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COEFFICIENT INEQUALITY FOR CERTAIN SUBCLASS OF p-VALENT ANALYTIC FUNCTIONS WHOSE RECIPROCAL DERIVATIVE HAS A POSITIVE REAL PART

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Abstract. The objective of this paper is to introduce certain new subclass of p-valent analytic functions in the open unit disc $E = \{z : |z| < 1\}$ and obtain sharp upper bound for the second Hankel determinant of functions belonging to this class, using Toeplitz determinants.

1. Introduction

Let A_p (p is a fixed integer ≥ 1) denote the class of functions f of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \cdots, \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in N = \{1, 2, 3, \dots\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [11] as follows, and has been extensively studied by several authors in the literature. For example, Ehrenborg [2] studied the Hankel determinant of exponential polynomials. Noonan and Thomas [8] studied the second Hankel determinant of areally mean p-valent functions. Noor [9] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary rotation. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [5].

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Further sharp upper bound for the functional $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$, when $q = 2$

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and $n = 2$, known as the second Hankel determinant (functional) was obtained for various subclasses of univalent and multivalent analytic functions by several authors. Janteng et al. [4] considered the functional $|a_2a_4 - a_3^2|$ and found a sharp upper bound for functions f in the subclass RT of S, consisting of functions whose derivative has a positive real part (also called bounded turning functions), extensively studied by Mac Gregor [7]. In their work, they have shown that if $f \in RT$ then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. Similarly, the same coefficient inequality was calculated for certain subclasses of univalent and multivalent analytic functions by many authors in the literature.

Motivated by the above result, in the present paper, we introduce certain new subclass of p-valent analytic functions and consider the Hankel determinant $H_2(p+1)$ in the case of $q = 2$ and $n = p+1$, given by

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2$$

and we seek sharp upper bound to the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for function f given in (1.1), when it belongs to the new subclass, defined as follows.

Definition 1.1. A function $f \in A_p$ is said to be in the class $\widetilde{RT}_p$ with $p \in N$, consisting of p-valent analytic functions, whose reciprocal derivative has a positive real part, if it satisfies the condition

$$\operatorname{Re}\left[\frac{pz^{p-1}}{f'(z)}\right] > 0, \ \forall z \in E.$$

Some preliminary lemmas required for proving our result are given in the next section.

2. Preliminary results

Let $\mathcal{P}$ denote the class of functions consisting of g, such that

$$g(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_nz^n, \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\operatorname{Re} g(z) > 0$ for any $z \in E$. Here $g(z)$ is called a Caratheòdory function [1].

Lemma 2.1. [10, 12] *If $g \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. [3] *The power series for g given in (2.1) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3 \ldots$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z)$ with $\sum_{k=1}^{\infty} \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [3] is due to Carathe`odory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, for some complex valued x with $|x| \leq 1$ and for some complex valued z with $|z| \leq 1$, we have

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.2}$$

$$\text{and } 4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z. \tag{2.3}$$

To obtain our result, we refer to the classical method initiated by Libera and Zlotkiewicz [6], used by several authors in the literature.

3. Main result

THEOREM 3.1. *If $f(z) \in \widetilde{RT}_p$ with $p \in N$, then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{2p}{p+2}\right]^2$$

and the inequality is sharp.

Proof. For the function $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \widetilde{RT}_p$, by virtue of Definition 1.1, there exists an analytic function $g \in \mathcal{P}$ in the open unit disc E with $g(0) = 1$ and $\operatorname{Re} g(z) > 0$ such that

$$pz^{p-1} = g(z)f'(z). \tag{3.1}$$

Using the series representations for $f'(z)$ and $g(z)$ in (3.1), we have

$$pz^{p-1} = \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\left\{pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right\}.$$

Upon simplification, we obtain

$$\begin{aligned} 0 = \{c_1 p + (p+1)a_{p+1}\} z^p + \{c_2 p + c_1(p+1)a_{p+1} + (p+2)a_{p+2}\} z^{p+1} + \\ \{c_3 p + c_2(p+1)a_{p+1} + c_1(p+2)a_{p+2} + (p+3)a_{p+3}\} z^{p+2} + \cdots . \end{aligned}$$

Equating the coefficients of like powers of z^p, z^{p+1} and z^{p+2} respectively in (3.2), we can now write

$$a_{p+1} = \frac{-c_1 p}{p+1};\ a_{p+2} = \frac{p(c_1^2 - c_2)}{p+2};\ a_{p+3} = \frac{-p(c_3 - 2c_1c_2 + c_1^3)}{p+3}. \tag{3.3}$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from (3.3) in the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function $f \in \widetilde{RT}_p$, upon simplification, we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2 \left|(p+2)^2 c_1 c_3 - 2c_1^2 c_2 - (p+1)(p+3)c_2^2 + c_1^4\right|}{(p+1)(p+2)^2(p+3)}, \tag{3.4}$$

which is equivalent to

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2 \left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|}{(p+1)(p+2)^2(p+3)}, \tag{3.5}$$

where

$$d_1 = (p+2)^2;\ d_2 = -2;\ d_3 = -(p+1)(p+3);\ d_4 = 1. \tag{3.6}$$

Substituting the values of c_2 and c_3 given in (2.2) and (2.3) respectively from Lemma 2.2 on the right-hand side of (3.5), we have

$$\begin{aligned}
&|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\
&\quad = |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} + \\
&\qquad\qquad d_2c_1^2 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} + d_3 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 + d_4c_1^4|.
\end{aligned}$$

Using the triangle inequality and the fact $|z| < 1$, which simplifies to

$$\begin{aligned}
&4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq |(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2d_1c_1(4-c_1^2) + \\
&\quad 2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x| - \left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\}(4-c_1^2)|x|^2|.
\end{aligned} \tag{3.7}$$

From (3.6), we can now write

$$d_1 + 2d_2 + d_3 + 4d_4 = 1;\ d_1 = (p+2)^2;\ d_1 + d_2 + d_3 = -1; \tag{3.8}$$

$$(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3 = c_1^2 + 2(p+2)^2c_1 + 4(p+1)(p+3). \tag{3.9}$$

Consider

$$\begin{aligned}
&c_1^2 + 2(p+2)^2c_1 + 4(p+1)(p+3) \\
&\quad = \left[\left\{c_1 + (p+2)^2\right\}^2 - (p+2)^4 + 4(p+1)(p+3)\right] \\
&\quad = \left[\left\{c_1 + (p+2)^2\right\}^2 - \left(\sqrt{p^4 + 8p^3 + 20p^2 + 16p + 4}\right)^2\right] \\
&\quad = \left[c_1 + \left\{(p+2)^2 + \left(\sqrt{p^4 + 8p^3 + 20p^2 + 16p + 4}\right)\right\}\right] \times \\
&\qquad\qquad \left[c_1 + \left\{(p+2)^2 - \left(\sqrt{p^4 + 8p^3 + 20p^2 + 16p + 4}\right)\right\}\right].
\end{aligned}$$

Since $c_1 \in [0,2]$, noting that $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a, b \geq 0$, on the right-hand side of the above expression, we get

$$\begin{aligned}
&-\left\{c_1^2 + 2(p+2)^2c_1 + 4(p+1)(p+3)\right\} \leq \\
&\qquad\qquad -\left\{c_1^2 - 2(p+2)^2c_1 + 4(p+1)(p+3)\right\}.
\end{aligned} \tag{3.10}$$

From the relations (3.9) and (3.10), we have

$$-\left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} \leq -\left\{c_1^2 - 2(p+2)^2c_1 + 4(p+1)(p+3)\right\}. \tag{3.11}$$

Substituting the calculated values from (3.8) and (3.11) on the right-hand side of (3.7), we have

$$\begin{aligned}
&4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq |c_1^4 + 2(p+2)^2c_1(4-c_1^2) - \\
&\qquad 2c_1^2(4-c_1^2)|x| - \left\{c_1^2 - 2(p+2)^2c_1 + 4(p+1)(p+3)\right\}(4-c_1^2)|x|^2|.
\end{aligned}$$

Choosing $c_1 = c \in [0, 2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we obtain

$$\begin{aligned}4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\le \left[c^4 + 2(p+2)^2c(4-c^2)+\right.\\ &\left. 2c^2(4-c^2)\mu + \left\{c^2 - 2(p+2)^2c + 4(p+1)(p+3)\right\}(4-c^2)\mu^2\right]\\ &= F(c,\mu)\ ,\ 0 \le \mu = |x| \le 1 \text{ and } 0 \le c \le 2,\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}F(c,\mu) = &\left[c^4 + 2(p+2)^2c(4-c^2) + 2c^2(4-c^2)\mu+\right.\\ &\left.\left\{c^2 - 2(p+2)^2c + 4(p+1)(p+3)\right\}(4-c^2)\mu^2\right].\end{aligned} \tag{3.13}$$

We next maximize the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ given in (3.13) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = 2\left[c^2 + \left\{c^2 - 2(p+2)^2c + 4(p+1)(p+3)\right\}\mu\right](4-c^2). \tag{3.14}$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and $p \in N$, from (3.14), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c, \mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Moreover, for fixed $c \in [0, 2]$, we have

$$\max_{0 \le \mu \le 1} F(c,\mu) = F(c,1) = G(c). \tag{3.15}$$

Therefore, replacing μ by 1 in $F(c, \mu)$, which simplifies to give

$$G(c) = -2c^4 - 4p(p+4)c^2 + 16(p+1)(p+3), \tag{3.16}$$

$$G'(c) = -8c\left\{c^2 + p(p+4)\right\}. \tag{3.17}$$

From (3.17), we observe that $G'(c) \le 0$, for every $c \in [0, 2]$ with $p \in N$. Therefore, $G(c)$ is a decreasing function of c in the interval $[0, 2]$, whose maximum value occurs at $c = 0$ only, from (3.16), it is given by

$$G_{max} = G(0) = 16(p+1)(p+3). \tag{3.18}$$

Simplifying the expressions (3.7) and (3.18), we get

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \le\ 4(p+1)(p+3). \tag{3.19}$$

From the relations (3.5) and (3.19), we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \le \left[\frac{2p}{p+2}\right]^2. \tag{3.20}$$

By setting $c_1 = c = 0$ and selecting $x = 1$ in the expressions (2.2) and (2.3), we find that $c_2 = 2$ and $c_3 = 0$ respectively. Substituting these values in (3.19) together with the values in (3.6), we observe that equality is attained, which shows that our result is sharp. For these values, from (2.1), we can derive the extremal function, given by

$$\frac{pz^{p-1}}{f'(z)} = 1 + 2z^2 + 2z^4 + \cdots = \frac{1+z^2}{1-z^2}.$$

This completes the proof of our theorem. ∎

Remark 3.2. It is observed that the sharp upper bound to the second Hankel determinant of a function whose derivative has a positive real part and a function whose reciprocal derivative has a positive real part for a p-valent function is the same.

Remark 3.3. Choosing $p = 1$ in (3.20), we get $|a_2a_4 - a_3^2| \leq \frac{4}{9}$, it coincides with that of Janteng et al. [7]. From this, we conclude that the sharp upper bound to the second Hankel determinant of a function whose derivative has a positive real part and a function whose reciprocal derivative has a positive real part is the same.

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REFERENCES

[**1**] P. L. Duren, *Univalent functions*, Vol. 259, Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 1983.

[**2**] R. Ehrenborg, *The Hankel determinant of exponential polynomials*, Amer. Math. Monthly, **107** (6) (2000), 557–560.

[**3**] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, 2nd ed., Chelsea Publishing Co., New York, 1984.

[**4**] A. Janteng, S. A. Halim and M. Darus, *Coefficient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math., **7** (2)(2006), 1–5.

[**5**] J. W. Layman, *The Hankel transform and some of its properties*, J. Integer Seq., **4** (1) (2001), 1–11.

[**6**] R. J. Libera and E. J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc., **87** (2) (1983), 251–257.

[**7**] T. H. Mac Gregor, *Functions whose derivative have a positive real part*, Trans. Amer. Math. Soc, **104** (3) (1962), 532–537.

[**8**] J. W. Noonan and D. K. Thomas, *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc., **223** (2) (1976), 337–346.

[**9**] K. I. Noor, *Hankel determinant problem for the class of functions with bounded boundary rotation*,Rev. Roumaine Math. Pures Appl., **28** (8) (1983), 731–739.

[**10**] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.

[**11**] Ch. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc., **41** (1966), 111–122.

[**12**] B. Simon, *Orthogonal polynomials on the unit circle, Part 1. Classical theory*, American Mathematical Society Colloquium Publications, 54, Part 1., American Mathematical Society, Providence, RI, 2005.

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