МАТЕМАТИЧКИ ВЕСНИК
68, 1 (2016), 58–65
March 2016

оригинални научни рад
research paper

DIRECTED PROPER CONNECTION OF GRAPHS

Colton Magnant, Patrick Ryan Morley, Sarabeth Porter, Pouria Salehi Nowbandegani and Hua Wang

Abstract. An edge-colored directed graph is called properly connected if, between every pair of vertices, there is a properly colored directed path. We study some conditions on directed graphs which guarantee the existence of a coloring that is properly connected. We also study conditions on a colored directed graph which guarantee that the coloring is properly connected.

1. Introduction

With the ever-increasing awareness of network security and the challenges it poses, there is always an effort to produce more reliable and better protected networks. A graph model of a computer network consists of a vertex for each node (computer) and an edge (possibly directed) between two vertices when there is a link directly between the two nodes. If we let different types of security measures on the links between nodes be represented by colors on the edges of the corresponding graph, it would make the network more secure if, in order to move from one computer to another within the network, one must pass through different types of security on each consecutive steps. The goal of this work is to study precisely this problem and to determine the minimum number of colors (types of security) that are required to achieve this property.

Unless otherwise noted, a coloring of a graph will be assumed to be an edge-coloring. All undefined terminology comes from [2].

A coloring of a graph is called *proper* if no two adjacent edges share a color. Given a colored (undirected) graph G, we say that G is *properly connected* if, between every pair of vertices, there is a properly colored path. In particular, if only two colors are used, then a properly colored path would be one that alternates between the two colors. This notion was defined and studied in [4] where the following result was proven.

2010 Mathematics Subject Classification: 05C15, 05C20
Keywords and phrases: directed graph; edge coloring; proper connection.
Partly supported by Simons Foundation (Grant #245307).

THEOREM 1. (Fujita et al. [4]) *If $n \geq 3$ and each vertex has at least $\frac{n}{2}$ different incident colors, then G is properly connected.*

Defined in [1] and further studied in [3], the *proper connection number* of a graph G, denoted $pc(G)$, is the minimum number of colors k needed so that there exists a k-coloring of the edges of G which is properly connected. Among other results, the following relation between connectivity and proper connection number was proven. Here we recall that $\kappa(G)$ denotes the vertex connectivity, the order of a smallest vertex cut set, of a graph.

THEOREM 2. (Borozan et al. [1]) *If $\kappa(G) \geq 2$, then $pc(G) \leq 3$ and if $\kappa(G) \geq 3$, then $pc(G) \leq 2$.*

In this work, we define a directed version of these concepts. First of all, a directed graph is called *strongly connected* if, for every ordered pair of vertices (u, v), there exists a directed path from u to v. We defined a colored digraph G to be *properly strong* if, for every ordered pair of vertices (u, v), there exists a properly colored directed path from u to v. The *directed proper connection number* $\overrightarrow{pc}(G)$ is then defined to be the minimum number of colors needed to color the edges of G so that G is properly strong. In what follows, our first main result is the following.

THEOREM 3. *If G is strongly connected, then $\overrightarrow{pc}(G) \leq 3$.*

Theorem 3 is proven in Section 3. Note here that a coloring can only be properly strong if the underlying digraph is strongly connected so Theorem 3 provides a bound for all nontrivial cases. Any directed odd cycle C certainly has $\overrightarrow{pc}(C) = 3$ so this result is also sharp.

In the case of a tournament, we get an even stronger result.

THEOREM 4. *Every strongly connected tournament T on $n \geq 4$ vertices has $\overrightarrow{pc}(T) = 2$.*

The proof of Theorem 4 is presented in Section 4. This proof relies on a structure much like that of Fact 1 (Section 2), an "almost spanning" strongly connected bipartite subgraph. Here by *almost spanning*, we mean that the subgraph spans all but at most two vertices. We wonder if such a substructure may be the only way to get $\overrightarrow{pc}(G) = 2$.

QUESTION 1. *Classify the digraphs G with $\overrightarrow{pc}(G) = 2$. Do they all have an almost spanning strongly connected bipartite subgraph?*

Here the assumption that the subgraph is almost spanning is critical since there are simple examples where the result is not true without this assumption. See, for example, the graph pictured in Figure 1, which consists of a directed C_4 sharing an edge with a directed C_5. This graph contains a strongly connected bipartite subgraph (the C_4) but requires three colors to produce a properly strong coloring.

As a special case of Question 1, we believe the following to be true.

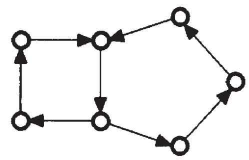

Fig. 1. Two directed cycles

Conjecture 1. *If G is a strongly connected digraph with no even cycle, then* $\overrightarrow{pc}(G) = 3$.

Finally, in Section 5, we discuss some notions of color degree and whether they imply that the graph is properly strong.

2. Preliminaries

We start this section with the following useful fact.

Fact 1. *If there exists a spanning, bipartite, strongly connected subgraph of a strongly connected digraph G, then* $\overrightarrow{pc}(G) = 2$.

Proof. Let $A \cup B$ be the assumed bipartition. Color all edges leaving A with red and all edges leaving B with blue. Since $A \cup B$ is strongly connected, there are directed paths between every pair of vertices but since $A \cup B$ is bipartite, these paths must alternate colors. This means $\overrightarrow{pc}(G) = 2$. ∎

One might be tempted to conjecture that this fact may be strengthened to provide a necessary and sufficient condition but this is not the case. Indeed, consider the directed graph constructed from a directed C_4, say using vertices u, v, w, x in this order around the cycle. To this cycle, we add a single vertex y with an edge in from u and out to v. If we color the edges $v \to w$ and $x \to u$ with red and all other edges with blue, this provides a properly connected 2-coloring but there is no spanning, bipartite, strongly connected subgraph of this graph. In fact, this coloring trick will be used in the proof of Theorem 4.

For our next result, we recall that $\kappa'(G)$ denotes the edge connectivity, the size of a minimum edge cut, of a graph.

Proposition 1. *If G is a graph with edge connectivity $\kappa'(G) \geq 3$, then there exists an orientation of the edges of G such that* $\overrightarrow{pc}(G) = 2$.

Proof. It is an easy exercise to show that if $\kappa'(G) \geq 3$, then G has a spanning 2-edge-connected bipartite subgraph B (see [1] for example). It is also straightforward to show that a 2-edge-connected graph has a strongly connected orientation. Such an orientation of B, provides a spanning, bipartite, strongly connected subgraph of an orientation of G. By Fact 1, there is an orientation of G with $\overrightarrow{pc}(G) = 2$. ∎

3. Proof of Theorem 3

An *ear decomposition* is a partition of the edges of a 2-edge-connected graph G into paths and cycles such that the graph G is constructed by starting with any single cycle and repeatedly adding one path or cycle at a time from the decomposition, at each step maintaining a 2-edge-connected graph on the vertices that have been used.

Since G is strongly connected, there is an ear decomposition of G into directed (possibly closed) ears starting at any directed cycle of G. We will produce a properly connected coloring of G using three colors by induction on the number of ears in this decomposition.

For a base, let $G = C_n$ be a directed cycle. A proper edge coloring of C_n with at most 3 colors is trivially properly connected.

Let G_i be this directed cycle after the addition and coloring of i directed ears. By induction, suppose we have a properly connected coloring of G_i using 3 colors with the additional property that all edges into each vertex have a single color. For convenience, color each vertex with the color of its incoming edges.

If the $(i+1)^{st}$ ear is a single directed edge, trivially color it with the color of its terminal vertex. Since G_i was properly connected, G_{i+1} is trivially properly connected as well. Note that the proper connectivity of G_{i+1}, does not depend on this edge so it can be avoided in future paths. Also, note that these single-edge ears are the only time an edge might go between two vertices of the same color. Thus, we may assume the $(i+1)^{st}$ ear contains at least one new vertex.

Let u and v be the start and end vertices of the $(i+1)^{st}$ ear respectively. Possibly u and v might be equal. Properly color the ear with the three available colors so that the last edge receives the color of v and the first edge is not the color of u. With three colors available, such a coloring is always possible. This produces a coloring of G_{i+1} in which each vertex has only one color on all incoming edges.

In G_i, there is a properly colored path Q from v to u. Let P be the newly added ear. Note that $Q \cup P$ is a properly colored directed cycle. Thus, each pair of vertices x and y with $x, y \in Q \cup P$ is connected by properly colored paths in both directions. Also, G_i is properly connected so we need only check pairs x, y where $x \in P$ and $y \in G_i \setminus Q$ or $x \in G_i \setminus Q$ and $y \in P$.

First suppose $x \in G_i \setminus Q$ and $y \in P$. Since G_i is properly connected, there exists a proper path from x to u. By the definition of the coloring, this path must end with an edge having the color of u. This path can then be extended along P to get to y and complete the proof in this case.

Finally, suppose $x \in P$ and $y \in G_i \setminus Q$, since G_i was properly connected, there exists a proper path, say R, from v to y, not using single-edge ears. In particular, this means the first edge R will not have the same color of v. Thus, $x \to P \to v \to R \to y$ is a properly colored path from x to y to complete the proof. ∎

4. Proof of Theorem 4

Theorem 4 follows almost immediately the following structural claim, which provides an almost spanning, bipartite, strongly connected subgraph, slightly weaker than the structure used in Fact 1.

CLAIM 1. *Every strongly connected tournament of order $n \geq 4$ contains a bipartite, strongly connected subgraph of order at least $n-2$. Moreover, if the bipartite subgraph is $A \cup B$, then the vertices outside the bipartite subgraph must have either Type I: all in-edges from A and all out-edges to B or Type II: all out-edges to A and all in-edges from B. Furthermore, there is at most one vertex of each type.*

Proof. Let T be a strongly connected tournament of order $n \geq 4$. The proof is by induction on n. For a base, if $n = 4$, every strongly connected tournament on 4 vertices contains a directed C_4. Since a directed C_4 is a spanning, bipartite, strongly connected subgraph, the base of the induction is complete. Now assume $n > 4$ and, by induction, there exists a bipartite, strongly connected subgraph $A \cup B$ of order at least $n-3$, say color A with red and B with blue to label the bipartition. Let $A \cup B$ be a largest bipartite, strongly connected subgraph of T.

Let v be a vertex outside $A \cup B$. If v has an in-edge from A, then it must have all edges from A being in-edges since otherwise we could color v with blue to contradict the choice of $A \cup B$ as a largest bipartite, strongly connected subgraph. By the same argument, v must have one of the following types:

Type I: all in-edges from A and all out-edges to B,

Type II: all out-edges to A and all in-edges from B,

Type III: all in-edges from A and all in-edges from B, or

Type IV: all out-edges to A and all out-edges to B.

Suppose there is a vertex of Type III. Since T is strongly connected, there must be a vertex w of Type III with an edge to a vertex that is not Type III. Without loss of generality, suppose w has an out-edge to x, which has an out-edge to B. Then the path $A \to w \to x \to B$ can be absorbed into $A \cup B$ by coloring w with blue and x with red, contradicting the maximality of $|A \cup B|$. Thus, there can be no vertex of Type III and symmetrically no vertex of Type IV.

Finally, suppose there are at least two vertices v, w of Type I. Without loss of generality, suppose the edge between v and w goes from v to w. Then the path $A \to v \to w \to B$ can be absorbed into $A \cup B$ by coloring v with blue and w with red, contradicting the maximality of $|A \cup B|$. Thus, there can be at most one vertex of Type I and symmetrically at most one of Type II, completing the proof of Claim 1. ■

By Claim 1, we may assume there is a bipartite subgraph of T, say $A \cup B$ that is strongly connected and misses at most two vertices of T. Color all edges from A to B with color 1 and all edges from B to A with color 2 as in the proof of Fact 1. If a vertex outside $A \cup B$ has Type I, color all edges to and from $A \cup B$ with color 1 and if a vertex outside has Type II, color all edges to and from $A \cup B$ with color 2. It is easy to see this coloring is properly connected so $pc(T) = 2$. ■

5. Color degree

We also consider using minimum color degree to force a colored digraph to be properly strong. Leaving out the assumption that the digraph is strongly connected almost trivializes the problem of just using out-degree since you need out-degree at least $\frac{n-1}{2}$ to guarantee that the digraph is strongly connected but then it is already a regular (strongly connected) tournament in which every vertex has all different colors on the out-edges. It might be natural to hope that this graph is properly strong as long as $n \geq 5$ (since $n = 3$ trivially fails) but in fact, this is not the case. For example, consider the cyclic (counter clockwise) tournament on 5 vertices and color some of the edges as shown in Figure 2. Remaining edges are colored arbitrarily to satisfy the color degree assumption. This coloring is not properly strong since there is no proper path from a to b. In fact, this shows that an original assumption that the graph was strongly connected would not have helped.

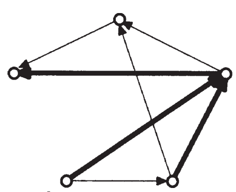

Fig. 2. Tournament that is not properly strong

More generally, consider the cyclic tournament T_n on $n = 2k + 1$ vertices in which the edges are colored with k colors. Prescribe consecutive vertices, b and a, in the cyclic ordering so the edge $b \to a$ is present. Label the vertices of T_n as $v_1 = a, v_2, v_3, \ldots, v_n = b$. To each vertex in $B = \{v_{k+1}, v_{k+1}, \ldots, v_{n-1}\}$, associate one of the available colors, say v_i becomes associated with color $i - k + 2$. For each vertex $v_i \in B$, let the edge $v_i b$ have the color $i - k + 2$ associated with v_i. Furthermore, color all incoming edges to each $v_i \in B$ with the associated color. All remaining edges of T_n are colored arbitrarily to satisfy the color degree condition. The last vertex on any path from a to b must be in B but since all incoming edges to each vertex $v_i \in B$ have the same color as the edge $v_i \to b$, there can be no properly colored path from a to b, meaning that T_n cannot be properly strong.

One aspect of this construction that allows it to work is that for several vertices, all in-edges have the same color. If this is not the case, we get the following result.

THEOREM 5. *If in a colored tournament T_n of odd order $n \geq 201$, each vertex has $\frac{n-1}{2}$ different colors on in-edges and $\frac{n-1}{2}$ different colors on out-edges, then T_n is properly strong.*

Here we choose $n \geq 201$ merely to simplify computations. We make no attempt to minimize this constant since it is unlikely that this approach will provide the desired result for all n. We believe this approach works for $n \geq 27$ and, on the other hand, we have verified the cases when $n = 5$ and 7 by hand.

Proof. First a simple fact that follows from the color degree assumption.

FACT 2. *Each vertex of T_n has no two in-edges in the same color and no two out-edges in the same color.*

For a contradiction, suppose there are two vertices a and b so that there is no directed properly colored path from a to b. In particular, this means that the edge ba is directed from b to a. Let A be the set of vertices with in-edges from a and let B be the set of vertices with out-edges to b. Since the out-degree of a is $\frac{n-1}{2}$ and the in-degree of b is $\frac{n-1}{2}$, we see that $|A \cap B| \geq 1$.

Suppose for a moment that $|A \cap B| \geq 2$ and let $x, y \in A \cap B$. In order to avoid a proper path from a to b, the edges ax and xb must have the same color and similarly ay and yb must also have the same color. Also, since b has $\frac{n-1}{2}$ different colored in-edges, we may assume these colors are different, say red and blue respectively. Without loss of generality, suppose the edge between x and y is directed from x to y. By Fact 2, the edge xy is neither red nor blue, say green. Then the path $axyb$ is rainbow (in particular, proper), a contradiction. Thus, $|A \cap B| = 1$. Let $c \in A \cap B$.

The set A induces a tournament so the graph induced on A has average out-degree $\frac{|A|-1}{2} = \frac{n-3}{4}$ assuming the appropriate divisibility. Thus, over all vertices in A, the average number of edges directed to vertices of $B \setminus \{c\}$ is at least

$$\frac{n-1}{2} - \frac{n-3}{4} - 1 = \frac{n-3}{4}$$

again assuming appropriate divisibility.

Since $n \geq 201$ and at least half the vertices of A have at least half the average number of edges to $B \setminus \{c\}$, there must exist two vertices in A which share at least 3 out-neighbors in B. Say we have $u, v \in A$ both adjacent with directed edges to all of $\{x, y, z\} \subseteq B$. By Fact 2, although the color of au may be the same as the color of uy, there is at least one vertex, say x, such that the colors satisfy $\mathrm{col}(au) \neq \mathrm{col}(ux)$ and $\mathrm{col}(av) \neq \mathrm{col}(vx)$. Then the edge xb cannot have the same color as both ux and vx, say $\mathrm{col}(xb) \neq \mathrm{col}(ux)$. This means $auxb$ is a proper path, a contradiction completing the proof. ■

REFERENCES

[**1**] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Montero, and Z. Tuza, *Proper connection of graphs*, Discrete Math. **312** (17) (2012), 2550–2560.

[**2**] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs*, CRC Press, Boca Raton, FL, fifth edition, 2011.

[**3**] S. Fujita, A. Gerek, and C. Magnant, *Proper connection with many colors*, J. Comb. **3** (4) (2012), 683–693.

[**4**] S. Fujita and C. Magnant, *Properly colored paths and cycles*, Discrete Appl. Math. **159** (14) (2011), 1391–1397.

(received 14.08.2015; in revised form 09.10.2015; available online 15.11.2015)

C.M., P.R.M., S.P., Hua Wang: Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, 30460, U.S.A.,

E-mail: cmagnant@georgiasouthern.edu, pmorley@georgiasouthern.edu, sarabeth_porter@georgiasouthern.edu, hwang@georgiasouthern.edu

P.S.N.: Pharmaceutical Research Center, Shiraz University of Medical Sciences, Shiraz, Iran,

E-mail: pouria.salehi@gmail.com