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SOME HOMOLOGICAL PROPERTIES OF AMALGAMATION

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Abstract. Let R and S be commutative rings, let J be an ideal of S and let $f\colon R \to S$ be a ring homomorphism. In this paper, we investigate some homological properties of the amalgamation of R with S along J with respect to f (denoted by $R \bowtie^f J$), introduced by D'Anna and Fontana in 2007. In addition, we deal with the strongly cotorsion properties of local cohomology module of $R \bowtie^f J$, when $R \bowtie^f J$ is a local Noetherian ring.

1. Introduction

Throughout this paper all rings are considered commutative with identity element, and all ring homomorphisms are unital. In [7], D'Anna and Fontana considered a construction obtained involving a ring R and an ideal $I \subset R$ that was denoted by $R \bowtie I$, called amalgamated duplication, and it was defined as the following subring of $R \times R$:

$$R \bowtie I = \{(r, r+i) \mid r \in R, i \in I\}.$$

This construction was studied from different points of view in [1, 3, 7, 10, 11, 13]. In [4], a systematic study of a new ring construction is initiated, called the "amalgamation of R with S along J with respect to f", for a given homomorphism of rings $f\colon R \to S$ and ideal J of S. This construction finds its roots in a paper by J.L. Dorroh appeared in [8] and provides a general frame for studying the amalgamated duplication of a ring along an ideal. The amalgamation of R with S along J with respect to f is a subring of $R \times S$ which is defined as follows:

$$R \bowtie^f J = \{(r, f(r)+j) \mid r \in R, j \in J\}.$$

This construction is a generalization of the amalgamated duplication of a ring along an ideal and other classical constructions, such as the Nagata's idealization are strictly related to it [4, Example 2.7 and Remark 2.8]. One of the key tools for studying $R \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [4]. This point of view allows to deepen the study initiated in [4] and continued in [5] and to provide an ample description of

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various properties of $R \bowtie^f J$, in connection with the properties of R, J and f. In [4], necessary and sufficient conditions are provided for $R \bowtie^f J$ to inherit the properties of Noetherian ring, integral domain, and reduced ring and characterized pullbacks that can be expressed as amalgamations. In [5], they provided a complete description of the prime spectrum of $R \bowtie^f J$ and gave bounds for its dimension. In [6], the authors studied in details its prime spectrum and, when $R \bowtie^f J$ is a local Noetherian ring, some of its invariants (like the embedding dimension) and relevant properties (like Cohen-Macaulayness and Gorensteinness). Indeed, in [6, Proposition 5.7], they stated necessary and sufficient conditions for the self-injectivity of the ring $R \bowtie^f J$. As a nice generalization of injectivity for modules, Enochs in [9] introduced the notion of cotorsion modules and as an special case of cotorsion modules Xu in [12] introduced the terminology of strongly cotorsion modules. In Theorem 2.2, we investigate the strongly cotorsion properties of $\mathrm{H}^{\dim R}_{\mathfrak{m}\bowtie^f J}(R \bowtie^f J)$ in connection with the strongly cotorsion properties of $\mathrm{H}^{\dim R}_{\mathfrak{m}}(R)$ and $\mathrm{H}^{\dim R}_{\mathfrak{m}}(J)$, when $R \bowtie^f J$ is a local Noetherian ring. In addition, we investigate some homological properties of the amalgamation.

2. Main results

Let R and S be commutative rings with unity, let J ba an ideal of S and let $f : R \to S$ be a ring homomorphism. In the following theorem we summarize some properties of $R \bowtie^f J$ from [4] and [6].

Theorem 2.1. *Let R and S be commutative rings, let J ba an ideal of S and let $f\colon R \to S$ be a ring homomorphism. The following statements hold.*

(*i*) *There exists the natural ring homomorphism $\varphi\colon R \to R \bowtie^f J$ defined by $\varphi(r) := (r, f(r))$, for all $r \in R$. The map φ is an embedding, making $R \bowtie^f J$ a ring extension of R. Furthermore, R has $(R \bowtie^f J)$-module structure by the natural projection $p_R\colon R \bowtie^f J \to R$.*

(*ii*) *$R \bowtie^f J$ is isomorphic as an R-module to $R \oplus J$.*

(*iii*) *$R \bowtie^f J$ is a local ring if and only if R is a local ring and $J \subseteq \mathrm{J}(S)$, where $\mathrm{J}(S)$ is the Jacobson radical of S. In particular, if $\mathfrak{m}$ is the unique maximal ideal of R, then $\mathfrak{m} \bowtie^f J = \{(m, f(m) + j) \mid m \in \mathfrak{m}, j \in J\}$ is the unique maximal ideal of $R \bowtie^f J$.*

(*iv*) *Let $(R, \mathfrak{m})$ be a local ring and let $J \subseteq \mathrm{J}(S)$ be finitely generated as an R-module. Then $\dim R = \dim(R \bowtie^f J) = \dim_R(R \bowtie^f J)$*

(*v*) *Let $(R, \mathfrak{m})$ be a local ring and let $J \subseteq \mathrm{J}(S)$ be finitely generated as an R-module. Then $R \bowtie^f J$ is a Cohen-Macaulay ring if and only if it is a Cohen-Macaulay R-module if and only if J is a maximal Cohen-Macaulay module.*

(*vi*) *Let $R \bowtie^f J$ be a local ring, where R is a Cohen-Macaulay ring. Assume that $f(R) + J$ satisfies Serre's condition (S_1) such that $\dim(f(R) + J) = \dim R$, and suppose that $J \neq 0$ such that $f^{-1}(J)$ is a regular ideal of R. Then the following conditions are equivalent:*

  (*a*) *$R \bowtie^f J$ is Gorenstein.*

(*b*) *$f(R)+J$ is a Cohen-Macaulay ring, J is a canonical module of $f(R)+J$ and $f^{-1}(J)$ is a canonical module of R.*

Note that Theorem 2.1(vi) provides the necessary and sufficient conditions of self-injectivity of the ring $R \bowtie^f J$. As a nice generalization of injectivity for modules, Enochs in [9] introduced the notion of cotorsion modules. An R-module M is called a cotorsion module if $\operatorname{Ext}^1_R(F, M) = 0$ for all flat R-modules F. Furthermore, as an special case of cotorsion modules Xu in [12] introduced the terminology of strongly cotorsion modules. An R-module M is called a strongly cotorsion module if $\operatorname{Ext}^1_R(F, M) = 0$ for all R-modules F with finite flat dimension. One can easily show that if M is a strongly cotorsion R-module, then $\operatorname{Ext}^i_R(F, M) = 0$ for all $i \geq 1$ and all R-modules F with finite flat dimension. In the following theorem we investigate the strongly cotorsion properties of $\mathrm{H}^{\dim R}_{\mathfrak{m}\bowtie^f J}(R \bowtie^f J)$ in connection with the strongly cotorsion properties of $\mathrm{H}^{\dim R}_{\mathfrak{m}}(R)$ and $\mathrm{H}^{\dim R}_{\mathfrak{m}}(J)$, when $R \bowtie^f J$ is a local Noetherian ring.

Theorem 2.2. *We preserve the assumptions of Theorem 2.1, and moreover we assume that $(R, \mathfrak{m})$ is a Noetherian local ring with dimension d and $0 \neq J \subseteq \mathrm{J}(S)$ is an ideal such that J is a finitely generated R-module. Then $\mathrm{H}^d_{\mathfrak{m}\bowtie^f J}(R \bowtie^f J)$ is a strongly cotorsion R-module if and only if $\mathrm{H}^d_{\mathfrak{m}}(R)$ and $\mathrm{H}^d_{\mathfrak{m}}(J)$ are strongly cotorsion R-modules.*

Proof. By Theorem 2.1(iv), R and $R \bowtie^f J$ have the same dimension d and $R \bowtie^f J$ is a local ring with maximal ideal $\mathfrak{m}_0 = \mathfrak{m} \bowtie^f J$. Then we have the following R-isomorphisms:

$$\mathrm{H}^d_{\mathfrak{m}_0}(R \bowtie^f J) \cong \mathrm{H}^d_{\mathfrak{m}}(R \bowtie^f J) \cong \mathrm{H}^d_{\mathfrak{m}}(R \oplus J) \cong \mathrm{H}^d_{\mathfrak{m}}(R) \oplus \mathrm{H}^d_{\mathfrak{m}}(J).$$

The first isomorphism follows from [2, Theorem 4.2.1] and the second one follows from Theorem 2.1(ii). Now assume that $\mathrm{H}^d_{\mathfrak{m}_0}(R \bowtie^f J)$ is a strongly cotorsion R-module. Therefore, for any R-module F with finite flat dimension we have

$$0 = \operatorname{Ext}^1_R(F, \mathrm{H}^d_{\mathfrak{m}_0}(R \bowtie^f J)) \cong \operatorname{Ext}^1_R(F, \mathrm{H}^d_{\mathfrak{m}}(R)) \oplus \operatorname{Ext}^1_R(F, \mathrm{H}^d_{\mathfrak{m}}(J)).$$

Hence, $\operatorname{Ext}^1_R(F, \mathrm{H}^d_{\mathfrak{m}}(J)) = \operatorname{Ext}^1_R(F, \mathrm{H}^d_{\mathfrak{m}}(R)) = 0$ for any R-module F with finite flat dimension and this implies that $\mathrm{H}^d_{\mathfrak{m}}(R)$ and $\mathrm{H}^d_{\mathfrak{m}}(J)$ are strongly cotorsion R-modules. The converse can be proven in a similar way. ■

Let R be a ring and let I be an ideal of R. The amalgamated duplication of R along I, denoted by $R \bowtie I$, is the special case of $R \bowtie^f I$ where $f: R \to R$ is an identity homomorphism, see [7]. Note that if $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d, then $R \bowtie I$ is a Noetherian local ring with maximal ideal $\mathfrak{m} \bowtie I = \{(m, m+i) \mid m \in \mathfrak{m}, i \in I\}$ of dimension d, see [7, Corollary 3.3 and Theorem 3.5]. Therefore we have the following result.

Corollary 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let $0 \neq I$ be an ideal of R. Then $\mathrm{H}^d_{\mathfrak{m}\bowtie I}(R \bowtie I)$ is a strongly cotorsion R-module if and only if $\mathrm{H}^d_{\mathfrak{m}}(R)$ and $\mathrm{H}^d_{\mathfrak{m}}(I)$ are strongly cotorsion R-modules.*

In the sequel we investigate some homological properties of the amalgamation.

Proposition 2.4. *Let $f\colon R \to S$ be a ring homomorphism and let J be a non-zero ideal of S which is a flat R-module. Then the following statements hold for any R-module M.*

(*i*) $\mathrm{fd}_R(M) = \mathrm{fd}_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J))$.

(*ii*) $\mathrm{pd}_R(M) = \mathrm{pd}_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J))$.

Proof. By Theorem 2.1(ii), the R-module $R \bowtie^f J$ is faithfully flat since J is flat as an R-module. First, suppose that $\mathrm{fd}_R(M) \leq n$ (resp. $\mathrm{pd}_R(M) \leq n$) and pick an n-step flat (resp. projective) resolution of M over R as follows:

$$(*) : 0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Applying the functor $- \otimes_R (R \bowtie^f J)$ to $(*)$, we obtain the exact sequence of $(R \bowtie^f J)$-modules:

$$0 \to F_n \otimes_R (R \bowtie^f J) \to \cdots \to F_0 \otimes_R (R \bowtie^f J) \to M \otimes_R (R \bowtie^f J) \to 0.$$

Thus, $\mathrm{fd}_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J)) \leq n$ (resp. $\mathrm{pd}_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J)) \leq n$). Conversely, suppose that $\mathrm{fd}_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J)) \leq n$ (resp. $\mathrm{pd}_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J)) \leq n$). Since $R \bowtie^f J$ is a flat R-module, we conclude that for any R-module N and each $i \geq 1$ we have:

$$(1) : \ \mathrm{Tor}_i^R(M, N \otimes_R (R \bowtie^f J)) \cong \mathrm{Tor}_i^{R\bowtie^f J}(M \otimes_R (R \bowtie^f J), N \otimes_R (R \bowtie^f J))$$

$$(2) : \ \mathrm{Ext}_R^i(M, N \otimes_R (R \bowtie^f J)) \cong \mathrm{Ext}^i_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J), N \otimes_R (R \bowtie^f J))$$

Furthermore, $\mathrm{Tor}_i^R(M, N)$ and $\mathrm{Ext}_R^i(M, N)$ are direct summands of $\mathrm{Tor}_i^R(M, N \otimes_R (R \bowtie^f J))$ and $\mathrm{Ext}_R^i(M, N \otimes_R (R \bowtie^f J))$ respectively. Then, we conclude that $\mathrm{fd}_R(M) \leq n$ (resp. $\mathrm{pd}_R(M) \leq n$). ■

Proposition 2.5. *Let $f\colon R \to S$ be a ring homomorphism and let J be a non-zero ideal of S which is a flat R-module. Then the following statements hold for every R-module M.*

(*i*) $\mathrm{id}_R(M) = \mathrm{id}_R(M \otimes_R (R \bowtie^f J))$

(*ii*) $\mathrm{fd}_R(M) = \mathrm{fd}_R(M \otimes_R (R \bowtie^f J))$

Proof. Note that $R \bowtie^f J$ is a faithfully flat R-module. (*i*) follows from [13, Corollary 2.9] and (*ii*) follows from [13, Corollary 2.11]. ■

Corollary 2.6. *We preserve the assumptions of Proposition 2.5. For every R-module M, we have*

$$\mathrm{fd}_R(M) = \mathrm{fd}_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J)) = \mathrm{fd}_R(M \otimes_R (R \bowtie^f J)).$$

Proof. By Proposition 2.4, we have $\mathrm{fd}_R(M) = \mathrm{fd}_{R\bowtie^f J}(M \otimes_R (R \bowtie^f J))$, and by Proposition 2.5, $\mathrm{fd}_R(M) = \mathrm{fd}_R(M \otimes_R (R \bowtie^f J))$. ■

Proposition 2.7. *Let $f\colon R \to S$ be a ring homomorphism and let J be a non-zero ideal of S. Then the following statements hold.*

(*i*) *If M is a (faithfully) injective R-module, then* $\operatorname{Hom}_R(R \bowtie^f J, M)$ *is a (faithfully) injective* $(R \bowtie^f J)$*-module.*

(*ii*) *Every injective* $(R \bowtie^f J)$*-module is a direct summand of the R-module* $\operatorname{Hom}_R(R \bowtie^f J, M)$*, where M is an injective R-module.*

Proof. (*i*) The following sequence of $(R \bowtie^f J)$-isomorphisms makes clear that if M is a (faithfully) injective R-module, then $\operatorname{Hom}_R(R \bowtie^f J, M)$ is a (faithfully) injective $(R \bowtie^f J)$-module.

$$\begin{aligned}\operatorname{Hom}_{R\bowtie^f J}(-, \operatorname{Hom}_R(R \bowtie^f J, M)) &\cong \operatorname{Hom}_R((R \bowtie^f J) \otimes_{R\bowtie^f J} -, M)\\ &\cong \operatorname{Hom}_R(-, M).\end{aligned}$$

Note that in the above sequence, the first isomorphism follows from Hom-tensor adjointness, and the second isomorphism is induced by tensor cancellation.

(*ii*) Let E be an injective $(R \bowtie^f J)$-module. It is enough to show that E is embeded into an R-module of the form $\operatorname{Hom}_R(R \bowtie^f J, M)$ where M is an injective R-module. Consider E as an R-module and embed it into an injective R-module M. Then use isomorphisms in part (*i*), to convert the monomorphism of R-modules $E \hookrightarrow M$ to a monomorphism of $(R \bowtie^f J)$-modules $E \hookrightarrow \operatorname{Hom}_R(R \bowtie^f J, M)$. ∎

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