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# COMMON FIXED POINTS OF COMMUTING MAPPINGS IN ULTRAMETRIC SPACES

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**Abstract.** In this paper, we will use implicit functions to obtain a general result about the existence of a unique common fixed point for commuting mappings in ultrametric spaces. This result enables us to improve some known fixed point theorems and enables us to obtain a relation between completeness and the existence of a unique fixed point for self-mappings in non-Archimedean metric spaces. By presenting some counterexamples, we will show that our results cannot be extended to general metric spaces.

## 1. Introduction

The fixed point theory is concerned with the conditions under which a certain selfmap $T$ of a set $X$ admits fixed points; that is a point $x \in X$ such that $Tx = x$.

The cornerstone of this theory is the Banach's contraction principle [4]. This statement turned out to be a basic tool for solving existence problems in many branches of mathematics. As a consequence many generalizations of it appeared until now; see [6, 8, 11–13, 16] and the references therein.

In 2008, T. Suzuki [22] proved the following conditional type generalization of the Banach contraction principle.

THEOREM 1.1. [22, Theorem 2] *Let $(X, d)$ be a complete metric space and let $T$ be a mapping on $X$. Define $\theta : [0, 1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1, & 0 < r < \frac{\sqrt{5}-1}{2} \\ (1-r)r^{-2}, & \frac{\sqrt{5}-1}{2} \le r < 2^{-1/2} \\ (1+r)^{-1}, & 2^{-1/2} \le r < 1. \end{cases}$$

*Assume that there exists $r \in [0, 1)$ such that*

$$\theta(r)d\big(x, Tx\big) \le d(x, y) \textit{ implies } d\big(Tx, Ty\big) \le rd(x, y)$$

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*for all $x, y \in X$. Then there exists a unique fixed point $z$ of $T$. Moreover $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.*

Using Banach iteration method, Jungck [10] proved a common fixed point theorem for commuting mappings. The idea of Theorem 1.1 suggests the following extension of Jungck's theorem.

Theorem 1.2. [15, Theorem 3] *Let $(X, d)$ be a complete metric space and let $\theta$ be as in Theorem 1.1. Suppose that $S$, $T$ are mappings on $X$ satisfying the following conditions:*

(a) *$S$ is continuous,*

(b) *$T(X) \subset S(X)$,*

(c) *$S$ and $T$ commute.*

*If there exists $r \in [0, 1)$ such that*

$$\theta(r)d(Sx, Tx) \le d(Sx, Sy) \quad \textit{implies} \quad d(Tx, Ty) \le rd(Sx, Sy)$$

*for all $x, y \in X$, then $S$ and $T$ have a unique common fixed point.*

In 2009, Popescu [18] improved the above result as follows.

Theorem 1.3. [18, Theorem 2.1] *Let $(X, d)$ be a complete metric space and $\theta$ be as in Theorem 1.1. Let $S$ and $T$ be mappings on $X$ satisfying the following.*

(a) *$S$ is continuous,*

(b) *$T(X) \subset S(X)$,*

(c) *$S$ and $T$ commute.*

*If there exists $r \in [0, 1)$ such that*

$$\theta(r)d(Sx, Tx) \le d(Sx, Sy) \quad \textit{implies} \quad d(Tx, Ty) \le rM_{S,T}(x, y)$$

*for all $x, y \in X$, where*

$$M_{S,T}(x, y) = \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), \frac{d(Sx, Ty) + d(Sy, Tx)}{2}\right\},$$

*then $S$ and $T$ have a unique common fixed point.*

Recall that a non-Archimedean metric space is a special kind of metric space in which the triangle inequality is replaced with $d(x, y) \le \max\{d(x, z), d(z, y)\}$. Sometimes the associated metric is also called a non-Archimedean metric or an ultra-metric. In a non-Archimedean metric space $X$, for any sequence $\{x_n\}$, we have

$$d(x_n, x_m) \le \max\{d(x_{j+1}, x_j) : m \le j \le n-1\} \quad (n > m).$$

The above inequality implies that a sequence $\{x_n\}$ is Cauchy in a non-Archimedean metric space if and only if $\{d(x_{n+1}, x_n)\}$ converges to zero.

Several mathematicians studied the existence of a fixed point for self-mapping on spherically complete non-Archimedean spaces; see for example [7, 14, 19]. The aim of this paper is to generalize the above results, when the underling space is non-Archimedean. More precisely, we generalize the method that was used in [3] to improve some results in [3, 18, 22] and others. We also show that our results enable us to characterize completeness in non-Archimedean metric spaces. By presenting some counterexamples, we will show that our results cannot be extended to general metric spaces.

## 2. Results

Implicit relations in metric spaces have been considered by several authors in connection with the existence of fixed points (see, for instance, [1–3, 17, 21] and the references therein). We give a new definition of this concept for non-Archimedean metric space as follows.

Let $\Phi$ denote the set of all continuous functions $g : [0,\infty)^6 \to \mathbb{R}$ satisfying the following conditions.

(a) For each $(t_1,t_2,t_3,t_4) \in [0,\infty)^4$ and $0 \le t \le t'$,

$$g(t_1,t_2,t_3,t_4,t',0) \le g(t_1,t_2,t_3,t_4,t,0) \quad \text{and}$$
$$g(t_1,t_2,0,t',t_3,t_4) \le g(t_1,t_2,0,t,t_3,t_4),$$

(b) there exists $r \in [0,1)$ such that

$$g(u,v,v,u,\max\{u,v\},0) \le 0 \text{ or } g(u,v,0,\max\{u,v\},u,v) \le 0$$
$$\text{or } g(u,v,v,v,v,v) \le 0$$

implies $u \le rv$,

(c) $g(u,u,0,0,u,u) > 0$, for all $u > 0$.

Example 2.1. Let $r \in [0,1)$ and $0 \le \alpha + 2\beta + 2\gamma < 1$. Define

(i) $g_1(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - rt_2$,

(ii) $g_2(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - r\max\{t_2,t_3,t_4,t_5,t_6\}$,

(iii) $g_3(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \alpha t_2 - \beta(t_3+t_4) - \gamma(t_5+t_6)$,

where $0 \le t_i < \infty$, $1 \le i \le 6$. A straightforward computation shows that $g_1, g_2, g_3 \in \Phi$.

Now, we are ready to state one of the main results of this section.

Theorem 2.2. *Let $(X,d)$ be a complete ultrametric space and let $T$ and $S$ be mappings on $X$ satisfying the following:*

*(i) $S$ is continuous,*

*(ii) $T(X) \subset S(X)$,*

*(iii) $S$ and $T$ commute.*

*Assume that there exists $g \in \Phi$ such that $d(Sx,Tx) \le d(Sx,Sy)$ implies that*

$$g\big(d(Tx,Ty), d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\big) \le 0$$

*for all $x, y \in X$. Then $S$ and $T$ have a unique common fixed point.*

*Proof.* Since $T(X) \subset S(X)$, we can define a mapping $f$ on $X$ such that $Sfx = Tx$ for all $x \in X$. Therefore $d(Sx, Tx) = d(Sx, Sfx) \le d(Sx, Sfx)$ and hence by assumption,

$$g(d(Tx, Tfx), d(Sx, Sfx), d(Sx, Tx), d(Sfx, Tfx), d(Sx, Tfx), d(Tx, Sfx)) \le 0.$$

Thanks to the property (a),

$$g(d(Sfx, Sffx), d(Sx, Sfx), d(Sx, Sfx), d(Sfx, Sffx), \max\{d(Sx, Sfx), d(Sfx, Sffx)\}, 0) \le 0.$$

By (b), there is some $r \in [0, 1)$ such that $d(Sfx, Sffx) \le rd(Sx, Sfx)$. Fix some $u \in X$ and define $u_n = f^n u$ for all $n \in \mathbb{N}$ and $u_0 = u$. Then $u_{n+1} = fu_n$ and $Su_{n+1} = Tu_n$. Therefore

$$\begin{aligned} d(Su_n, Su_{n+1}) &= d(Sfu_{n-1}, Sffu_{n-1}) \le rd(Su_{n-1}, Sfu_{n-1}) \\ &= rd(Su_{n-1}, Su_n) \le \cdots \le r^n d(Su_0, Su_1). \end{aligned}$$

Thus $\lim_{n\to\infty} d(Su_n, Su_{n+1}) = 0$, that is, $\{Su_n\}$ is a Cauchy sequence. Since $X$ is complete, there is some $z \in X$ such that $Su_n \to z$. We will show that $z$ is a fixed point of $S$. Two alternatives are possible.

(1) $\sharp\{n : d(Sx_n, Tx_n) > d(Sx_n, SSx_n)\} = \infty$ or

(2) $\sharp\{n : d(Sx_n, Tx_n) > d(Sx_n, SSx_n)\} < \infty$.

In the first case, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$d(Sx_{n_j}, Tx_{n_j}) > d(Sx_{n_j}, SSx_{n_j}) \quad (j \in \mathbb{N}).$$

Since $S$ is continuous,

$$\begin{aligned} d(Sz, z) &= \lim_{j\to\infty} d(SSu_{n_j}, z) \le \lim_{j\to\infty} \big(\max\{d(SSu_{n_j}, Su_{n_j}), d(Su_{n_j}, z)\}\big) \\ &\le \lim_{j\to\infty} \big(\max\{d(Su_{n_j}, Tu_{n_j}), d(Su_{n_j}, z)\}\big) \\ &= \lim_{j\to\infty} \big(\max\{d(Su_{n_j}, Su_{n_j+1}), d(Su_{n_j}, z)\}\big) = 0. \end{aligned}$$

Therefore $Sz = z$. In the second case, there exists $l \in \mathbb{N}$ such that for each $n \ge l$ we have $d(Su_n, Tu_n) \le d(Su_n, SSu_n)$. Thus

$$g(d(Tu_n, TSu_n), d(Su_n, SSu_n), d(Su_n, Tu_n), d(SSu_n, TSu_n), d(Su_n, TSu_n), d(SSu_n, Tu_n)) \le 0.$$

Since $S$ and $T$ commute and $Su_{n+1} = Tu_n$,

$$g(d(Su_{n+1}, SSu_{n+1}), d(Su_n, SSu_n), d(Su_n, Su_{n+1}), d(SSu_n, SSu_{n+1}), d(Su_n, SSu_{n+1}), d(SSu_n, Su_{n+1})) \le 0.$$

By letting $n \to \infty$, we have

$$g(d(z, Sz), d(z, Sz), 0, 0, d(z, Sz), d(Sz, z)) \le 0.$$

By (c), $d(z, Sz) = 0$. That is, $z$ is a fixed point of $S$. Next, we will prove that

$$d(T^n z, T^{n+1} z) \le r^n d(Tz, z) \quad (n \in \mathbb{N}). \tag{2.1}$$

Since for each $n > 1$,

$$d(ST^{n-1}z, T^n z) \le d(ST^{n-1}z, T^n z) = d(ST^{n-1}z, T^n Sz) = d(ST^{n-1}z, ST^n z),$$

we have

$$g\big(d(T^n z, T^{n+1}z), d(ST^{n-1}z, ST^n z), d(ST^{n-1}z, T^n z), d(ST^n z, T^{n+1}z),\\ d(ST^{n-1}z, T^{n+1}z), d(ST^n z, T^n z)\big) \le 0.$$

Hence from (iii),

$$g(d(T^n z, T^{n+1}z), d(T^{n-1}z, T^n z), d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z),\\ d(T^{n-1}z, T^{n+1}z), 0) \le 0.$$

According to (a),

$$g(d(T^n z, T^{n+1}z), d(T^{n-1}z, T^n z), d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z),\\ \max\{d(T^{n-1}z, T^n z), d(T^n z, T^{n+1}z)\}, 0) \le 0.$$

By (b), we have $d(T^n z, T^{n+1}z) \le r d(T^{n-1}z, T^n z)$. So that (2.1) is proved.

Next, we will show that

$$d(Tx, z) \le r d(Sx, z) \quad (Sx \ne z). \tag{2.2}$$

For $x \in X$ with $Sx \ne z$, there exists $n_0 \in \mathbb{N}$ such that $d(Su_n, z) < \dfrac{1}{3} d(z, Sx)$ for all $n \ge n_0$. If $n \ge n_0$, we have

$$\begin{aligned} d(Su_n, Tu_n) = d(Su_n, Su_{n+1}) &\le \max\{d(Su_n, z), d(Su_{n+1}, z)\} \\ &< \frac{2}{3} d(Sx, z) = d(Sx, z) - \frac{1}{3} d(Sx, z) \\ &\le d(Sx, z) - d(Su_n, z) \le d(Su_n, Sx). \end{aligned}$$

By assumption,

$$g\big(d(Tu_n, Tx), d(Su_n, Sx), d(Su_n, Tu_n), d(Sx, Tx),\\ d(Su_n, Tx), d(Sx, Tu_n)\big) \le 0.$$

for all $n \ge n_0$. That is,

$$g\big(d(Su_{n+1}, Tx), d(Su_n, Sx), d(Su_n, Su_{n+1}), d(Sx, Tx),\\ d(Su_n, Tx), d(Sx, Su_{n+1})\big) \le 0.$$

By continuity of $g$, it follows that

$$g\big(d(z, Tx), d(z, Sx), 0, d(Sx, Tx), d(z, Tx), d(Sx, z)\big) \le 0.$$

The property (a) implies that

$$g\big(d(z, Tx), d(z, Sx), 0, \max\{d(Sx, z), d(z, Tx)\}, d(z, Tx), d(Sx, z)\big) \le 0.$$

It follows from the property (b)and the above inequality that (2.2) holds.

By induction we will show that

$$d(T^n z, Tz) \le r d(Tz, z) \tag{2.3}$$

for $n \geq 2$. For $n = 2$, by (2.1), we obtain $d(T^2z, Tz) \leq rd(Tz, z)$. Assume that (2.3) holds for some $n \geq 2$. Then

$$\begin{aligned} d(T^{n+1}z, Tz) &\leq \max\{d(T^nz, Tz), d(T^nz, T^{n+1}z)\} \\ &\leq \max\{rd(z, Tz), r^nd(z, Tz)\} = rd(z, Tz). \end{aligned}$$

Hence (2.3) is true.

According to (2.1), $\{T^nz\}$ is a Cauchy sequence in $(X, d)$. If $T^nz = z$ for some $n$, then by (2.3), $Tz = z$ in this case. Otherwise, we can assume that $T^mz \neq z$ for all $m \in \mathbb{N}$. In the latter case, by (2.2) we have

$$d(T^{m+1}z, z) \leq r^md(Tz, z) \quad (m \in \mathbb{N}). \tag{2.4}$$

Therefore $\{T^nz\}$ converges to $z$. Since $d(T^nz, Tz) \leq rd(Tz, z)$, by letting $n \to \infty$, we obtain $d(z, Tz) \leq rd(Tz, z)$. This is a contradiction. Therefore $Tz = z$.

We will prove that $z$ is a unique common fixed point. Suppose that $y$ is another common fixed point of $S$ and $T$. Then $d(Sz, Tz) = 0 \leq d(Sz, Sy)$. By our hypothesis,

$$g(d(Tz, Ty), d(Sz, Sy), d(Sz, Tz), d(Sy, Ty), d(Sz, Ty), d(Sy, Tz)) \leq 0.$$

That is,

$$g(d(z, y), d(z, y), d(z, z), d(y, y), d(z, y), d(y, z)) \leq 0.$$

Hence

$$g(d(z, y), d(z, y), 0, 0, d(z, y), d(y, z)) \leq 0.$$

By (c), we have $d(y, z) = 0$. Therefore $y = z$. ■

The following result generalizes [3, Theorem 3.1], when the metric is non-Archimedean.

Corollary 2.3. *Let $(X, d)$ be a complete ultrametric space and let $T$ be a mapping on $X$. Assume that there exists $g \in \Phi$ such that $d(x, Tx) \leq d(x, y)$ implies*

$$g(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0$$

*for all $x, y \in X$. Then $T$ has a unique fixed point.*

*Proof.* Let $S$ be the identity function on $X$. Then the result follows from Theorem 2.2. ■

We are also able to extend Theorem 1.2 in ultrametric spaces.

Corollary 2.4. *Let $(X, d)$ be a complete ultrametric space and let $T$ and $S$ be mappings on $X$ satisfying the following:*

(i) *$S$ is continuous,*

(ii) *$T(X) \subset S(X)$,*

(iii) *$S$ and $T$ commute.*

*Assume that $d(Sx, Tx) \leq d(Sx, Sy)$ implies $d(Tx, Ty) \leq rd(Sx, Sy)$ for all $x, y \in X$. Then $S$ and $T$ have a unique common fixed point.*

*Proof.* Let $g = g_1$ in Example 2.1. Then $g$ satisfies the above condition. So that the result follows from Theorem 2.2. ■

The following example shows that the conditions of Theorem 2.2 cannot be weakened, that is $\leq$ cannot be replaced by $<$.

EXAMPLE 2.5. Let $X = \{\pm 1\}$ with discrete metric and $0 < r < 1$. Define $T : X \to X$ by $Tx = -x$ and $Sx = x$ on $X$. Clearly, $(X, d)$ is a complete ultrametric space. We have $d(x, Tx) = 1 \geq d(x, y) = d(Sx, Sy)$ for all $x, y \in X$. Hence

$$d(x, Tx) < d(Sx, Sy) \text{ implies } d(Tx, Ty) \leq r d(Sx, Sy)$$

for all $x, y \in X$. But $T$ does not have a fixed point. That is, $S$ and $T$ do not have a common fixed point.

The following example is due to Suzuki [22, Theorem 3] which shows that in general metric spaces, Corollary 2.4 and hence Theorem 2.2 is not true.

EXAMPLE 2.6. Define a complete subset $X$ of the Euclidean space $\mathbb{R}$ as follows: $X = \{0, 1\} \cup \big\{x_n : n \in \mathbb{N} \cup \{0\}\big\}$, where $x_n = (\frac{1}{4})(-\frac{3}{4})^n$ for $n \in \mathbb{N} \cup \{0\}$. Define a mapping $T$ on $X$ by $T0 = 1$, $T1 = x_0$ and $Tx_n = x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly, $T$ does not have a fixed point. We claim that

$$d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \frac{3}{4} d(x, y).$$

for all $x, y \in X$. Indeed,

- $d(T0, T1) = \frac{3}{4} d(0, 1)$,
- $d(Tx_m, Tx_n) = \frac{3}{4} d(x_m, x_n)$ for $m, n \in \mathbb{N} \cup \{0\}$,
- $d(0, T0) > d(0, x_n)$ for $n \in \mathbb{N} \cup \{0\}$.

Also, we have

$$\begin{aligned} d(T1, Tx_n) - \frac{3}{4} d(1, x_n) &= \frac{1}{4} - \frac{1}{4}\left(-\frac{3}{4}\right)^{n+1} - \frac{3}{4}\left(1 - \frac{1}{4}\left(-\frac{3}{4}\right)^n\right) \\ &= -\frac{1}{2} - \frac{1}{2}\left(-\frac{3}{4}\right)^{n+1} < 0, \end{aligned}$$

for each $n \in \mathbb{N}$.

We are also able to give the following generalization of Ćirić fixed point theorem [5] in ultrametric spaces.

COROLLARY 2.7. *Let $(X, d)$ be a complete ultrametric space and let $T$ and $S$ be mappings on $X$ satisfying the following:*

(i) *$S$ is continuous,*

(ii) *$T(X) \subset S(X)$,*

(iii) *$S$ and $T$ commute.*

*Assume that there exists $r \in [0, 1)$ such that $d(Sx, Tx) \leq d(Sx, Sy)$ implies*

$$d(Tx, Ty) \leq r \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)\}$$

*for all $x, y \in X$. Then $S$ and $T$ have a unique common fixed point.*

*Proof.* Let $g = g_2$ in Example 2.1. By Theorem 2.2, $S$ and $T$ have a common fixed point. ∎

Theorem 2.2 also enables us to extend Hardy-Rogers fixed point theorem [9] for ultrametric spaces.

Corollary 2.8. *Let $(X, d)$ be a complete ultrametric space and let $T$ and $S$ be mappings on $X$ satisfying the following:*

(i) *$S$ is continuous,*

(ii) *$T(X) \subset S(X)$,*

(iii) *$S$ and $T$ commute.*

*Let $\sum_{i=1}^{5} \alpha_i < 1$ and $\alpha_i \geq 0$ for $1 \leq i \leq 5$. Suppose that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha_1 d(Sx, Sy) + \alpha_2 d(Sx, Tx) + \alpha_3 d(Sy, Ty) + \alpha_4 d(Sx, Ty) + \alpha_5 d(Sy, Tx). \quad (2.5)$$

*Then $S$ and $T$ have a unique common fixed point.*

*Proof.* By symmetry, we have

$$d(Tx, Ty) \leq \alpha_1 d(Sx, Sy) + \alpha_2 d(Sy, Ty) + \alpha_3 d(Sx, Tx) + \alpha_4 d(Sy, Tx) + \alpha_5 d(Sx, Ty). \quad (2.6)$$

It follows from (2.5) and (2.6) that

$$d(Tx, Ty) \leq \alpha_1 d(Sx, Sy) + \frac{\alpha_2 + \alpha_3}{2}\big(d(Sx, Tx) + d(Sy, Ty)\big) + \frac{\alpha_4 + \alpha_5}{2}\big(d(Sx, Ty) + d(Sy, Tx)\big).$$

Let $\alpha = \alpha_1$, $\beta = \frac{\alpha_2+\alpha_3}{2}$, $\gamma = \frac{\alpha_4+\alpha_5}{2}$ and $g = g_3$ in Example 2.1. Then the result follows from Theorem 2.2. ∎

The following result shows that when the underling space is non-Archimedean, we may assume $\theta \equiv 1$ in Theorem 1.1.

Corollary 2.9. *Let $(X, d)$ be a complete non-Archimedean metric space and $T : X \to X$. Let for some $0 < r < 1$,*

$$d(x, Tx) \leq d(x, y) \ \ \text{implies} \ \ d(Tx, Ty) \leq rd(x, y) \ \ \text{for all } x, y \in X. \qquad (*)$$

*Then $T$ has a unique fixed point $z$ and for every $x \in X$, $\lim_{n\to\infty} T^n(x) = z$.*

*Proof.* Let $S$ be the identity function on $X$. Then the result follows from Corollary 2.4. ∎

The following example shows that our results are genuine generalization of Suzuki's fixed point theorem provided that the underling space is non-Archimedean. In fact, we give an example of a mapping on a complete ultrametric space which satisfies the conditions of Corollary 2.9 but Theorem 1.1 cannot be applied.

EXAMPLE 2.10. Let $X = \{a, b, c, e\}$ and $d(a, c) = d(a, e) = d(b, c) = d(b, e) = 1$ and $d(a, b) = d(c, e) = \frac{3}{4}$. It is easy to verify that $X$ is a complete ultrametric space. Define $T : X \to X$ by $T(a) = T(b) = T(c) = a$ and $T(e) = b$. For $r = \frac{3}{4}$, we have $\theta(r) = \frac{4}{7}$. Since $\theta(r)d(c, Tc) = \frac{4}{7} \le \frac{3}{4} = d(c, e)$ and $d(Tc, Te) = \frac{3}{4} > \frac{9}{16} = rd(c, e)$, $T$ does not satisfy in assumption of Theorem 1.1. We will show that

$$d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le \frac{3}{4}d(x, y).$$

for all $x, y \in X$. Since $d(Ta, Tb) = d(Ta, Tc) = d(Tb, Tc) = 0$, we have $d(Tx, Ty) \le \frac{3}{4}d(x, y)$ for $x, y \in \{a, b, c\}$. Also,

$$d(Ta, Te) = \frac{3}{4} \le \frac{3}{4} = \frac{3}{4}d(a, e), \quad d(Tb, Te) = \frac{3}{4} \le \frac{3}{4} = \frac{3}{4}d(b, e).$$

Since $d(e, Te) = 1 > \frac{3}{4} = d(c, e)$, the proof is completed.

The next result gives a characterization for completeness in non-Archimedean metric spaces.

THEOREM 2.11. *Suppose that $(X, d)$ is a non-Archimedean metric space such that for some $0 < r < 1$, every self mapping $T : X \to X$ with the property $(*)$ has a fixed point. Then $(X, d)$ is complete.*

*Proof.* Let $(X, d)$ be incomplete non-Archimedean metric space and $0 < r < 1$. Then there is a Cauchy sequence $\{x_n\}$ in $X$ which is not convergent. Define $f : X \to [0, \infty)$ by $f(x) = \lim_{n\to\infty} d(x, x_n)$ for all $x \in X$. Since $\{d(x_n, x)\}$ is a Cauchy sequence in $\mathbb{R}$, it is convergent. Hence $f$ is well-defined. It follows from the definition that

(i) $f(x) - f(y) \le d(x, y) \le \max\{f(x), f(y)\}$ for all $x, y \in X$.
(ii) $f(x) > 0$ for all $x \in X$ and
(iii) $\lim_{n\to\infty} f(x_n) = 0$.

It follows from (ii) and (iii) that for each $x \in X$, there is some $n_x \in X$ such that $f(x_{n_x}) < \frac{r}{4}f(x)$. Define $T : X \to X$ by $Tx = x_{n_x}$ for all $x \in X$. Then $f(Tx) \le \frac{r}{4}f(x)$ for all $x \in X$. Hence $Tx \ne x$ for all $x \in X$. We will show that $(*)$ holds. Let for some $x, y \in X$, $d(x, Tx) \le d(x, y)$. Two cases may happen.

(a) $f(y) > 2f(x)$. In this case by (i) we have,

$$\begin{aligned} d(Tx, Ty) &\le \max\big\{f(Tx), f(Ty)\big\} \\ &\le \max\big\{\frac{r}{4}f(x), \frac{r}{4}f(y)\big\} \le \frac{r}{2}f(y) \le r\big(f(y) - f(x)\big) \le rd(x, y). \end{aligned}$$

(b) $f(y) \le 2f(x)$. We have

$$d(x, y) \ge d(x, Tx) \ge f(x) - f(Tx) \ge (1 - \frac{r}{4})f(x) \ge \frac{1}{2}f(x).$$

Since

$$\begin{aligned} d(Tx, Ty) &\le \max\big\{f(Tx), f(Ty)\big\} \\ &\le \max\big\{\frac{r}{4}f(x), \frac{r}{4}f(y)\big\} \le \frac{r}{2}f(x), \end{aligned}$$

$(*)$ also holds in this case. This completes our proof. ∎

The following result follows immediately from Corollary 2.9 and Theorem 2.11.

Corollary 2.12. *Let $(X, d)$ be a non-Archimedean metric space. Then the following are equivalent:*

(a) *$(X, d)$ is complete.*

(b) *There is some $0 < r < 1$ such that every self mapping $T : X \to X$ which satisfies $(*)$ has a unique fixed point.*

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REFERENCES


[**1**] S. M. A. Aleomraninejad, Sh. Rezapour and N. Shahzad, *On fixed point generalizations of Suzukis method*, Appl. Math. Lett. **24** (2011), 1037–1040.

[**2**] A. Aliouche and V. Popa, *General common fixed point theorems for occasionally weakly compatible hybrid mappings and applications*, Novi Sad J. Math. **39** (1), (2009), 89–109.

[**3**] I. Altun and A. Erduran, *A Suzuki type fixed-point theorem*, Int. J. Math. Math. Sci., Article ID 736063, (2011), 9 pages.

[**4**] S. Banach, *Sur les opérations dans les ensemble abstraits et leur application aux équations intégrales,* Fund. Mat. **3** (1922), 133–181.

[**5**] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **43** (2), (1974), 267–273.

[**6**] M. Edelstein, *An extension of Banach contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7–10.

[**7**] Lj. Gajić, *On ultra metric spaces*, Novi Sad J. Math., **31** (2), (2001), 69–71.

[**8**] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[**9**] G. E. Hardy and T. D. Rogers, *A generalization of the theorem of Reich*, Canad. Math. Bull. Vol. **16** (2), (1973), 201–206.

[**10**] G. Jungck, *Commuting mappings and fixed points,* Amer. Math. Montly (1976), 261–263.

[**11**] R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc. **60** (1968), 71–76.

[**12**] R. Kannan, *Some results on fixed points-II*, Amer. Math. Monthly **76** (1969), 405–408.

[**13**] R. Kannan, *Some results on fixed points-III,* Fund. Math. **70** (1971), 169–177.

[**14**] J. Kubiaczyk and N. Mostafa Ali, *A multivalued fixed point theorem in non-Archimedean vector spaces*, Novi Sad J. Math. **26** (2), (1996), 111–116.

[**15**] M. Kikkawa and T. Suzuki, *Three fixed point theorems for generalized contractions with constants in complete metric spaces,* Nonlinear Anal. **69** (2008) 2942-2949.

[**16**] W. A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Kluwer Academic, London, 2001.

[**17**] V. Popa and M. Imdad and J. Ali, *Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces*, Bull. Malays. Math. Sci. Soc. (2), **33** (1), (2010), 105–120.

[**18**] O. Popescu, *Two fixed point theorems for generalized contractions with constants in complete metric space*, Cent. Eur. J. Math. **7** (3), (2009), 529–538.

[**19**] C. Petalas and F. Vidalis, *A fixed point theorem in non-Archimedaen vector spaces*, Proc. Amer. Math. Soc., **118** (1993), 819–821.

[**20**] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **236** (1997), 257–290.

[**21**] B. Samet, *Fixed point results for implicit contractions on spaces with two metrics*, J. Ineq. Appl., **2014**:84 (2014).

[**22**] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness,* Proc. Amer. Math. Soc. **136** (5), (2008), 1861–1869.

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