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S-PARACOMPACTNESS IN IDEAL TOPOLOGICAL SPACES

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Abstract. In this paper, we study the notion of S-paracompact spaces in ideal topological spaces. We provide some characterizations of these spaces and investigate relationships to other classes of spaces. Moreover, we study the invariance of such spaces under some special types of functions.

1. Introduction and preliminaries

In 2006, Al-Zoubi [2] introduced a weaker version of paracompactness called S-paracompactness. A space is said to be S-paracompact if every open cover of the space has a locally finite semi-open refinement. In this paper, we introduce and investigate a new class of spaces, namely $\mathcal{I}$-S-paracompact spaces, which are defined on an ideal topological space. This class contains S-paracompact and $\mathcal{I}$-paracompact spaces [9].

Recall that an ideal $\mathcal{I}$ on a nonempty set X is a nonempty collection of subsets of X closed under the operations of subset and finite union. Although the use of ideals on topological spaces has been considered since 1930 (see [11]), in recent years this theory has taken a leading role in the generalization of some topological notions such as regularity, compactness, paracompactness and nearly paracompactness (see [7, 9, 14, 19]).

Throughout this paper, (X, τ) always denotes a topological space on which no separation axioms are assumed unless explicitly stated. If A is a subset of (X, τ), then we denote the closure of A and the interior of A by $\mathrm{Cl}\,(A)$ and $\mathrm{Int}\,(A)$, respectively. Also, we denote by $\wp(X)$ the class of all subsets of X. A subset A of (X, τ) is said to be *semi-open* [12] if there exists $U \in \tau$ such that $U \subset A \subset \mathrm{Cl}\,(U)$. This is equivalent to say that $A \subset \mathrm{Cl}\,(\mathrm{Int}\,(A))$. A subset A of (X, τ) is called *regular open* (resp. *α-open* [15]) if $A = \mathrm{Int}\,(\mathrm{Cl}\,(A))$ (resp. $A \subset \mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(A)))$). The complement of a semi-open (resp. regular open) set is called a *semi-closed*

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(resp. *regular closed*) set. The *semi-closure* of A, denoted by $\mathrm{sCl}\,(A)$, is defined by the intersection of all semi-closed sets containing A. The collection of all semi-open (resp. regular open, α-open) sets of a topological space (X,τ) is denoted by $\mathrm{SO}\,(X,\tau)$ (resp. $\mathrm{RO}\,(X,\tau)$, τ^α). It is known that τ^α forms a topology on X such that $\tau \subset \tau^\alpha \subset \mathrm{SO}\,(X,\tau)$ and $\mathrm{SO}\,(X,\tau^\alpha) = \mathrm{SO}\,(X,\tau)$ [15]. A space (X,τ) is said to be *extremally disconnected* (briefly e.d.) if the closure of every open set in (X,τ) is open. A collection $\mathcal{V}$ of subsets of a space (X,τ) is said to be *locally finite* (resp. *s-locally finite* [1]), if for each $x \in X$ there exists $U_x \in \tau$ (resp. $U_x \in \mathrm{SO}\,(X,\tau)$) containing x and U_x intersects at most finitely many members of $\mathcal{V}$. A space (X,τ) is said to be *paracompact* [3] (resp. *S-paracompact* [2]), if every open cover of X has a locally finite open (resp. semi-open) refinement which covers to X (we do not require a refinement to be a cover). A space (X,τ) is said to be *almost-paracompact* [20] if every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ such that the collection $\{\mathrm{Cl}\,(V) : V \in \mathcal{V}\}$ is a cover of X.

LEMMA 1.1. [16] *If (X,τ) is e.d., then $\mathrm{Cl}\,(U) = \mathrm{sCl}\,(U)$ for each $U \in \mathrm{SO}\,(X,\tau)$.*

LEMMA 1.2. [1] *If $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a collection s-locally finite of subsets of a space (X,τ), then:*

(1) *The collection $\mathcal{B} = \{\mathrm{sCl}\,(V_\lambda) : \lambda \in \Lambda\}$ is s-locally finite.*

(2) $\mathrm{sCl}\,(\bigcup\{V_\lambda : \lambda \in \Lambda\}) = \bigcup\{\mathrm{sCl}\,(V_\lambda) : \lambda \in \Lambda\}$.

An *ideal* $\mathcal{I}$ on a nonempty set X is a nonempty collection of subsets of X which satisfies the following two properties:

(1) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$;

(2) $A \in \mathcal{I}$ and $B \in I$ implies $A \cup B \in \mathcal{I}$.

In this paper, the triple $(X,\tau,\mathcal{I})$ denotes a topological space (X,τ) together with an ideal $\mathcal{I}$ on X and will be simply called a space. Given a space $(X,\tau,\mathcal{I})$, a set operator $(\cdot)^\star : P(X) \to P(X)$, called the *local function* [11] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\mathcal{I},\tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I},\tau)$. In general, $X^\star$ is a proper subset of X. The hypothesis $X = X^\star$ is equivalent to the hypothesis $\tau \cap \mathcal{I} = \emptyset$. According to [14], we call the ideals which satisfy this hypothesis *τ-boundary* ideals. Note that $\mathrm{Cl}^\star(A) = A \cup A^\star$ defines a Kuratowski closure for a topology $\tau^\star(\mathcal{I})$, finer than τ. A basis $\beta(I,\tau)$ for $\tau^\star(\mathcal{I})$ can be described as follows: $\beta(\mathcal{I},\tau) = \{V - J : V \in \tau$ and $J \in \mathcal{I}\}$ [10]. When there is no chance for confusion, we will simply write $\tau^\star$ for $\tau^\star(\mathcal{I})$ and β for $\beta(I,\tau)$. If $\beta = \tau^\star$, then the ideal $\mathcal{I}$ is said to be *τ-simple* [9]. A sufficient condition for $\mathcal{I}$ to be τ-simple is the following: given $A \subset X$, if for every $x \in A$ there exists $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. If $(X,\tau,\mathcal{I})$ satisfies this condition, then $\mathcal{I}$ is said to be τ-local [9]. Given a space $(X,\tau,\mathcal{I})$ we say that $\mathcal{I}$ is *weakly τ-local* [9] if $A^\star = \emptyset$ implies $A \in \mathcal{I}$. Also, $\mathcal{I}$ is called *τ-locally finite* [9] if the union of each τ-locally finite collection contained in $\mathcal{I}$ belongs in $\mathcal{I}$. It is known that if an ideal $\mathcal{I}$ is τ-local then $\mathcal{I}$ is weakly τ-local and, if $\mathcal{I}$ is weakly τ-local then $\mathcal{I}$ is τ-locally finite [9].

A space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{I}$-paracompact* [22], or *paracompact with respect to $\mathcal{I}$*, if every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. A collection $\mathcal{V}$ of subsets of X such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$ is called an *$\mathcal{I}$-cover* [9] of X. A space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{I}$-regular* [7], if for each closed set F and a point $p \notin F$, there exist disjoint open sets U and V such that $p \in U$ and $F - V \in \mathcal{I}$. The following lemma will be useful in the sequel.

Lemma 1.3. *If a cover $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of a space $(X, \tau, \mathcal{I})$ has a locally finite semi-open refinement which is an $\mathcal{I}$-cover of X, then there exists a locally finite precise semi-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}$ which is an $\mathcal{I}$-cover of X ("precise" means that $\mathcal{U}$ and $\mathcal{V}$ have the same index set Λ and $V_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$).*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be a cover of $(X, \tau, \mathcal{I})$ and let $\mathcal{W} = \{W_\mu : \mu \in \Delta\}$ be a locally finite semi-open refinement of $\mathcal{U}$ which is an $\mathcal{I}$-cover of X. Then, there exists a function $\Psi : \Delta \to \Lambda$ such that $W_\mu \subset U_{\Psi(\mu)}$ if $\mu \in \Delta$. Put $V_\lambda = \bigcup\{W_\mu : \Psi(\mu) = \lambda\}$. Observe that the collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a semi-open $\mathcal{I}$-cover of X and each $V_\lambda \subset U_\lambda$. Finally we show that $\mathcal{V}$ is locally finite. Let $x \in X$, then there exists an open set G such that $x \in G$ and $\Delta_0 = \{\mu \in \Delta : G \cap W_\mu \neq \emptyset\}$ is a finite set. Observe that $G \cap V_\lambda \neq \emptyset$ if and only if $\lambda = \Psi(\mu)$ for some $\mu \in \Delta_0$. Thus, $\{\lambda \in \Lambda : G \cap V_\lambda \neq \emptyset\}$ is a finite set and hence, $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is locally finite. ∎

2. $\mathcal{I}$-S-paracompact spaces

Definition 2.1. A space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{I}$-S-paracompact*, or *S-paracompact with respect to $\mathcal{I}$*, if every open cover $\mathcal{U}$ of X has a locally finite semi-open refinement $\mathcal{V}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. Equivalently, $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact, if for every open cover $\mathcal{U}$ of X, there exist $I \in \mathcal{I}$ and a locally finite semi-open refinement $\mathcal{V}$ such that $X = \bigcup\{V : V \in \mathcal{V}\} \cup I$.

Clearly, every S-paracompact spaces is $\mathcal{I}$-S-paracompact, but the converse is not necessarily true as we can see in the following example.

Example 2.1. Let $X = \mathbb{R}^+ \cup \{p\} \cup \{q\}$, where $\mathbb{R}^+ = [0, +\infty)$, $p \notin \mathbb{R}^+$, $q \notin \mathbb{R}^+$ and $p \neq q$. Topologize X as follows: $\mathbb{R}^+$ has the usual topology and is an open subspace of X; a basic neighborhood of $p \in X$ takes the form

$$O_n(p) = \{p\} \cup \bigcup_{i=n}^{\infty} (2i, 2i+1), \text{ where } n \in \mathbb{N};$$

a basic neighborhood of $q \in X$ takes the form

$$O_m(q) = \{q\} \cup \bigcup_{i=m}^{\infty} (2i-1, 2i), \text{ where } m \in \mathbb{N}.$$

Now, if we take $\mathcal{I} = \mathcal{N}$, the ideal of nowhere dense subsets in X, then X is $\mathcal{I}$-S-paracompact but X is not S-paracompact.

THEOREM 2.1. *If $(X, \tau, \mathcal{I})$ is a space, the following properties hold:*

(1) *If $\mathcal{I} = \{\emptyset\}$, then (X, τ) is S-paracompact if and only if (X, τ) is $\mathcal{I}$-S-paracompact.*

(2) *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact and $\mathcal{J}$ is an ideal on X with $\mathcal{I} \subseteq \mathcal{J}$, then $(X, \tau, \mathcal{J})$ is $\mathcal{J}$-S-paracompact.*

(3) *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-paracompact, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact.*

Proof. (1) and (2) are obvious. (3) Follows from the fact that every open set is semi-open. ■

REMARK 2.1. Considering the ideal $\mathcal{I} = \{\emptyset\}$ in Example 2.3 of [13], we have a space $(X, \tau, \mathcal{I})$ that is $\mathcal{I}$-S-paracompact but is not $\mathcal{I}$-paracompact [13]. This shows that, in Theorem 2.1.(3), the converse is not true.

THEOREM 2.2. *If $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$-S-paracompact space and $\mathcal{I}$ is τ-boundary, then (X, τ) is almost-paracompact.*

Proof. Let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be an open cover of X. By hypothesis, $\mathcal{U}$ has a locally finite semi-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $X - \bigcup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. For each $\lambda \in \Lambda$ there exists an open subset G_λ of X such that $G_\lambda \subset V_\lambda \subset \mathrm{Cl}\,(G_\lambda)$. The collection $\mathcal{W} = \{G_\lambda : \lambda \in \Lambda\}$ is an open refinement of $\mathcal{U}$, since for each $\lambda \in \Lambda$ there exists $\mu \in \Delta$ such that $G_\lambda \subset V_\lambda \subset U_\mu$. Let $x \in X$. Then as $\mathcal{V}$ is locally finite, there exists an open set O_x such that $x \in O_x$ and $\{\lambda \in \Lambda : O_x \cap V_\lambda \neq \emptyset\}$ is a finite set. Since $\{\lambda \in \Lambda : O_x \cap G_\lambda \neq \emptyset\} \subset \{\lambda \in \Lambda : O_x \cap V_\lambda \neq \emptyset\}$, we have the collection $\mathcal{W}$ is locally finite. Because $X - \bigcup\{\mathrm{Cl}\,(G_\lambda) : \lambda \in \Lambda\} \subset X - \bigcup\{V_\lambda : \lambda \in \Lambda\}$, then $X - \bigcup\{\mathrm{Cl}\,(G_\lambda) : \lambda \in \Lambda\} \in \mathcal{I}$. Now using the fact that $\mathcal{I}$ is τ-boundary, we have $\emptyset = \mathrm{Int}\,(X - \bigcup\{\mathrm{Cl}\,(G_\lambda) : \lambda \in \Lambda\}) = X - \bigcup\{\mathrm{Cl}\,(G_\lambda) : \lambda \in \Lambda\}$, it follows that $X = \bigcup\{\mathrm{Cl}\,(G_\lambda) : \lambda \in \Lambda\}$. Therefore, (X, τ) is almost-paracompact. ■

REMARK 2.2. It is shown in [9, Theorem II.1(1)] that if $\mathcal{I} = \mathcal{N}$, the ideal of nowhere dense subsets in (X, τ), then:

$$(X, \tau) \text{ is almost-paracompact} \;\Leftrightarrow\; (X, \tau) \text{ is } \mathcal{I}\text{-paracompact.}$$

Now, using Theorem 2.1 and the fact that $\mathcal{I} = \mathcal{N}$ is τ-boundary, then (by Theorem 2.1) it follows that:

$$\begin{aligned}(X, \tau) \text{ is almost-paracompact} \;&\Leftrightarrow\; (X, \tau) \text{ is } \mathcal{I}\text{-paracompact}\\ &\Leftrightarrow\; (X, \tau) \text{ is } \mathcal{I}\text{-}S\text{-paracompact.}\end{aligned}$$

Example 2.1 shows that, in Theorem 2.1(1), the equivalence is not true if $\mathcal{I} \neq \{\emptyset\}$.

LEMMA 2.1. [17] *If (X, τ) is a space with an ideal $\mathcal{I}$, then the following properties are equivalent:*

(1) *$\mathcal{I}$ is τ-boundary.*

(2) $\mathrm{SO}\,(X, \tau) \cap \mathcal{I} = \{\emptyset\}$.

In the case that $\mathcal{I}$ is τ-simple, τ-boundary and $\tau^\star \subset \mathrm{SO}(X,\tau)$, we obtain the following result.

Theorem 2.3. *Let $(X,\tau,\mathcal{I})$ a space where $\mathcal{I}$ is τ-simple and τ-boundary. If $\tau^\star \subset \mathrm{SO}(X,\tau)$ and $(X,\tau^\star)$ is $\mathcal{I}$-S-paracompact, then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact.*

Proof. Let $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ be a τ-open cover of X. Then, $\mathcal{U} = \{U_\mu : \mu \in \Delta\}$ is a $\tau^\star$-open cover of X and hence, $\mathcal{U}$ has a $\tau^\star$-locally finite $\tau^\star$-semi-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $X - \bigcup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Since $\mathcal{I}$ is τ-boundary and $\tau^\star \subset \mathrm{SO}(X,\tau)$, then (by [21, Corollary 2.4]) it follows that $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a collection of τ-semi-open sets. Let $x \in X$, there exists a $\tau^\star$-open set O_x containing x such that $O_x \cap V_\lambda = \emptyset$ for $\lambda \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Since $\mathcal{I}$ is τ-simple, we have $O_x = G_x - I_x$ for some $G_x \in \tau$ and $I_x \in \mathcal{I}$. Therefore, $(G_x - I_x) \cap V_\lambda = \emptyset$ for $\lambda \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Because $(G_x - I_x) \cap V_\lambda = (G_x \cap V_\lambda) - I_x$ it follows that $(G_x \cap V_\lambda) - I_x = \emptyset$ for $\lambda \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. This implies $G_x \cap V_\lambda = \emptyset$ for $\lambda \notin \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Otherwise $G_x \cap V_\lambda$ is a nonempty τ-semi-open subset of I_x contradicting the fact that $\mathrm{SO}(X,\tau) \cap \mathcal{I} = \{\emptyset\}$ by Lemma 2.1. Consequently, the collection $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a τ-locally finite τ-semi-open refinement of $\mathcal{U}$ such that $X - \bigcup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Therefore, $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact. ∎

The converse of Theorem 2.3 is not necessarily true as we can see in the following example.

Example 2.2. Let $X = \mathbb{R}$ be the real number set topologized with the topology $\tau = \{\emptyset, X, \{1\}\}$. If we take $\mathcal{I} = \mathcal{N}$, then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact but $(X,\tau^\star)$ is not $\mathcal{I}$-S-paracompact.

Lemma 2.2. *Let $(X,\tau,\mathcal{I})$ be a space where $\mathcal{I}$ is τ-boundary. If $A \in \mathrm{SO}(X,\tau)$, then $A - I \in \mathrm{SO}(X,\tau^\star)$ for each $I \in \mathcal{I}$.*

Proof. Assume $A \in \mathrm{SO}(X,\tau)$ and $I \in \mathcal{I}$. There exists $U \in \tau$ such that $U \subset A \subset \mathrm{Cl}(U)$, it follows that $U - I \subset A - I \subset \mathrm{Cl}(U) - I$. Since $\mathcal{I}$ is τ-boundary, then by [10, Theorem 6.1], we have $U \subset U^\star$ and so, $U - I \subset A - I \subset \mathrm{Cl}(U) - I \subset \mathrm{Cl}(U^\star) - I = U^\star - I \subset U^\star$. Now by [10, Theorem 2.3 (h)], we obtain $U^\star = (U - I)^\star$ and hence, $U - I \subset A - I \subset (U - I)^\star \subset \mathrm{Cl}^\star(U - I)$. Since $U - I$ is a basic $\tau^\star$-open, it follows that $A - I \in \mathrm{SO}(X,\tau^\star)$. ∎

In the case that $\mathcal{I}$ is τ-boundary and weakly-τ-local, we obtain the following result.

Theorem 2.4. *If $\mathcal{I}$ is τ-boundary, weakly τ-local, and $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact, then $(X,\tau^\star)$ is $\mathcal{I}$-S-paracompact.*

Proof. Let $\mathcal{U}' = \{U_\lambda - I_\lambda : \lambda \in \Lambda, U_\lambda \in \tau, I_\lambda \in \mathcal{I}\}$ be a basic $\tau^\star$-open cover of X. Then $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ is a τ-open cover of X and so, by Lemma 1.3, $\mathcal{U}$ has a τ-locally finite precise τ-semi-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ which is an $\mathcal{I}$-cover of X. Observe that $\{V_\lambda \cap I_\lambda : \lambda \in \Lambda\}$ is a τ-locally finite subset of $\mathcal{I}$ and as $\mathcal{I}$ is weakly τ-local. Then $\bigcup\{V_\lambda \cap I_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Hence, $X - \bigcup\{V_\lambda - I_\lambda : \lambda \in \Lambda\} \subset (X - \bigcup\{V_\lambda : \lambda \in \Lambda\}) \cup (\bigcup\{V_\lambda \cap I_\lambda : \lambda \in \Lambda\}) \in \mathcal{I}$.

Since $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is τ-locally finite, then $\mathcal{V}' = \{V_\lambda - I_\lambda : \lambda \in \Lambda\}$ is τ-locally finite. Since $\tau \subset \tau^\star$, we have $\mathcal{V}' = \{V_\lambda - I_\lambda : \lambda \in \Lambda\}$ is $\tau^\star$-locally finite. Finally, by Lemma 2.2, $\mathcal{V}' = \{V_\lambda - I_\lambda : \lambda \in \Lambda\}$ is a collection of $\tau^\star$-semi-open sets which refines $\mathcal{U}'$. This shows that $(X, \tau^\star)$ is $\mathcal{I}$-S-paracompact. ∎

Corollary 2.1. *If $\mathcal{I}$ is τ-local, τ-boundary and $\tau^\star \subset \mathrm{SO}(X, \tau)$, then (X, τ) is $\mathcal{I}$-S-paracompact if and only if $(X, \tau^\star)$ is $\mathcal{I}$-S-paracompact.*

Proof. This is an immediate consequence of Theorems 2.3 and 2.4. ∎

Theorem 2.5. *Let $(X, \tau, \mathcal{I})$ be a space and consider the following statements:*

(1) *$(X, \tau, \mathcal{I})$ is Hausdorff $\mathcal{I}$-S-paracompact.*

(2) *For every closed subset A of X and every $x \notin A$, there exist disjoint sets $U \in \mathrm{RO}(X, \tau)$ and $V \in \mathrm{SO}(X, \tau)$ such that $x \in U$ and $A - V \in \mathcal{I}$.*

(3) *For every open subset G of X and every $x \in G$, there exist $U \in \mathrm{RO}(X, \tau)$ such that $x \in U$ and $\mathrm{sCl}(U) - G \in \mathcal{I}$.*

Then, the following implications hold (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): Let x be any point of X and A be a closed subset of X such that $x \notin A$. Since X is a Hausdorff space, for each $y \in A$ there exists an open set W_y such that $y \in W_y$ and $x \notin \mathrm{Cl}(W_y)$. Observe that $A \subseteq \bigcup\{W_y : y \in A\}$ and the collection $\mathcal{W} = \{W_y : y \in A\} \cup \{X - A\}$ is an open cover of X. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact, by Lemma 1.3, $\mathcal{W}$ has a locally finite precise semi-open refinement $\mathcal{M} = \{M_y : y \in A\} \cup \{G\}$ such that $M_y \subseteq W_y$ for each $y \in A$, $G \subset X - A$ and $\mathcal{M}$ is an $\mathcal{I}$-cover of X. Note that if $y \in A$, then $M_y \subset W_y$, consequently $\mathrm{Cl}(M_y) \subset \mathrm{Cl}(W_y)$, hence $x \notin \mathrm{Cl}(M_y)$. Let $V = \bigcup\{M_y : y \in A\}$, then V is a semi-open set such $A - V = A - \bigcup\{M_y : y \in A\} \subset A - \bigcup\{M : M \in \mathcal{M}\} \in \mathcal{I}$. Since $\mathcal{M}$ is locally finite, we have $\mathrm{Cl}(V) = \mathrm{Cl}(\bigcup\{M_y : y \in A\}) = \bigcup\{\mathrm{Cl}(M_y) : y \in A\}$. Therefore, if $U = X - \mathrm{Cl}(V)$ then U is a regular open set such that $x \in U$ and $U \cap V = (X - \mathrm{Cl}(V)) \cap V = \emptyset$.

(2) $\Rightarrow$ (3): Let G be an open subset of X and $x \in G$, then $A = X - G$ is closed set and $x \notin A$. By (2), there exist disjoint set $U \in \mathrm{RO}(X, \tau)$ and $V \in \mathrm{SO}(X, \tau)$ such that $x \in U$ and $A - V \in \mathcal{I}$, it follows that $x \in U \subset \mathrm{sCl}(U)$. To show $\mathrm{sCl}(U) - G \in \mathcal{I}$, observe that $U \cap V = \emptyset$ implies $\mathrm{sCl}(U) \cap V = \emptyset$, hence $\mathrm{sCl}(U) \subset X - V$ and so, $A \cap \mathrm{sCl}(U) \subset A \cap (X - V)$. Then, $(X - G) \cap \mathrm{sCl}(U) \subset (X - G) \cap (X - V)$ and this implies $\mathrm{sCl}(U) - G \subset (X - G) - V = A - V \in \mathcal{I}$. Therefore, $\mathrm{sCl}(U) - G \in \mathcal{I}$.

(3) $\Rightarrow$ (2): Let A be a closed subset of X and let $x \in X$ such that $x \notin A$, then $G = X - A$ is an open subset of X with $x \in G$. By (3), there exists $U \in \mathrm{RO}(X, \tau)$ such that $x \in U$ and $\mathrm{sCl}(U) - G \in \mathcal{I}$. Thus, the set $V = X - \mathrm{sCl}(U) \in \mathrm{SO}(X, \tau)$ and $U \cap V = U \cap (X - \mathrm{sCl}(U)) \subset U \cap (X - U) = \emptyset$. Moreover, $A - V = A - (X - \mathrm{sCl}(U)) = A \cap \mathrm{sCl}(U) = (X - G) \cap \mathrm{sCl}(U) = \mathrm{sCl}(U) - G \in \mathcal{I}$, it follows that $A - V \in \mathcal{I}$. ∎

Corollary 2.2. *If $(X, \tau, \mathcal{I})$ is an e.d. Hausdorff $\mathcal{I}$-S-paracompact space, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-regular.*

Proof. Let U be an open subset of X and $x \in U$. By Theorem 2.5, there exists $V \in \mathrm{RO}(X,\tau) \subseteq \tau$ such that $x \in V$ and $\mathrm{sCl}(V) - U \in \mathcal{I}$. Since X is e.d., by Lemma 1.1, we have $\mathrm{sCl}(V) = \mathrm{Cl}(V)$ and so, there exists $V \in \tau$ such that $x \in V$ and $\mathrm{Cl}(V) - U \in \mathcal{I}$. Now by [18, Lemma 2.8], it follows that $(X,\tau,\mathcal{I})$ is an $\mathcal{I}$-regular space. ∎

In the following theorem, we show that under some conditions every open cover of X has a locally finite open refinement which is an $\mathcal{I}$-cover of X.

Theorem 2.6. *Let $(X,\tau,\mathcal{I})$ be an e.d. regular space. If every open cover of X has a s-locally finite semi-open refinement which is an $\mathcal{I}$-cover of X, then every open cover of X has a locally finite open refinement which is an $\mathcal{I}$-cover of X.*

Proof. The proof is similar to the proof of [2, Theorem 2.5]. It is only necessary to prove that $\mathcal{H} = \{\mathrm{Cl}(H_\lambda) : \lambda \in \Lambda\}$ is an $\mathcal{I}$-cover of X. Observe that $X - \bigcup\{\mathrm{Cl}(H_\lambda) : \lambda \in \Lambda\} = X - \bigcup\{\mathrm{Cl}(W_\lambda) : \lambda \in \Lambda\} \subset X - \bigcup\{W_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Thus, $\mathcal{H} = \{\mathrm{Cl}(H_\lambda) : \lambda \in \Lambda\}$ is an $\mathcal{I}$-cover of X. ∎

Corollary 2.3. *If $(X,\tau,\mathcal{I})$ is an e.d. regular $\mathcal{I}$-S-paracompact space, then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of X, using the $\mathcal{I}$-S-paracompactness of $(X,\tau,\mathcal{I})$, $\mathcal{U}$ has a locally finite semi-open refinement $\mathcal{V}$ which is an $\mathcal{I}$-cover of X. Since every locally finite collection is s-locally finite, then $\mathcal{V}$ is s-locally finite and, by Theorem 2.6, $\mathcal{U}$ has a locally finite open refinement which is an $\mathcal{I}$-cover of X. This shows that $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-paracompact. ∎

Recall that a space (X,τ) is said to be *countably S-closed* [4], if every countable semi-open cover of X has a finite subfamily the closures of whose members cover X. Al-Zoubi [1] has shown that, in a countably S-closed space, every s-locally finite collection of semi-open sets is finite. According to [14], a space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-compact, or compact with respect to $\mathcal{I}$, if every open cover $\mathcal{U}$ of X contains a finite subcollection $\mathcal{V}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. Clearly, every $\mathcal{I}$-compact space is $\mathcal{I}$-paracompact and hence $\mathcal{I}$-S-paracompact. In the following theorem, we shows that if the space is countably S-closed, then the converse also is true.

Theorem 2.7. *If $(X,\tau,\mathcal{I})$ is an $\mathcal{I}$-S-paracompact countably S-closed space, then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-compact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X. Since $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact, $\mathcal{U}$ has a locally finite semi-open refinement $\mathcal{V} = \{V_\mu : \mu \in \Delta\}$ which is an $\mathcal{I}$-cover of X. T hen, $\mathcal{V}$ is a s-locally finite collection of semi-open sets and since $(X,\tau,\mathcal{I})$ is countably S-closed space, we have by [1, Theorem 2.7], $\mathcal{V}$ is finite. Without loss of generality, assume $\mathcal{V} = \{V_{\mu_i} : i = 1,2,\ldots,n\}$. Now, since $\mathcal{V}$ refines $\mathcal{U}$, then for each $i = 1,2,\ldots,n$, there exist $U_{\lambda(i)} \in \mathcal{U}$ such that $V_{\mu_i} \subset U_{\lambda(i)}$. Thus, $X - \bigcup\{U_{\lambda(i)} : i = 1,\ldots,n\} \subset X - \bigcup\{V_{\mu_i} : i = 1,\ldots,n\} = X - \bigcup\{V : V \in \mathcal{V}\} \in I$, it follows that $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-compact. ∎

The converse of Theorem 2.7 is not necessarily true as we can see in the following example.

EXAMPLE 2,3. Let D be an infinite set with the discrete topology. Let (X,τ) be the Alexandroff compactification of D, where $X = D \cup \{a\}$ and $a \notin D$ is the only isolated point of (X,τ). If we take $\mathcal{I} = \{\emptyset\}$, then $(X,\tau,\mathcal{I})$ is an $\mathcal{I}$-compact space and hence $\mathcal{I}$-S-paracompact, but it is not countably S-closed.

Recall that $\tau \subset \tau^\alpha$. The following theorem shows that, if (X,τ^α) is $\mathcal{I}$-S-paracompact then (X,τ) is $\mathcal{I}$-S-paracompact for any ideal $\mathcal{I}$.

THEOREM 2.8. *If $(X,\tau^\alpha,\mathcal{I})$ is an $\mathcal{I}$-S-paracompact space, then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact.*

Proof. Let $\mathcal{U}$ be a τ-open cover of X. Since $\tau \subset \tau^\alpha$, then $\mathcal{U}$ is a τ^α-open cover of X. Since $(X,\tau^\alpha,\mathcal{I})$ is an $\mathcal{I}$-S-paracompact space, $\mathcal{U}$ has a τ^α-locally finite τ^α-semi-open refinement $\mathcal{V}$ which is an $\mathcal{I}$-cover of X. Because $\mathrm{SO}\,(X,\tau) = SO(X,\tau^\alpha)$, we have $\mathcal{V}$ is a τ-semi-open refinement of $\mathcal{U}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. To show that $\mathcal{V}$ is τ-locally finite , let $x \in X$. Since $\mathcal{V}$ is τ^α-locally finite, there exists a τ^α-open set G containing x such that G intersects at most finitely many members $V_1, V_2, \ldots, V_n$ of $\mathcal{V}$. Now, for each $V \in \mathcal{V}$ there exists $W_V \in \tau$ such that $W_V \subset V \subset \mathrm{Cl}\,(W_V)$, since $G \in \tau^\alpha$, then $x \in G \subset \mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G)))$. Thus, $\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G)))$ is a τ-open set containing x with the property $\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G))) \cap V = \emptyset$ for each $V \in \mathcal{V} - \{V_1, V_2, \ldots, V_n\}$. To see this last claim, assume $\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G))) \cap V \neq \emptyset$, then $\emptyset \neq \mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G))) \cap V \subset \mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G))) \cap \mathrm{Cl}\,(W_V)$, it follows that $\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G))) \cap \mathrm{Cl}\,(W_V) \neq \emptyset$. Consequently, $\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G))) \cap W_V \neq \emptyset$, since $\emptyset \neq \mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(G))) \cap W_V \subset \mathrm{Cl}\,(G) \cap W_V$, then $\mathrm{Cl}\,(G) \cap W_V \neq \emptyset$, which implies that $G \cap W_V \neq \emptyset$. Therefore, $G \cap V \neq \emptyset$ and $V \in \{V_1, V_2, \ldots, V_n\}$. Thus, we obtain $\mathcal{V}$ is a τ-locally finite τ-semi-open refinement of $\mathcal{U}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$ and hence, $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact. ∎

The converse of Theorem 2.8 is not necessarily true as we can see in the following example.

EXAMPLE 2.4. Let $X = \mathbb{R}$ be the real number set topologized with the topology $\tau = \{\emptyset, X, \{1\}\}$. If we take $\mathcal{I} = \{\emptyset\}$, then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact but $(X,\tau^\alpha,\mathcal{I})$ is not $\mathcal{I}$-S-paracompact.

Given a space (X,τ), we denote by τ_{SO} the topology on X which has $\mathrm{SO}\,(X,\tau)$ as a subbase. It is well known that the collection $\mathrm{SO}\,(X,\tau)$ is a topology on X if and only if (X,τ) is e.d. [15]; in this case $\tau_{\mathrm{SO}} = \mathrm{SO}\,(X,\tau)$.

COROLLARY 2.4. *Let $(X,\tau,\mathcal{I})$ an e.d. space. If $(X,\tau_{\mathrm{SO}},\mathcal{I})$ is $\mathcal{I}$-S-paracompact, then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact.*

Proof. Follows from Theorem 2.8 and the fact that in an e.d. space it is satisfied that $\tau^\alpha = \mathrm{SO}\,(X,\tau) = \tau_{\mathrm{SO}}$. ∎

THEOREM 2.9. *Let $(X,\tau,\mathcal{I})$ be a regular space. Then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-S-paracompact if and only if every open cover $\mathcal{U}$ of X has a locally finite regular closed refinement $\mathcal{V}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$.*

Proof. The sufficiency follows directly from the fact that every regular closed set is semi-open. To show necessity, let $\mathcal{U}$ be an open cover of X. For each $x \in X$

there exists $U_x \in \mathcal{U}$ such that $x \in U_x$ and, since $(X, \tau, \mathcal{I})$ is a regular space, then there exists a set $W_x \in \tau$ such that $x \in W_x \subset \mathrm{Cl}\,(W_x) \subset U_x$. Thus, the collection $\mathcal{W} = \{W_x : x \in X\}$ is an open cover of X and, by hypothesis, $\mathcal{W}$ has a locally finite semi-open refinement $\mathcal{W}' = \{G_\lambda : \lambda \in \Lambda\}$ which is an $\mathcal{I}$-cover of X. Observe that $\mathrm{Cl}\,(G_\lambda)$ is a regular closed set for each $\lambda \in \Lambda$, hence $\mathcal{V} = \{\mathrm{Cl}\,(G_\lambda) : \lambda \in \Lambda\}$ is a locally finite collection of regular closed sets. Since $\mathcal{W}'$ refines $\mathcal{W}$, for each $\lambda \in \Lambda$ there exists $W_x \in \mathcal{W}$ such that $G_\lambda \subseteq W_x$, and so, for each $\lambda \in \Lambda$ we have $G_\lambda \subset W_x \subset \mathrm{Cl}\,(W_x) \subset U_x$ for some $U_x \in \mathcal{U}$. Therefore, for each $\lambda \in \Lambda$, $\mathrm{Cl}\,(G_\lambda) \subset \mathrm{Cl}\,(W_x) \subset U_x$ for some $U_x \in \mathcal{U}$. Thus, the collection $\mathcal{V} = \{\mathrm{Cl}\,(G_\lambda) : \lambda \in \Lambda\}$ is a refinement of $\mathcal{U}$. Finally, note that $X \setminus \bigcup\{\mathrm{Cl}\,(G_\lambda : \lambda \in \Lambda\} \subset X \setminus \bigcup\{G_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$, it follows that $X \setminus \bigcup\{V : V \in \mathcal{V}\} \subset X \setminus \bigcup\{\mathrm{Cl}\,(G_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. ∎

Recall that a space $(X, \tau, \mathcal{I})$ is said to be countably $\mathcal{I}$-compact [14], or countably compact with respect to $\mathcal{I}$, if every countable open cover $\mathcal{U}$ of X contains a finite subcollection $\mathcal{V}$ such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. According to [8], a subset $A \subseteq X$ is said to be a *non-ideal* set if $A \notin \mathcal{I}$.

Lemma 2.3. [8] *Let $(X, \tau, \mathcal{I})$ be a space. Then $(X, \tau, \mathcal{I})$ is countably $\mathcal{I}$-compact if and only if every collection locally finite of non-ideal sets is finite.*

Clearly, any $\mathcal{I}$-compact space is countably $\mathcal{I}$-compact, but the converse, in general, is not true. In Example 3.10.16 of [5], it is shown that the space W_0 of all countable ordinal number is countably $\{\emptyset\}$-compact but is not $\{\emptyset\}$-compact. In the following theorem, we show that under some conditions the notions of $\mathcal{I}$-compact space and countably $\mathcal{I}$-compact space are equivalent.

Theorem 2.10. *Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}$-S-paracompact space where $\mathcal{I}$ is τ-boundary. Then, $(X, \tau, \mathcal{I})$ is countably $\mathcal{I}$-compact if and only if $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-compact.*

Proof. Suppose that $(X, \tau, \mathcal{I})$ is a countably $\mathcal{I}$-compact space and let $\mathcal{U}$ be an open cover of X. Since $(X, \tau, \mathcal{I})$ is an $\mathcal{I}$-S-paracompact space, $\mathcal{U}$ has a locally finite refinement $\mathcal{V}$ of nonempty semi-open sets such that $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. By Lemma 2.1, we have $\mathrm{SO}\,(X, \tau) \cap \mathcal{I} = \{\emptyset\}$ and since $(X, \tau, \mathcal{I})$ is countably $\mathcal{I}$-compact, then by Lemma 2.3, we have $\mathcal{V}$ is finite. Thus, every open cover $\mathcal{U}$ of X has a refinement $\mathcal{V}$ which is finite and $X - \bigcup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. Therefore, $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-compact. ∎

3. Preservation by functions

In [6], it was shown that the open and perfect functions both preserve and inversely preserve S-paracompact spaces. In this section, we give a more general result than those mentioned above. First, we present the following two lemmas that will be useful.

Lemma 3.1. [6, Lemma 3.1] *Let $f : X \to Y$ be an open and continuous function. If U is an semi-open subset of Y and V is an open subset of X, then $f^{-1}(U) \cap V$ is a semi-open subset of X.*

LEMMA 3.2. [5, Lemma 3.10.11] *If $f : (X, \tau) \to (Y, \sigma)$ is a perfect function and $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a locally finite collection of subsets of X, then $f(\mathcal{V}) = \{f(V_\lambda) : \lambda \in \Lambda\}$ is a locally finite collection of subsets of Y.*

It is well known that if $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a surjective, continuous function and $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact then $(Y, \sigma, \mathcal{J})$ is not $\mathcal{J}$-S-paracompact. In the next theorem we give sufficient conditions in order to prove that if $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact then $(Y, \sigma, \mathcal{J})$ is $\mathcal{J}$-S-paracompact.

THEOREM 3.1. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be an open and perfect function with $f(\mathcal{I}) \subset \mathcal{J}$. If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact, then $(Y, \sigma, \mathcal{J})$ is $\mathcal{J}$-S-paracompact.*

Proof. The proof is similar to that of [6, Theorem 2.4]. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of Y. By the continuity of f, the collection $f^{-1}(\mathcal{U}) = \{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is an open cover of X and hence, there exists a locally finite precise refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $f^{-1}(\mathcal{U})$ such that $X - \bigcup\{V_\lambda : \lambda \in \Lambda\} = I \in \mathcal{I}$. For each $\lambda \in \Lambda$ there exists an open subset $G_\lambda \subset V_\lambda \subset \mathrm{Cl}\,(G_\lambda)$ and hence, $f(G_\lambda) \subset f(V_\lambda) \subset f(\mathrm{Cl}\,(G_\lambda)) \subset \mathrm{Cl}\,(f(G_\lambda))$ for each $\lambda \in \Lambda$. Now, using the fact that f is open, the collection $f(\mathcal{V}) = \{f(V_\lambda) : \lambda \in \Lambda\}$ is a precise semi-open refinement of $\mathcal{U}$. Also, $f(\mathcal{V})$ is an $\mathcal{J}$-cover of Y, since $Y = f(X) = f\left(\bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I\right) = \bigcup\{f(V_\lambda) : \lambda \in \Lambda\}) \cup f(I)$ which implies that $Y - \bigcup\{f(V_\lambda) : \lambda \in \Lambda\} \subset f(I) \in \mathcal{J}$. Finally, by Lemma 3.2, we have the collection $f(\mathcal{V}) = \{f(V_\lambda) : \lambda \in \Lambda\}$ is locally finite in Y. This shows that $(Y, \sigma, \mathcal{J})$ is $\mathcal{J}$-S-paracompact. ■

According to [9], if $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ is a function, then $\langle f^{-1}(\mathcal{J})\rangle = \{A : A \subset f^{-1}(J)$ for some $J \in \mathcal{J}\}$ is an ideal on X. The following result shows that $\mathcal{I}$-S-paracompactness is inversely preserved by open and perfect functions.

THEOREM 3.2. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be an open and perfect function with $\langle f^{-1}(\mathcal{J})\rangle \subset \mathcal{I}$. If $(Y, \sigma, \mathcal{J})$ is $\mathcal{J}$-S-paracompact, then $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact.*

Proof. Let $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of X. For each $y \in Y$, we have $f^{-1}(y) \subset X = \bigcup_{\lambda\in\Lambda} U_\lambda$, so that $\mathcal{U}$ is an open cover of $f^{-1}(y)$ for each $y \in Y$. Since $f^{-1}(y)$ is a compact subset of X, for each $y \in Y$, there exists a finite subcollection $\mathcal{U}_y = \{U_{\lambda_1}(y), U_{\lambda_2}(y), \dots, U_{\lambda_n}(y)\}$ of $\mathcal{U}$ such that $f^{-1}(y) \subset \bigcup_{i=1}^n U_{\lambda_i}(y)$. Now, because f is a closed function and $\bigcup_{i=1}^n U_{\lambda_i}(y)$ is an open set in X, there exists an open set $V_y \subset Y$ such that $y \in V_y$ and $f^{-1}(V_y) \subset \bigcup_{i=1}^n U_{\lambda_i}(y)$. Since $Y = \bigcup_{y\in Y} V_y$, the collection $\mathcal{V} = \{V_y : y \in Y\}$ is an open cover of Y and as $(Y, \sigma, \mathcal{J})$ is $\mathcal{J}$-S-paracompact, by Lemma 3.1, there exists a locally finite precise semi-open refinement $\mathcal{H} = \{H_y : y \in Y\}$ of $\mathcal{V}$ such that $Y - \bigcup\{H_y : y \in Y\} \in \mathcal{J}$. Put $\mathcal{F}_y = \{U_{\lambda_i}(y) \cap f^{-1}(H_y) : i = 1, \dots, n\}$ for each $y \in Y$. By Lemma 3.1, we have $\mathcal{F} = \bigcup_{y\in Y} \mathcal{F}_y$ is a collection of semi-open subsets of X which obviously refines $\mathcal{U}$. We claim that $\mathcal{F}$ have the following two properties:

CLAIM 1. $\mathcal{F}$ is an $\mathcal{I}$-cover of X. Let $x \in X$. Then $y = f(x) \in H_{y_0} \cup J$ for some $y_0 \in Y$ and some $J \in \mathcal{J}$, which implies that $x \in f^{-1}(y) \subset f^{-1}(H_{y_0}) \cup f^{-1}(J) \subset f^{-1}(V_{y_0}) \cup f^{-1}(J) \subset \bigcup_{i=1}^n U_{\lambda_i}(y_0) \cup f^{-1}(J)$ and hence, there exists some

$U_{\lambda_i}(y_0) \in \mathcal{U}_{y_0}$ such that $x \in U_{\lambda_i}(y_0) \cup f^{-1}(J)$. Then $x \in [f^{-1}(H_{y_0}) \cup f^{-1}(J)] \cap [U_{\lambda_i}(y_0) \cup f^{-1}(J)] = [f^{-1}(H_{y_0}) \cap U_{\lambda_i}(y_0)] \cup f^{-1}(J)$ and $f^{-1}(H_{y_0}) \cap U_{\lambda_i}(y_0) \in \mathcal{F}_{y_0}$. Thus, $X = \bigcup_{y\in Y} \mathcal{F}_y \cup f^{-1}(J)$, which implies that $X - \bigcup_{y\in Y} \mathcal{F}_y \subset f^{-1}(J)$, it follows that $\mathcal{F} = \bigcup_{y\in Y} \mathcal{F}_y$ is an $\mathcal{I}$-cover of X.

Claim 2. $\mathcal{F}$ is locally finite. Let $x \in X$. Then $y = f(x) \in Y$ and because $\mathcal{H} = \{H_y : y \in Y\}$ is locally finite in Y, there exists an open set $O_y \subset Y$ such that $y \in O_y$ and $Y_0 = \{y \in Y : H_y \cap O_y \neq \emptyset\}$ is a finite set. Observe that $O_x = f^{-1}(O_y)$ is an open subset of X such that $x \in O_x$ and $O_x \cap f^{-1}(H_y) = f^{-1}(O_y) \cap f^{-1}(H_y) = f^{-1}(O_y \cap H_y) \neq \emptyset$ if and only if $O_y \cap H_y \neq \emptyset$, it follows that O_x intersects at most finitely many members of the collection $f^{-1}(\mathcal{H}) = \{f^{-1}(H_y) : y \in Y\}$. Since every member $F \in \mathcal{F}$ is of the form $F = U_{\beta_i}(y) \cap f^{-1}(H_y)$ for some $y \in Y$, and there exists only a finite number of $U_{\lambda_i}(y)$ for each H_y, then O_x intersects at most finitely many members of $\mathcal{F}$. Thus $\mathcal{F}$ is locally finite.

Consequently, from both claims, it follows that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-S-paracompact. ■

In the following result, we consider the cartesian product $X \times Y$ of two spaces X and Y, endowed with the product topology. It is well known that the projection function $\pi_2 : X \times Y \to Y$ is surjective, continuous and open, and if X is compact then π_2 is perfect.

Corollary 3.1. *Let (X, τ) be a compact space and let $(Y, \sigma, \mathcal{I})$ be an $\mathcal{I}$-S-paracompact space. If $\mathcal{J}$ is an ideal on $X \times Y$ such that $\langle \pi_2^{-1}(\mathcal{I}) \rangle \subset \mathcal{J}$, then $X \times Y$ is $\mathcal{J}$-S-paracompact.*

Proof. The proof follows from Theorem 3.2, using the fact that π_2 is a perfect function. ■

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