) is normaloid (i.e., $\|T\|$ equals the spectral radius $r(T)$). $T \in \mathcal{HN}$ is totally hereditarily normaloid, $T \in \mathcal{THN}$, if also the inverse of every invertible part of T is normaloid and T is completely totally hereditarily normaloid (abbr. $T \in \mathcal{CHN}$), if either $T \in \mathcal{THN}$ or $T - \lambda I \in \mathcal{HN}$ for every complex number λ. The class $\mathcal{CHN}$ is large. In particular, let H be a Hilbert space and $T \in L(H)$ be a Hilbert space operator. If T is hyponormal ($T^*T \geq TT^*$) or p-hyponormal ($(T^*T)^p) \geq (TT^*)^p$) for some ($0 < p \leq 1$) or w-hyponormal ($(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|$), then T is in $\mathcal{THN}$. Again, totaly *-paranormal operators ($\|(T-\lambda I)^*x\|^2 \leq \|(T-\lambda I)x\|^2$ for every unit vector x) are $\mathcal{HN}$-operators and paranormal operators ($\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all unit vector x) are $\mathcal{THN}$-operators. It is proved in[11] that if $A, B^* \in L(H)$ are hyponormal, then the generalized Weyl's theorem holds for $f(\delta_{A,B})$ for every $f \in \mathcal{H}(\sigma(\delta_{A,B}))$, where $\mathcal{H}(\sigma(\delta_{A,B}))$ is the set of all analytic functions defined on a neighborhood of $\sigma(\delta_{A,B})$. This result was extended to log-hyponormal or p-hyponormal operators in [14] and [22]. Also, in [10] and [23], it is shown that if $A, B^* \in L(H)$ are w-hyponormal operators, then Weyl's theorem holds for $f(\delta_{A,B})$ for every $f \in \mathcal{H}(\sigma(\delta_{A,B}))$. Let $\mathcal{H}_c(\sigma(T))$ denote the space of all analytic functions defined on a neighborhood of $\sigma(T)$ which is non constant on each of the components of its domain. In the next results we can give more.

Theorem 5.1. *Suppose that $A, B \in L(H)$ are $\mathcal{CHN}$ operators; then $\delta_{A,B}$ satisfies a-Browder's theorem.*

Proof. Since A and B are $\mathcal{CHN}$-operators, it follows from [13, Corollary 2.10] that A has SVEP at $\mu \in \sigma_a(A)\backslash\sigma_{SF_+}(A)$ and B has SVEP at $\mu \in \sigma_a(B^*)\backslash\sigma_{SF_-}(B)$. Then by Theorem 4.4, a-Browder's theorem holds for $\delta_{A,B}$. ■

Corollary 5.2. *If $A, B \in L(H)$ are $\mathcal{CHN}$ operators, then*

i) *$\delta_{A,B}$ has SVEP at $\lambda \notin \sigma_{UBW}(\delta_{A,B})$,*

ii) *$\delta_{A,B}$ satisfies a-Browder's theorem.*

iii) *$\delta_{A,B}$ satisfies a-Weyl's theorem.*

iv) *$\delta_{A,B}$ satisfies generalized a-Weyl's theorem.*

v) *$\sigma_{aw}(\delta_{A,B}) = (\sigma_{aw}(A) - \sigma_s(B)) \cup (\sigma_a(A) - \overline{\sigma_{aw}(B^*)})$.*

Proof. Since A and B are $\mathcal{CHN}$-operators, it follows from [13, Corollary 2.15] that A, B, A^* and B^* satisfy a-Browder's theorem. By [13, Proposition 2.1], we conclude that A and B^* are left polaroids. The assertions follows from Corollary 4.3. ■

Corollary 5.3. *Suppose that $A, B \in L(H)$ are $\mathcal{CHN}$-operators. Then $f(\delta_{A,B})$ satisfies generalized a-Browder's theorem, for every $f \in \mathcal{H}_c(\sigma(\delta_{A,B}))$.*

Proof. By Corollary 5.2 and [16, Corollary 3.5], we get that generalized a-Browder's theorem holds for $f(\delta_{A,B})$. ∎

COROLLARY 5.4. *Suppose that* $A, B \in L(H)$ *are* $\mathcal{CHN}$*-operators. Then* $f(\delta_{A,B})$ *satisfies generalized a-Weyl's theorem, for every* $f \in \mathcal{H}_c(\sigma(\delta_{A,B}))$.

Proof. By [13, Proposition 2.1] and Theorem 3.4, we get that $\delta_{A,B}$ is a left polaroid and from Corollary 5.2 we have that $\delta_{A,B}$ satisfies generalized a-Weyl's theorem. Applying [16, Theorem 3.14] we get that generalized a-Weyl's's theorem holds for $f(\delta_{A,B})$. ∎

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