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AN ENGEL CONDITION OF GENERALIZED DERIVATIONS WITH ANNIHILATOR ON LIE IDEAL IN PRIME RINGS

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Abstract. Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ extended centroid of R, F a nonzero generalized derivation of R, L a noncentral Lie ideal of R and $k \geq 2$ a fixed integer. Suppose that there exists $0 \neq a \in R$ such that $a[F(u^{n_1}), u^{n_2}, \dots, u^{n_k}] = 0$ for all $u \in L$, where $n_1, n_2, \dots, n_k \geq 1$ are fixed integers. Then either there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, or R satisfies s_4, the standard identity in four variables.

1. Introduction

Let R be an associative ring. For $x, y \in R$, the commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. A Lie ideal L of R is an additive subgroup of R such that $[L, R] \subseteq L$. The Engel type identity is defined by $[x, y]_k = [[x, y]_{k-1}, y]$ for all $x, y \in R$, where $k \geq 2$ is an integer. We denote $[x_1, x_2, \dots, x_n] = [[x_1, x_2, \dots, x_{n-1}], x_n]$ for all $x_1, x_2, \dots, x_n \in R$, for every positive integer $n \geq 2$. The standard polynomial identity s_4 in four variables is defined as $s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$ where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or odd permutation in symmetric group S_4.

Throughout this paper, unless specifically stated, R will always represent a prime ring with center $Z(R)$, extended centroid C and U is its Utumi quotient ring. For the properties of U and C, we refer the reader to [1]. By d we mean a derivation of R.

A well known result proved by Posner [14] states that if the commutator $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. The result of Posner was generalized in many directions by a number of authors.

Lanski generalized the Posner's theorem by considering Engel condition in [9]. He proved that if L is a noncommutative Lie ideal of R such that $[d(x), x]_k = 0$ for all $x \in L$, where $k \geq 1$ is a fixed integer, then char $(R) = 2$ and $R \subseteq M_2(K)$ for a field K.

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Later in [8], Lanski studied the more general situation $[d(x^{t_0}), x^{t_1}, \dots, x^{t_n}] = 0$ for all $x \in I$, where I is a nonzero left ideal of semiprime ring R and $t_0, \dots, t_n \geq 1$ are fixed integers. In particular, Lanski proved that if R is prime ring and d is nonzero, then R must be commutative.

In [5], Dhara et al. generalized the Lanski's result [8] replacing derivation by a generalized derivation. An additive map $F : R \to R$ is called generalized derivation, if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. A significant example is a map of the form $F(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner. In [5], Dhara et al. proved that if $[F(u^{n_1}), u^{n_2}, \dots, u^{n_k}] = 0$ holds for all $u \in L$, where L is a noncentral Lie ideal of R, and $k \geq 2$, $n_1, \dots, n_k \geq 1$ are fixed integers, then there exists $\alpha \in C$ such that $F(x) = \alpha x$ for all $x \in R$, unless R satisfies s_4, the standard identity in four variables.

In [17], Shiue studied the left annihilator of the set $\{[d(u), u]_k = 0, u \in L\}$, where L is a noncentral Lie ideal of R, $d \neq 0$ and $k \geq 1$. In case the annihilator is not zero, the conclusion is that R satisfies s_4 and $char(R) = 2$. Moreover, Shiue [18] obtained the same conclusion in case the left annihilator of the set $\{[d(u^n), u^n]_k = 0, u \in L\}$ is nonzero, where L is a noncentral Lie ideal of R, $d \neq 0$ and $k \geq 1$, $n \geq 1$. Recently, in [15] Scudo proved that if for some $0 \neq a \in R$, $a[F(x), x]_k \in Z(R)$ for all $x \in L$, where L is a noncentral Lie ideal, F a generalized derivation of R and $k \geq 1$ fixed integer, then one of the following holds: (1) there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$; (2) char $(R) = 2$ and R satisfies s_4; (3) R satisfies s_4 and there exist $q \in U$ and $\gamma \in C$ such that $F(x) = qx + xq + \gamma x$ for all $x \in R$.

Following this line of investigation, in this paper we prove the following theorems.

Theorem 1.1. *Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ extended centroid of R, F a nonzero generalized derivation of R, L a noncentral Lie ideal of R and $k \geq 2$ a fixed integer. Suppose that there exists $0 \neq a \in R$ such that $a[F(u^{n_1}), u^{n_2}, \dots, u^{n_k}] = 0$ for all $u \in L$, where $n_1, n_2, \dots, n_k \geq 1$ are fixed integers. Then either there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, or R satisfies s_4, the standard identity in four variables.*

Theorem 1.2. *Let R be a prime ring of characteristic different from 2, with its Utumi ring of quotient U, C the extended centroid of R, F a nonzero generalized derivation of R and $k \geq 2$ a fixed integer. Suppose that there exists $0 \neq a \in R$ such that $a[F(x^{n_1}), x^{n_2}, x^{n_3}, \dots, x^{n_k}] = 0$ for all $x \in R$, where $n_1, n_2, \dots, n_k \geq 1$ are fixed integers. Then there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$.*

In [16], Shiue studied the situation $[d(u^m)u^n - u^p\delta(u^q), u^r]_k = 0$ for all $u \in L$, where m, n, p, q, k are fixed positive integers and d, δ two derivations of R and obtained that either R satisfies s_4 or $d = \delta = 0$. Our next theorem investigate the situation with left annihilator condition.

Theorem 1.2. *Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ extended centroid of R, d and δ two nonzero derivations of R and*

L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a[d(u^{n_1})u^{n_2} - u^{n_3}\delta(u^{n_4}), u^{n_5}, \dots, u^{n_k}] = 0$ for all $u \in L$, where $k \geq 5$ and $n_1, n_2, \dots, n_k \geq 1$ are fixed integers. Then either $d = \delta = 0$, or R satisfies s_4, the standard identity in four variables.

We need the following remarks:

REMARK 1. Let R be a prime ring and L a noncentral Lie ideal of R. If char $(R) \neq 2$, by [2, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If char $(R) = 2$ and $dim_C RC > 4$ i.e., char $(R) = 2$ and R does not satisfy s_4, then by [10, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. Thus if either char $(R) \neq 2$ or R does not satisfy s_4, then we may conclude that there exists a nonzero ideal I of R such that $[I, I] \subseteq L$.

REMARK 2. Let R be a prime ring and U be the Utumi quotient ring of R and $C = Z(U)$, the center of U (see [1] for more details). It is well known that any derivation of R can be uniquely extended to a derivation of U. In [11, Theorem 3], Lee proved that every generalized derivation g on a dense right ideal of R can be uniquely extended to a generalized derivation of U. Furthermore, the extended generalized derivation g has the form $g(x) = ax + d(x)$ for all $x \in U$, where $a \in U$ and d is a derivation of U.

REMARK 3. Let R be a prime ring and U be its Utumi quotient ring and $C = Z(U)$. Let $X = \{x_1, \dots, x_n, \dots\}$, the countable set consisting of the noncommuting indeterminates $x_1, \dots, x_n, \dots$. Consider $T = U *_C C\{X\}$, the free product over C of the C-algebra U and the free C-algebra $C\{X\}$.

The elements of T are called the generalized polynomials with coefficients in U. By a nontrivial generalized polynomial, we mean a nonzero element of T. An element $m \in T$ of the form $m = q_0 y_1 q_1 y_2 q_2 \dots y_n q_n$, where $\{q_0, q_1, \dots, q_n\} \subseteq U$ and $\{y_1, y_2, \dots, y_n\} \subseteq X$, is called a monomial. $q_0, q_1, \dots, q_n$ are called the coefficients of m. Each $f \in T$ can be represented as a finite sum of monomials.

Note that if I is a non-zero ideal of R, then I, R and U satisfy the same generalized polynomial identities with coefficients in U. For more details about these objects we refer the reader to [1] and [3].

2. Main Results

We begin with two lemmas.

LEMMA 2.1. *Let R be a prime ring with extended centroid C and $a, b, c \in R$. If $a \neq 0$ such that*

$$a\Big[[b, [x_1, x_2]^{n_1}][x_1, x_2]^{n_2} - [x_1, x_2]^{n_3}[c, [x_1, x_2]^{n_4}], [x_1, x_2]^{n_5}, \dots, [x_1, x_2]^{n_k}\Big] = 0$$

for all $x_1, x_2 \in R$, where $n_1, n_2, \dots, n_k \geq 1$ are fixed integers, then either R satisfies a nontrivial generalized polynomial identity (GPI) or $b, c \in C$.

Proof. Assume that R does not satisfy any nontrivial GPI. Let $T = U *_C C\{x_1, x_2\}$, the free product of U and $C\{x_1, x_2\}$, the free C-algebra in noncommuting indeterminates x_1 and x_2. If R is commutative, then R satisfies trivially a nontrivial GPI, a contradiction. So, R must be noncommutative.

Then,

$$a\Big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2}-[x_1,x_2]^{n_3}[c,[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}\Big]=0\in T. \tag{1}$$

If $c\notin C$, then c and 1 are linearly independent over C. Thus, (1) implies

$$a[x_1,x_2]^{n_3+n_4+\dots+n_k}c=0$$

in T implying $c=0$, since $a\neq 0$, a contradiction. Therefore, we conclude that $c\in C$ and hence (1) reduces to

$$a\Big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2},[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}\Big]=0 \tag{2}$$

in T. If $b\notin C$, then (2) implies

$$a[x_1,x_2]^{n_1+n_5+n_6+\dots+n_k}b[x_1,x_2]^{n_2}=0$$

in T again implying $b=0$, a contradiction. Therefore, $b\in C$. ∎

Lemma 2.2. *Let R be a noncommutative prime ring with extended centroid C and $b,c\in R$. Suppose that there exists $0\neq a\in R$ such that*

$$a\Big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2}-[x_1,x_2]^{n_3}[c,[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}\Big]=0$$

for all $x_1,x_2\in R$, where $n_1,n_2,\dots,n_k\geq 1$ are all fixed integers. Then either $b,c\in C$, or R satisfies s_4.

Proof. Suppose that R does not satisfy s_4. We have that R satisfies generalized polynomial identity

$$f(x_1,x_2)=$$
$$a\Big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2}-[x_1,x_2]^{n_3}[c,[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}\Big]. \tag{3}$$

If R does not satisfy any nontrivial GPI, by Lemma 2.1, we obtain $b,c\in C$ and we are done. So, we assume that R satisfies a nontrivial GPI. Since R and U satisfy the same generalized polynomial identities (see [3]), U satisfies $f(x_1,x_2)$. In case C is infinite, we have $f(x_1,x_2)=0$ for all $x_1,x_2\in U\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Moreover, both U and $U\otimes_C\overline{C}$ are prime and centrally closed algebras [4]. Hence, replacing R by U or $U\otimes_C\overline{C}$ according to C finite or infinite, without loss of generality we may assume that $C=Z(R)$ and R is C-algebra centrally closed. By Martindale's theorem [13], R is then a primitive ring having nonzero socle $soc(R)$ with C as the associated division ring. Hence, by Jacobson's theorem [6, p. 75], R is isomorphic to a dense ring of linear transformations of a vector space V over C.

If $\dim_C V=2$, then $R\cong M_2(C)$. This implies that R satisfies s_4, a contradiction. So let $\dim_C V\geq 3$.

We show that for any $v \in V$, v and cv are linearly C-dependent. Suppose that v and cv are linearly independent for some $v \in V$. Since $\dim_C V \geq 3$, there exists $u \in V$ such that v, cv, u are linearly C-independent set of vectors. By density, there exist $x_1, x_2 \in R$ such that

$$x_1 v = v, \quad x_1 cv = 0, \quad x_1 u = cv; \quad x_2 v = 0, \quad x_2 cv = u, \quad x_2 u = 0.$$

Then

$$\begin{aligned} 0 &= a\Big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2} - [x_1,x_2]^{n_3}[c,[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\ldots,[x_1,x_2]^{n_k}\Big]v \\ &= acv. \end{aligned}$$

This implies that if $acv \neq 0$, then by contradiction we may conclude that v and cv are linearly C-dependent. Now choose $v \in V$ such that v and cv are linearly C-independent. Set $W = Span_C\{v, cv\}$. Then $acv = 0$. Let $ac \neq 0$. Then, there exists $w \in V$ such that $acw \neq 0$ and then $ac(v-w) = acw \neq 0$. By the previous argument we have that w, cw are linearly C-dependent and $(v-w), c(v-w)$ too. Thus there exist $\alpha, \beta \in C$ such that $cw = \alpha w$ and $c(v-w) = \beta(v-w)$. Then $cv = \beta(v-w) + cw = \beta(v-w) + \alpha w$ i.e., $(\alpha - \beta)w = cv - \beta v \in W$. Now $\alpha = \beta$ implies that $cv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$. Again, if $u \in V$ with $acu = 0$ then $ac(w+u) \neq 0$. So, $w + u \in W$ forcing $u \in W$. Thus it is observed that $w \in V$ with $acw \neq 0$ implies $w \in W$ and $u \in V$ with $acu = 0$ implies $u \in W$. This implies that $V = W$ i.e., $\dim_C V = 2$, a contradiction.

Hence, v and cv are linearly C-dependent for all $v \in V$, unless $ac = 0$. Thus for each $v \in V$, $cv = \alpha_v v$ for some $\alpha_v \in C$. It is very easy to prove that α_v is independent of the choice of $v \in V$. Thus we can write $cv = \alpha v$ for all $v \in V$ and $\alpha \in C$ fixed. Now let $r \in R$, $v \in V$. Since $cv = \alpha v$,

$$[c,r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[c,r]v = 0$ for all $v \in V$ i.e., $[c,r]V = 0$. Since $[c,r]$ acts faithfully as a linear transformation on the vector space V, $[c,r] = 0$ for all $r \in R$. Therefore, $c \in Z(R)$, unless $ac = 0$. Now let $ac = 0$. Since $\dim_C V \geq 3$, there exists $w \in V$ such that v, cv, w are linearly C-independent set of vectors. By density, there exist $x_1, x_2 \in R$ such that

$$x_1 v = v, \quad x_1 cv = 0, \quad x_1 w = v + cv; \quad x_2 v = 0, \quad x_2 cv = w, \quad x_2 w = 0.$$

Then

$$\begin{aligned} 0 &= a\Big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2} - [x_1,x_2]^{n_3}[c,[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\ldots,[x_1,x_2]^{n_k}\Big]v \\ &= av. \end{aligned}$$

Then by the above argument, since $a \neq 0$, $c \in C$.

Now our hypothesis (3) becomes

$$a\Big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2},[x_1,x_2]^{n_5},\ldots,[x_1,x_2]^{n_k}\Big] = 0$$

for all $x_1, x_2 \in R$. Let for any $v \in V$, v and bv are linearly C-independent. Since $\dim_C V \geq 3$, there exists $w \in V$ such that v, bv, w are linearly C-independent set of vectors. By density, there exist $x_1, x_2 \in R$ such that

$$x_1 v = 0, \quad x_1 bv = v, \quad x_1 w = bv; \quad x_2 v = bv, \quad x_2 bv = w, \quad x_2 w = 0,$$

which implies $0 = a\big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2},[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}\big]v = abv$. By the same argument as earlier we have either $b \in C$ or $ab = 0$.

Let $ab = 0$. Again by density, there exist $x_1, x_2 \in R$ such that

$$x_1v = 0,\quad x_1bv = v,\quad x_1w = v + bv;\quad x_2v = bv,\quad x_2bv = w,\quad x_2w = 0.$$

Then $[x_1,x_2]v = (x_1x_2 - x_2x_1)v = v$, $[x_1,x_2]bv = (x_1x_2 - x_2x_1)bv = v$ and hence

$$0 = a\Big[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2},[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}\Big]v = -av.$$

Again, by the same argument as earlier we conclude either $b \in C$ or $a = 0$. Since $a \neq 0$, $b \in C$. ■

Proof of Theorem 1.1. Suppose that R does not satisfy s_4. Since L is a noncentral Lie ideal of R, by Remark 1, there exists a nonzero ideal I of R such that $[I,I] \subseteq L$. Hence, by our assumption, we have,

$$a[F([x_1,x_2]^{n_1}),[x_1,x_2]^{n_2},\dots,[x_1,x_2]^{n_k}] = 0$$

for all $x_1, x_2 \in I$. Since I, R and U satisfy the same generalized polynomial identities (see [3]) as well as the same differential identities (see [12]), they also satisfy the same generalized differential identities. Hence,

$$a[F([x_1,x_2]^{n_1}),[x_1,x_2]^{n_2},\dots,[x_1,x_2]^{n_k}] = 0$$

for all $x_1, x_2 \in U$. By Remark 2, there exist $b \in U$ and a derivation d of U such that $F(x) = bx + d(x)$ for all $x \in U$. Hence, U satisfies

$$a[b[x_1,x_2]^{n_1} + d([x_1,x_2]^{n_1}),[x_1,x_2]^{n_2},\dots,[x_1,x_2]^{n_k}] = 0. \tag{4}$$

Now we divide the proof into two cases:

Case I: Let for some $p \in U$, $d(x) = [p,x]$ for all $x \in U$ that is, d is an inner derivation of U. Then from (4), we obtain that U satisfies

$$a[(b+p)[x_1,x_2]^{n_1} - [x_1,x_2]^{n_1}p,[x_1,x_2]^{n_2},\dots,[x_1,x_2]^{n_k}] = 0$$

that is

$$a[[(b+p),[x_1,x_2]^{n_2}][x_1,x_2]^{n_1} - [x_1,x_2]^{n_1}[p,[x_1,x_2]^{n_2}],[x_1,x_2]^{n_3},\dots,[x_1,x_2]^{n_k}] = 0.$$

By Lemma 2.2, since R and so U does not satisfy s_4, we have $b+p, p \in C$. This implies $F(x) = bx$ for all $x \in U$ and so for all $x \in R$, where $b \in C$. Thus the conclusion is obtained.

Case II: Next assume that d is not inner derivation of U. Then by Kharchenko's theorem [7], we have from (4) that U satisfies

$$a[b[x_1,x_2]^{n_1} + \sum_{i=0}^{n_1-1}[x_1,x_2]^i([y_1,x_2]+[x_1,y_2])[x_1,x_2]^{n_1-1-i},[x_1,x_2]^{n_2},\dots,[x_1,x_2]^{n_k}] = 0. \tag{5}$$

Since R and so U is noncommutative, there exists some $q \in U$ such that $q \notin C$. Now replacing y_1 with $[q,x_1]$ and y_2 with $[q,x_2]$ in (5), where $q \notin C$, we can write that U satisfies

$$a[(b+q)[x_1,x_2]^{n_1} - [x_1,x_2]^{n_1}q,[x_1,x_2]^{n_2},\dots,[x_1,x_2]^{n_k}] = 0.$$

Then by same argument as earlier of inner derivation case, we have $q \in C$, a contradiction. Thus the proof of theorem is complete. ∎

Proof of Theorem 1.2. By Theorem 1.1, we consider only the case when R satisfies s_4. In this case R is a PI-ring, and so there exists a field K such that $R \subseteq M_2(K)$ and both R and $M_2(K)$ satisfy the same GPI. Let F be inner generalized derivation on R. Then $F(x) = bx + xc$ for all $x \in R$. So our hypothesis becomes

$$a[bx^{n_1} + x^{n_1}c, x^{n_2}, x^{n_3}, \ldots, x^{n_k}] = 0.$$

Here R is a dense ring of K-linear transformations over a vector space V. Our first aim is to show that for any $v \in V$, cv and v are linearly K-dependent. If not, Assume there exists $v \neq 0$, such that $\{v, cv\}$ is linearly K-independent. By the density of R, there exists $x \in R$ such that $xv = 0$ and $xcv = cv$. So we have

$$0 = a[bx^{n_1} + x^{n_1}c, x^{n_2}, x^{n_3}, \ldots, x^{n_k}]v = acv.$$

Of course for any $u \in V$, $\{u, v\}$ linearly K-dependent implies $acu = 0$. Let $ac \neq 0$. Then there exists $w \in V$ such that $acw \neq 0$ and so $\{w, v\}$ are linearly K-independent. Also $ac(w + v) = acw \neq 0$ and $ac(w - v) = acw \neq 0$. By the above argument, it follows that w and cw are linearly K-dependent, as are $\{w+v, c(w+v)\}$ and $\{w - v, c(w - v)\}$. Therefore, there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in K$ such that

$$cw = \alpha_w w, \quad c(w + v) = \alpha_{w+v}(w + v), \quad c(w - v) = \alpha_{w-v}(w - v).$$

In other words, we have

$$\alpha_w w + cv = \alpha_{w+v} w + \alpha_{w+v} v \tag{6}$$

and

$$\alpha_w w - cv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{7}$$

By comparing (6) with (7) we get both

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \tag{8}$$

and

$$2cv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \tag{9}$$

By (8), and since $\{w, v\}$ are K-independent and $char(K) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Thus by (9) it follows $2cv = 2\alpha_w v$. This leads to a contradiction with the fact that $\{v, cv\}$ is linear K-independent. Therefore, v and cv are linearly K-dependent for all $v \in V$, unless $ac = 0$.

Now let $ac = 0$. By the density of R, there exists $x \in R$ such that $xv = 0$ and $xcv = v + cv$. Then we have $0 = a[bx^{n_1} + x^{n_1}c, x^{n_2}, x^{n_3}, \ldots, x^{n_k}]v = av$. By the same argument as above, since $a \neq 0$, v and cv are linearly K-dependent for all $v \in V$. Thus in any case we have v and cv are linearly K-dependent for all $v \in V$. Then for each $v \in V$, $cv = \alpha_v v$ for some $\alpha_v \in K$. It is very easy to prove that α_v is independent of the choice of $v \in V$. Thus we can write $cv = \alpha v$ for all $v \in V$, where $\alpha \in K$ is fixed. Now let $r \in R$, $v \in V$. Since $cv = \alpha v$,

$$[c, r]v = (cr)v - (rc)v = c(rv) - r(cv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[c, R]V = 0$. Since $[c, R]$ acts faithfully as a linear transformation on the vector space V, $[c, R] = 0$. Therefore, $c \in Z(R)$.

Therefore our identity reduces to

$$a[b, x^{n_2}, x^{n_3}, \ldots, x^{n_k}]x^{n_1} = 0 \tag{10}$$

for all $x \in R$. Let us assume that there exists $0 \neq v \in V$ such that bv and v are linearly K-independent. By density of R there exists $x \in R$ such that $xv = v, xbv = 0$. Then we have $0 = a[b, x^{n_2}, x^{n_3}, \ldots, x^{n_k}]x^{n_1} = abv$. By the same argument, either $b \in Z(R)$ or $ab = 0$.

Now let $ab = 0$. In this case we put $x = e_{11}$ in (10). Then we have $0 = a[b, e_{11}, e_{11}, \ldots, e_{11}]e_{11} = a[b, e_{11}]e_{11} = ae_{11}be_{11}$, since $ab = 0$. This implies $a_{21}b_{11} = 0 = a_{11}b_{11}$. Similarly, by putting $x = e_{22}$ in (10), we get $a_{22}b_{22} = 0 = a_{12}b_{22}$. Moreover, $ab = 0$ implies

$$\begin{aligned} a_{11}b_{11} + a_{12}b_{21} &= 0, \\ a_{11}b_{12} + a_{12}b_{22} &= 0, \\ a_{21}b_{11} + a_{22}b_{21} &= 0, \\ a_{21}b_{12} + a_{22}b_{22} &= 0. \end{aligned}$$

Using these facts we get from above that $a_{12}b_{21} = a_{11}b_{12} = a_{22}b_{21} = a_{21}b_{12} = 0$. Now we assert that b is diagonal. If not, then at least one of non-diagonal elements of b must be nonzero. Without loss of generality, let us assume that $b_{12} \neq 0$. Then $a_{11} = a_{21} = 0$. For any automorphism θ of R, $\theta(a)$ and $\theta(b)$ satisfy the same property of a and b. Let $\theta(x) = (1+e_{21})x(1-e_{21})$. Denote by $\theta(a)_{ij}$ the (i, j)-entry of $\theta(a)$ and by $\theta(b)_{ij}$ the (i, j)-entry of $\theta(b)$. Now $\theta(b)_{12} = b_{12} \neq 0$ implies that $\theta(a)_{11} = \theta(a)_{21} = 0$, that is $0 = \theta(a)_{11} = -a_{12}$ and $0 = \theta(a)_{21} = -a_{12} - a_{22} = -a_{22}$. Thus $a = 0$, a contradiction. Therefore, b is a diagonal matrix.

Now since we have

$$\theta(a)[\theta(b), x^{n_2}, x^{n_3}, \ldots, x^{n_k}]x^{n_1} = 0$$

with $\theta(a) \neq 0$ and $\theta(a)\theta(b) = \theta(ab) = 0$, $\theta(b)$ is also diagonal that is, $0 = \theta(b)_{21} = b_{11} - b_{22}$ implying $b_{11} = b_{22}$. Therefore $b \in Z(R)$. Moreover, in this case $b = 0$, since if $b \neq 0$ then $ab = 0$ implies $a = 0$, which is a contradiction. Therefore, $F(x) = cx$ for all $x \in R$, where $c \in C$.

Next assume that $F(x) = bx + d(x)$, where d is not inner derivation of R. In this case our hypothesis reduces to

$$a[bx^{n_1} + d(x^{n_1}), x^{n_2}, x^{n_3}, \ldots, x^{n_k}] = 0$$

that is

$$a[bx^{n_1} + \sum_{i=0}^{n_1-1} x^i d(x) x^{n_1-i-1}, x^{n_2}, x^{n_3}, \ldots, x^{n_k}] = 0$$

for all $x \in R$. By Kharchenko's theorem [7], R satisfies

$$a[bx^{n_1} + \sum_{i=0}^{n_1-1} x^i y x^{n_1-i-1}, x^{n_2}, x^{n_3}, \ldots, x^{n_k}] = 0.$$

Now replacing y by $[q,x]$, where $q\notin C$, we have

$$a[bx^{n_1}+[q,x^{n_1}],x^{n_2},x^{n_3},\dots,x^{n_k}]=0$$

that is

$$a[(b+q)x^{n_1}-x^{n_1}q,x^{n_2},x^{n_3},\dots,x^{n_k}]=0$$

for all $x\in R$. Then by above arguments, $q\in C$, which is a contradiction. ■

Proof of Theorem 1.3. Suppose that R does not satisfy s_4. Since L is a noncentral Lie ideal of R, by Remark 1, there exists a nonzero ideal I of R such that $[I,I]\subseteq L$. Hence, by our assumption, we have,

$$a[d([x_1,x_2]^{n_1})[x_1,x_2]^{n_2}-[x_1,x_2]^{n_3}\delta([x_1,x_2]^{n_4}),[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}]=0$$

for all $x_1,x_2\in I$. Since I, R and U satisfy the same differential identities (see [12]),

$$a[d([x_1,x_2]^{n_1})[x_1,x_2]^{n_2}-[x_1,x_2]^{n_3}\delta([x_1,x_2]^{n_4}),[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}]=0 \quad (11)$$

for all $x_1,x_2\in U$.

Now we divide the proof into two following cases:

Case I: Let $d(x)=[b,x]$ for all $x\in U$ and $\delta(x)=[c,x]$ for all $x\in U$, are two inner derivations of U, where $b,c\in U$. Then from (11), we have

$$a[[b,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2}-[x_1,x_2]^{n_3}[c,[x_1,x_2]^{n_4})],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}]=0$$

for all $x_1,x_2\in U$. By Lemma 2.2, we conclude that $b,c\in C$, implying $d=\delta=0$.

Case II: Let d and δ are not both inner derivations of U.

Sub-case i: Let d and δ be C-dependent modulo inner derivations of U. Then there exist $\alpha,\beta\in C$ such that $\alpha d+\beta\delta=ad_p$, where $ad_p(x)=[p,x]$ for all $x\in U$.

If $\alpha\neq 0$, then $d=\lambda\delta+ad_q$, where $\lambda=-\beta\alpha^{-1}$ and $q=p\alpha^{-1}$. Then (11) gives

$$a[\lambda\delta([x_1,x_2]^{n_1})[x_1,x_2]^{n_2}+[q,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2} \\ -[x_1,x_2]^{n_3}\delta([x_1,x_2]^{n_4}),[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}]=0$$

for all $x_1,x_2\in U$. Applying Kharchenko's theorem [7], we can write

$$a[\lambda(\sum_{i=0}^{n_1-1}[x_1,x_2]^i([y,x_2]+[x_1,z])[x_1,x_2]^{n_1-i-1})[x_1,x_2]^{n_2}+[q,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2} \\ -[x_1,x_2]^{n_3}(\sum_{i=0}^{n_4-1}[x_1,x_2]^i([y,x_2]+[x_1,z])[x_1,x_2]^{n_4-i-1}),[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}] \\ =0 \quad (12)$$

for all $x_1,x_2\in U$. Since L is noncentral, U must be noncommutative and hence there exits $q'\in U$ such that $q'\notin C$. Now replacing y with $[q',x_1]$ and z with $[q',x_2]$ in (12), we get

$$a[[\lambda q'+q,[x_1,x_2]^{n_1}][x_1,x_2]^{n_2} \\ -[x_1,x_2]^{n_3}[q',[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}]=0$$

for all $x_1,x_2\in U$. By Lemma 2.2, we get $q'\in C$, a contradiction.

If $\alpha = 0$, then $\delta = ad_{p'}$, where $p' = p\beta^{-1}$. Then (11) becomes

$$a[d([x_1,x_2]^{n_1})[x_1,x_2]^{n_2} - [x_1,x_2]^{n_3}[p',[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}] = 0$$

for all $x_1, x_2 \in U$. Since δ is inner, d can not be inner. Hence by Kharchenko's theorem [7], we can write

$$a[(\sum_{i=0}^{n_1-1}[x_1,x_2]^i([y,x_2]+[x_1,z])[x_1,x_2]^{n_1-i-1})[x_1,x_2]^{n_2} - [x_1,x_2]^{n_3}[p',[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}] = 0 \quad (13)$$

for all $x_1, x_2, y, z \in U$. Now replacing y with $[q', x_1]$ and z with $[q', x_2]$ in (13), for some $q' \notin C$, we get

$$a[[q',[x_1,x_2]^{n_1}][x_1,x_2]^{n_2} - [x_1,x_2]^{n_3}[p',[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}] = 0$$

for all $x_1, x_2 \in U$. Then by Lemma 2.2, we get $q' \in C$, a contradiction.

Sub-case ii: Let d and δ be C-independent modulo inner derivations of U. Then applying Kharchenko's theorem [7] to (11), we have

$$a[(\sum_{i=0}^{n_1-1}[x_1,x_2]^i([y,x_2]+[x_1,z])[x_1,x_2]^{n_1-i-1})[x_1,x_2]^{n_2} - [x_1,x_2]^{n_3}(\sum_{i=0}^{n_4-1}[x_1,x_2]^i([u,x_2]+[x_1,v])[x_1,x_2]^{n_4-i-1}),[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}] = 0 \quad (14)$$

for all $x_1, x_2, y, z, u, v \in U$. Then again replacing y and u with $[q', x_1]$ and z and v with $[q', x_2]$ in (12), for some $q' \notin C$, (14) becomes

$$a[[q',[x_1,x_2]^{n_1}][x_1,x_2]^{n_2} - [x_1,x_2]^{n_3}[q',[x_1,x_2]^{n_4}],[x_1,x_2]^{n_5},\dots,[x_1,x_2]^{n_k}] = 0$$

for all $x_1, x_2 \in U$. Then again by Lemma 2.2, we get $q' \in C$, a contradiction. ∎

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