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CONSTRUCTION OF COSPECTRAL REGULAR GRAPHS

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Abstract. Graphs G and H are called cospectral if they have the same characteristic polynomial, equivalently, if they have the same eigenvalues considering multiplicities. In this article we introduce a construction to produce pairs of cospectral regular graphs. We also investigate conditions under which the graphs are integral.

1. Introduction

We consider simple graphs, that is, graphs without loops or parallel edges. For basic notions in graph theory we refer to [6], whereas for preliminaries on graphs and matrices, see [1]. By the eigenvalues of a graph G, we mean the eigenvalues of its adjacency matrix $A(G)$. Graphs G and H are said to be cospectral if they have the same eigenvalues, counting multiplicities, or equivalently, they have the same characteristic polynomial. There are considerable literatures on construction of cospectral graphs [4, 5].

A graph with only integer eigenvalues is termed an integral graph. In Section 2 of this paper we first describe a construction starting with r copies each of regular graphs G and H. It is possible to explicitly describe the eigenvalues of the resulting graph. Under certain conditions on the parameters, we get an infinite family of regular integral graphs. The construction allows us to obtain cospectral integral regular graphs. In certain cases we obtain biregular graphs (that is, graphs with two possible degrees) and graphs which have the same eigenvalues, but with different multiplicities.

2. The construction

In this section we present a new method of constructing cospectral graphs. Graph $T^r(G, H)$ from $r > 1$ copies of given graphs G and H by adding three types of edges is defined. Formulas for eigenvalues of the resulting graph, conditions for being regular (Corollary 2.3), and integral (Proposition 2.9) have been derived. It

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is shown in what way infinitely many pairs of cospectral (regular) graphs can be constructed. We also provide some figures and examples of constructed graphs.

A square matrix is said to be regular if all its row sums and column sums are equal. The common value of the row and column sum is called the regularity of the matrix. Clearly, in this case the regularity is an eigenvalue with the all ones vector as an eigenvector. The next result is known when A and B are adjacency matrices of graphs (see [3, Theorem 2.8, p. 57]). We present a proof of the more general statement for completeness.

THEOREM 2.1. *Let A and B be symmetric, regular matrices of orders p, s and regularity q, w, respectively. If $q, \mu_2, \cdots, \mu_p$ and $w, \lambda_2, \cdots, \lambda_s$, are respectively the eigenvalues of A and B, then the eigenvalues of the matrix*

$$T = \begin{pmatrix} A & J_{p\times s} \\ J_{s\times p} & B \end{pmatrix}$$

are $\mu_2, \cdots, \mu_p, \lambda_2, \cdots, \lambda_s$ and $[q + w \pm \sqrt{(q+w)^2 + 4(ps - wq)}]/2$.

Proof. First suppose that $U = [u_1, \ldots, u_p]'$ is an eigenvector of A corresponding to $\mu_i \neq q$. Since $\mathbf{1}$ is an eigenvector of A corresponding to q, we may assume that $\mathbf{1}'U = 0$. We see that

$$\begin{pmatrix} A & J_{n\times s} \\ J_{s\times n} & B \end{pmatrix} \begin{pmatrix} U \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mu_i \begin{pmatrix} U \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and thus μ_i is an eigenvalue of T for $i = 2, \ldots p$. Similarly we can show that λ_i, for $i = 1, \ldots, s$, is an eigenvalue of T. Let the remaining two eigenvalues of T be x and y. Since

$$\begin{aligned} x + y + \lambda_2 + \cdots + \lambda_p + \mu_2 + \cdots + \mu_s &= \mathrm{tr}(T) = \mathrm{tr}(B) + \mathrm{tr}(A) \\ w + \lambda_2 + \cdots + \lambda_p &= \mathrm{tr}(B) \\ q + \mu_2 + \cdots + \mu_s &= \mathrm{tr}(A), \end{aligned}$$

then $x + y = q + w$. It is well-known that the sum of all 2×2 principal minors of a square matrix, say H, is equal to the second elementary symmetric function of the eigenvalues of H, which we denote by $\sigma_2(H)$. We have

$$\begin{aligned} \sigma_2(T) = xy &+ x(\mu_2 + \cdots + \mu_p) + x(\lambda_2 + \cdots + \lambda_s) \\ &+ y(\mu_2 + \cdots + \mu_p) + y(\lambda_2 + \cdots + \lambda_s) \\ &+ \sum_{2\le i<j} \mu_i\mu_j + \sum_{2\le i<j} \lambda_i\lambda_j + \sum_{2\le i,j} \lambda_i\mu_j. \end{aligned}$$

Hence

$$\begin{aligned} \sigma_2(T) = xy &+ (x+y)(\mathrm{tr}(B) + \mathrm{tr}(A) - q - w) \\ &+ \sum_{2\le i<j} \mu_i\mu_j + \sum_{2\le i<j} \lambda_i\lambda_j + (\mathrm{tr}(B) - w)(\mathrm{tr}(A) - q). \end{aligned}$$

Similarly,

$$\sigma_2(A) = q(\mu_2 + \cdots + \mu_p) + \sum_{2\le i<j} \mu_i\mu_j = q(\mathrm{tr}(A) - q) + \sum_{2\le i<j} \mu_i\mu_j$$
$$\sigma_2(B) = w(\lambda_2 + \cdots + \lambda_s) + \sum_{2\le i<j} \lambda_i\lambda_j = w(\mathrm{tr}(B) - w) + \sum_{2\le i<j} \lambda_i\lambda_j.$$

Note that every 2×2 principal minor of T is either a 2×2 principal minor of A or B, or has the form

$$\begin{vmatrix} a_{ii} & 1 \\ 1 & b_{jj} \end{vmatrix} = a_{ii}b_{jj} - 1,\ i = 1,\dots,p,\ j = 1,\dots,s,$$

where a_{ii} and b_{jj} are diagonal entries of A and B, respectively. Therefore,

$$\begin{aligned}\sigma_2(T) &= \sigma_2(A) + \sigma_2(B) + \sum_{j=1}^{s}\sum_{i=1}^{p}(a_{ii}b_{jj} - 1)\\ &= \sigma_2(A) + \sigma_2(B) + \mathrm{tr}(A)\,\mathrm{tr}(B) - ps.\end{aligned}$$

From the previous relations we get

$$\begin{aligned}&xy + (x+y)(\mathrm{tr}(B) + \mathrm{tr}(A) - q - w) + (\mathrm{tr}(B) - w)(\mathrm{tr}(A) - q)\\ &\quad = q(\mathrm{tr}(A) - q) + w(\mathrm{tr}(B) - w) + \mathrm{tr}(A)\,\mathrm{tr}(B) - ps.\end{aligned}$$

This, together with $x + y = q + w$ yields $xy = qw - ps$, giving the values of x and y as $[w + q \pm \sqrt{(q+w)^2 + 4(ps - wq)}]/2$. ∎

LEMMA 2.2. *Suppose that X and Y are square matrices of the same order. Let*

$$T = \begin{pmatrix} X & Y & \dots & Y \\ Y & X & \dots & Y \\ \vdots & \vdots & \ddots & \vdots \\ Y & Y & \dots & X \end{pmatrix} \tag{1}$$

be an $r\times r$ block matrix. Then the eigenvalues of T are the eigenvalues of $X - Y$, $r - 1$ times, and the eigenvalues of $X + (r-1)Y$.

Proof. By transforming T into a triangular block matrix we can compute the characteristic polynomial of T as product of the characteristic polynomials of the diagonal block matrices, and then we see that the eigenvalues of T are the same as the eigenvalues of the diagonal block entries. To this end, to the first column add all the other columns. Then we have

$$\left(\begin{array}{c|ccc} X+(r-1)Y & Y & \dots & Y \\ \hline X+(r-1)Y & X & \dots & Y \\ \vdots & \vdots & \ddots & \vdots \\ X+(r-1)Y & Y & \dots & X \end{array}\right) \tag{2}$$

Now subtract the first row from the other rows to get

$$\left(\begin{array}{c|ccc} X+(r-1)Y & Y & \dots & Y \\ \hline 0 & X-Y & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & X-Y \end{array}\right). \tag{3}$$

Since $T - \lambda I$ has a form similar to T, we conclude that $|T - \lambda I| = |X + (r-1)Y - \lambda I| \underbrace{|X - Y - \lambda I| \dots |X - Y - \lambda I|}_{r-1}$. It follows that the eigenvalues of T are the eigenvalues of $X - Y$, $r - 1$ times, and the eigenvalues of $X + (r-1)Y$. ■

We now introduce the definition of graph in accordance with the title of this section. Let G and H be graphs of p and s vertices respectively. Take $r > 1$ copies of G, say $G_1, \dots, G_r$ and r copies of H, say $H_1, \dots, H_r$. Then $T^r(G, H)$ is defined as the graph with vertex set $\bigcup_{i=1}^{r}(V(G_i) \cup V(H_i))$ and edge set defined as follows:

(i) Each vertex of H_i is adjacent to each vertex of G_i, $i = 1, \dots, r$.

(ii) Each vertex of G_i is adjacent to the corresponding vertex of G_j for all $i \neq j$.

(iii) Each vertex of G_i is adjacent to the corresponding neighbors in G_j for all $i \neq j$.

Note that the adjacency matrix of $T^r(G, H)$, denoted $A(T^r(G, H))$, is in the block form given in (1) with

$$X =: \left(\begin{array}{c|ccc} & 1 & \dots & 1 \\ A(G) & \vdots & & \vdots \\ & 1 & \dots & 1 \\ \hline 1 \ \dots \ 1 & & & \\ \vdots \quad\ \ \vdots & & A(H) & \\ 1 \ \dots \ 1 & & & \end{array}\right) \text{ and } Y =: \left(\begin{array}{c|ccc} & 0 & \dots & 0 \\ A(G)+I & \vdots & & \vdots \\ & 0 & \dots & 0 \\ \hline 0 \ \dots \ 0 & 0 & \dots & 0 \\ \vdots \quad\ \ \vdots & \vdots & & \vdots \\ 0 \ \dots \ 0 & 0 & \dots & 0 \end{array}\right). \tag{4}$$

Figure 1 illustrates some graphs constructed by this definition.

THEOREM 2.3. *Let G be a q-regular graph and H a w-regular graph. If $q, \mu_2, \dots, \mu_p$ are eigenvalues of $A(G)$ and $w, \lambda_2, \dots, \lambda_s$ are eigenvalues of $A(H)$, then the eigenvalues of $A(T^r(G, H))$ are as follows.*

eigenvalue	*multiplicity*
$r\mu_i + r - 1 \ \ (i = 2, \ldots, p)$	1
$\lambda_i \ \ (i = 2, \ldots, s)$	r
-1	$(r-1)(p-1)$
$\frac{rq+r-1+w \pm \sqrt{(rq+r-1+w)^2+4(ps-(rq+r-1)w)}}{2}$	1
$\frac{w-1 \pm \sqrt{(w-1)^2+4(ps+w)}}{2}$	$r-1$

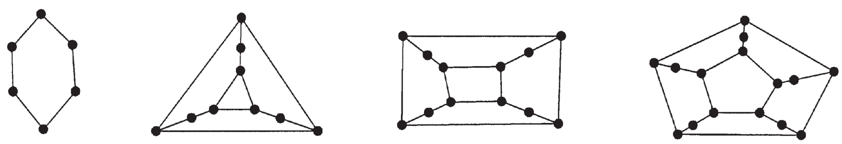

Fig. 1. $T^r(2K_1, K_1)$ for $r = 2, 3, 4$ and 5

Proof. In view of Lemma 2.2 the eigenvalues of $A(T^r(G,H))$ are the eigenvalues of

$$X + (r-1)Y = \left(\begin{array}{ccc|ccc} & & & 1 & \ldots & 1 \\ & rA(G)+(r-1)I & & \vdots & & \vdots \\ & & & 1 & \ldots & 1 \\ \hline 1 & \ldots & 1 & & & \\ \vdots & & \vdots & & A(H) & \\ 1 & \ldots & 1 & & & \end{array}\right), \tag{5}$$

and the eigenvalues of

$$X - Y = \left(\begin{array}{ccc|ccc} & & & 1 & \ldots & 1 \\ & -I & & \vdots & & \vdots \\ & & & 1 & \ldots & 1 \\ \hline 1 & \ldots & 1 & & & \\ \vdots & & \vdots & & A(H) & \\ 1 & \ldots & 1 & & & \end{array}\right) \tag{6}$$

of multiplicity $r-1$. On the other hand, the matrices $rA(G)+(r-1)I$, B and $-I$ are regular, hence by Theorem 2.1, we can compute the eigenvalues of $X+(r-1)Y$ and $X-Y$ which are listed as follows:

eigenvalues of $X+(r-1)Y$	multiplicity
$(r-1)+r\mu_i \ \ (i=2,\ldots,p)$	1
$\lambda_i \ \ (i=2,\ldots,s)$	1
$\frac{rq+r-1+w\pm\sqrt{(rq+r-1+w)^2+4(ps-(rq+r-1)w)}}{2}$	1

Also

eigenvalues of $X-Y$	multiplicity
-1	$(p-1)(r-1)$
$\lambda_i \ \ (i=2,\ldots,s)$	$r-1$
$\frac{w-1\pm\sqrt{(w-1)^2+4(ps+w)}}{2}$	$r-1$

Hence the proof is complete. ■

COROLLARY 2.4. *$T^r(G,H)$ is a regular graph if and only if G is q-regular and H is w-regular for some q and w such that*

$$s+rq+(r-1)=p+w.$$

In this case, the eigenvalues of $A(T^r(G,H))$ are as follows.

eigenvalue	*multiplicity*
$\lambda_i \ \ (i=2,\ldots,s)$	r
$r\mu_i+r-1 \ \ (i=2,\ldots,p)$	1
-1	$(r-1)(p-1)$
$p+w$	1
$w-s$	1
$\frac{w-1\pm\sqrt{(w-1)^2+4(ps+w)}}{2}$	$r-1$

(7)

Proof. First suppose that $T^r(G,H)$ is regular. By considering the matrices X, Y and $A(T^r(G,H))$ in the definition of $T^r(G,H)$, it is not hard to see that $A(G)$ and $A(H)$ are regular matrices, and so G and H are regular graphs. Thus G is a $q-$regular graph and H is a $w-$regular graph for some q and w. Since every row sum of $A(T^r(G,H))$ is equal to $p+w$ or $q+s+(r-1)(q+1)$, the regularity of $T^r(G,H)$ implies $s+rq+(r-1)=p+w$.

The converse statement is clear.

To see the second statement, assume $T^r(G,H)$ is regular. Using $s+rq+(r-1)=p+w$ we see after some simplification that

$$\frac{rq+r-1+w+\sqrt{(rq+r-1+w)^2+4(ps-(rq+r-1)w)}}{2}=p+w$$

and

$$\frac{rq+r-1+w-\sqrt{(rq+r-1+w)^2+4(ps-(rq+r-1)w)}}{2}=w-s.$$

To complete the proof, update the table in Theorem 2.3. ∎

In view of Corollary 2.4, if G and H are regular graphs, then $T^r(G,H)$ is either a regular or a biregular graph.

REMARK 2.5 (i) Let G and H be graphs. Set $T_0^r(G,H):=T^r(G,H)$, and for $n\geq 1$ set $T_n^r(G,H):=T^r(T_{n-1}^r(G,H),H)$. If p_{n+1} is the number of vertices of $T_{n+1}^r(G,H)$ then $p_{n+1}=r(p_n+s)$. The solution of this recurrence relation with the initial condition $p_0=:p$ is $p_n=pr^n+\frac{rs}{r-1}(r^n-1)$.

(ii) Suppose that G,G' and H,H' are pairs of cospectral graphs. Then for any n, $T_n^r(G,H)$ and $T_n^r(G',H')$ are also cospectral graphs. In this manner we can construct infinitely many pairs of cospectral graphs. If we take G,G' to be cospectral and $H=H'$ to be a complete graph then we obtain infinitely many pairs of cospectral regular graphs in view of the following result.

THEOREM 2.6. *Suppose that the hypotheses of Theorem 2.3 hold. If $T_m^r(G,H)$ is regular for some $m\geq 2$, then H is a complete graph and $T_n^r(G,H)$ is a regular graph for all n.*

Proof. Suppose that $T_m^r(G,H)$ is regular where $m\geq 2$. Since $T^r(T_{n-1}^r(G,H),H)=T_n^r(G,H)$, in view of Corollary 2.4, $T_n^r(G,H)$ is regular for all $n\leq m$. In particular, $T_1^r(G,H)$ and $T_2^r(G,H)$ are regular.

Let $T_1^r(G,H)$ be q_1-regular with p_1 vertices. Then $q_1=p+w$ and $p_1=r(p+s)$. By Corollary 2.4 we have

$$s=p_1+w-rq_1-r+1=r(p+s)+w-r(p+w)-r+1, \tag{8}$$

and it follows that $w=s-1$. Therefore H is a complete graph.

We proceed by induction on n. So assume that $T_n^r(G,H)$ is regular for $n=k\geq 2$. Let q_k be the regularity of $T_k^r(G,H)$. Since $q_n=rq_{n-1}+r-1$, for $n=1,\dots,k$, then Corollary 2.4 implies that

$$s=p_{n-1}+w-rq_{n-1}-r+1, \text{ for } n=1,\dots,k. \tag{9}$$

Now apply $w=s-1$ in (9) and deduce that

$$p_{n-1}=r(q_{n-1}+1) \text{ for } n=1,\dots,k. \tag{10}$$

In view of the fact that H is complete we can use (10) as a necessary and sufficient condition for the regularity of $T_n^r(G,H)$. Hence to complete the proof we need to verify that (10) is also valid for $n=k+1$. To achieve this, first note that the recurrence relation $q_n=rq_{n-1}+r-1$ implies $q_k=\frac{s+r-1}{r-1}(r^k-1)+qr^k$, where

$q_0 = q$. Now assume $n = k$. Hence

$$\begin{aligned} r(q_k+1) &= r\left(\frac{s+r-1}{r-1}(r^k-1)+qr^k+1\right) \\ &= r\left(\left(\frac{s}{r-1}+1\right)(r^k-1)+qr^k+1\right) \\ &= \frac{rs}{r-1}(r^k-1)+rr^k-r+qr^k+r \\ &= \frac{rs}{r-1}(r^k-1)+rr^k(q+1) \\ &= \frac{rs}{r-1}(r^k-1)+rp^k \\ &= p_k. \end{aligned}$$

Therefore $T_n^r(G,H)$ is regular for all n. ■

In view of Theorem 2.6, that $T_1^r(G,H)$ is regular does not guarantee the regularity of $T_2^r(G,H)$, whereas if $T_2^r(G,H)$ is regular, then $T_n^r(G,H)$ is regular for all n.

COROLLARY 2.7. *Suppose that for all $n \geq 0$ the graph $T_n^r(G,H)$ is q_n-regular with eigenvalues $q_n, \mu_2^n, \ldots, \mu_{p_n}^n$. Then the eigenvalues of $T_{n+1}^r(G,H)$ are given as follows.*

eigenvalue	*multiplicity*
$r\mu_i^n + r - 1 \ \ (i = 2, \ldots, p_n)$	1
-1	$(r-1)(p_n-1)+r(s-1)+1$
q_{n+1}	1
$\frac{w-1\pm\sqrt{(w-1)^2+4(p_n s+w)}}{2}$	$r-1$

Proof. Since $T_n^r(G,H)$ is a regular graph for $n \geq 0$, by Theorem 2.6, H is a complete graph, say $H = K_s$. Hence in Table (7), we have $\lambda_i = -1$ for $i = 2, \ldots, s$, appearing a total of r times. Note that $w - s = -1$ appearing once. Thus -1 will appear as many as $(r-1)(p_n-1)+r(s-1)+1$ times. Furthermore, q_{n+1}, as regularity of $T_{n+1}^r(G,H)$ equals $p+w$ in (7). The remaining eigenvalues come from those of $T_n^r(G,H)$ which has p_n vertices. ■

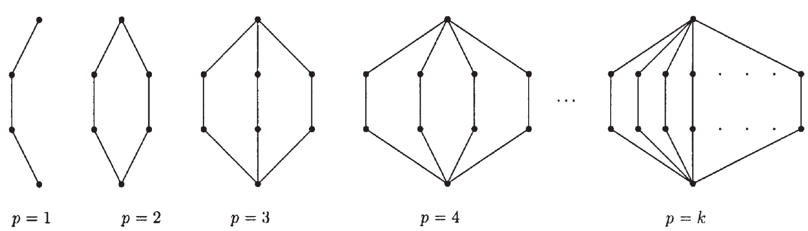

Fig. 2. The graph of $T_1^2(pK_1, K_1)$ for $p = 1, \ldots, k$

EXAMPLE 2.8. Let pK_1 denote p copies of K_1, that is, an empty graph with p vertices. For $p = 1, 2, 3, \ldots, k$, Figure 2 illustrates the graph $T_1^2(pK_1, K_1)$. If we put $r = 2$, $s = 1$, $w = 0$ and $q = 0$ in Theorem 2.3, then eigenvalues are as below.

eigenvalue		*multiplicity*	
$\lambda_i (i = 2, \ldots s)$	$-$	$-$	r
$r\mu_i + r - 1 (i = 2, \ldots, p)$	1	$p-1$	1
-1	-1	$p-1$	$(r-1)(p-1)$
$\frac{rq+r-1+w\pm\sqrt{(rq+r-1+w)^2+4(ps-(rq+r-1)w)}}{2}$	$\frac{1\pm\sqrt{1+4p}}{2}$	1	1
$\frac{w-1\pm\sqrt{(w-1)^2+4(ps+w)}}{2}$	$\frac{-1\pm\sqrt{1+4p}}{2}$	1	$r-1$

Note that $T_1^2(pK_1, K_1)$ is always bipartite, and it is an integral graph if and only if $p = m(m+1)$ for some integer m. In this case the eigenvalues are

eigenvalue	multiplicity
1	$p-1$
-1	$p-1$
$\pm m, \pm(m+1)$	1

For the general case we have the following proposition.

PROPOSITION 2.9. *$T_1^r(pK_1, sK_1)$ is integral if and only if one of the following holds.*

(i) *$r = 2k$ and there exist integers m and n such that $ps = n(n+1) = (m+k)(m-k+1)$.*

(ii) *$r = 2k+1$ and there exist integers m and n such that $ps = n(n+1) = (m+k)(m-k)$.*

Furthermore, in this case the eigenvalues are

eigenvalue	*multiplicity*
$0 (i = 2, \ldots, s)$	r
$r - 1 (i = 2, \ldots, p)$	1
-1	$(r-1)(p-1)$
$\begin{cases} k+m, k-m-1 & \text{if } r = 2k \\ k+m, k-m & \text{if } r = 2k+1 \end{cases}$	1
$n, -1-n$	$r-1$

Proof. Suppose that $T_1^r(pK_1, sK_1)$ is integral. Since $w = q = 0$, in Theorem 2.3 we have

$$\frac{rq+r-1+w\pm\sqrt{(rq+r-1+w)^2+4(ps-(rq+r-1)w)}}{2} = \frac{r-1\pm\sqrt{(r-1)^2+4ps}}{2} \quad (11)$$

and

$$\frac{w-1\pm\sqrt{(w-1)^2+4(ps+w)}}{2}=\frac{-1\pm\sqrt{1+4ps}}{2}. \tag{12}$$

By our assumption, (11) and (12) must be integers. We have two cases.

CASE 1. $r=2k$ for some k. Then $A:=\sqrt{(r-1)^2+4ps}$ is an odd integer, say $A=2m+1$ for some m. This shows that

$$4ps=(2m+1)^2-(r-1)^2=(2m+1+2k-1)(2m+1-2k+1)=4(m+k)(m-k+1),$$

and so $ps=(m+k)(m-k+1)$.

On the other hand, $B:=\sqrt{1+4ps}$ is also an odd integer, say $B=2n+1$ for some n. Again we have $ps=n(n+1)$. Hence $n(n+1)=(m+k)(m-k+1)$. In this case, relation (11) reduces to $\frac{2k-1\pm(2m+1)}{2}$ and (12) reduces to $\frac{-1\pm(2n+1)}{2}$.

CASE 2. $r=2k+1$ for some k. Then $r-1\pm\sqrt{(r-1)^2+4ps}=2k\pm\sqrt{(r-1)^2+4ps}$ is an even integer. This means that $\sqrt{(r-1)^2+4ps}$ must be an even integer, say $2m$ for some m. Hence $4ps=(2m)^2-(2k)^2$, and so $ps=(m-k)(m+k)$. Also as in Case 1, we must have $ps=n(n+1)$ for some n. Hence $n(n+1)=(m-k)(m+k)$. In this case relation (11) reduces to $k\pm m$ and (12) reduces to $\frac{-1\pm(2n+1)}{2}$.

The proof of the converse is straightforward. ■

REMARK 2.10. (i) In view of Proposition 2.9, whether $T_1^r(pK_1,sK_1)$ is integral or not just depends on the value of ps and not on the individual values of p and s. Using Maple we explored parameters (r,ps,m,n,k) such that $T_1^r(pK_1,sK_1)$ is integral. In Table 1 we have listed all possible values for $r\le 12$ and $ps\le 42$ such that $T_1^r(pK_1,sK_1)$ is integral.

r	2										5	6	7	8	9	10	11	12	
ps	2	6	12	20	30	42	56	72	90	110	12	6	72	30	20	90	56	12	42
m	1	2	3	4	5	6	7	8	9	10	4	6	9	6	6	10	9	6	8
n	1	2	3	4	5	6	7	8	9	10	3	3	8	5	4	9	7	3	6
k	1										2	3	3	4	4	5	5	6	6

Table 1. All possible values for $r\le 12$ and $ps\le 42$ such that $T_1^r(pK_1,sK_1)$ is integral

(ii) Suppose that $p,s>1$ and that $T_1^r(pK_1,sK_1)$ is integral. If $ps=t_1t_2$ such that $t_1,t_2>1$, then in view of Proposition 2.9, $T_1^r(pK_1,sK_1)$ and $T_1^r(t_1K_1,t_2K_1)$ have the same eigenvalues regardless of multiplicity. Furthermore, $T_1^r(pK_1,sK_1)$ and $T_1^r(t_1K_1,t_2K_1)$ have the same number of vertices whenever $p+s=t_1+t_2$. Consequently, $T_1^r(pK_1,sK_1)$ and $T_1^r(pK_1,sK_1)$ have the same number of vertices and have the same eigenvalues regardless of multiplicity. Note that $T_1^r(pK_1,sK_1)$ is never an integral graph if $p=s$ because in this case ps is a perfect square that by Proposition 2.9 must satisfy $ps=n(n+1)$ for some n, which is impossible.

Example 2.11. By Table 1, if $ps = 12$ and $r = 2$, then $T_1^2(3K_1, 4K_1)$, $T_1^2(4K_1, 3K_1)$, $T_1^2(2K_1, 6K_1)$ and $T_1^2(6K_1, 2K_1)$ are all integral biregular graphs, and regardless of multiplicity they have the same eigenvalues. By Proposition 2.9, these eigenvalues are $0, 1, -1, 3, -3, 4$ and -4. See Figure 3.

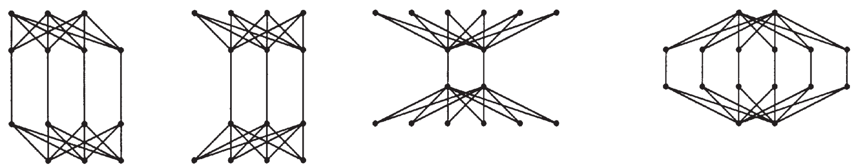

Fig. 3. $T_1^2(3K_1, 4K_1)$, $T_1^2(4K_1, 3K_1)$, $T_1^2(2K_1, 6K_1)$ and $T_1^2(6K_1, 2K_1)$

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