

STRONG MIXED AND GENERALIZED FRACTIONAL CALCULUS FOR BANACH SPACE VALUED FUNCTIONS

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Abstract. We present here a strong mixed fractional calculus theory for Banach space valued functions of generalized Canavati type. Then we establish several mixed fractional Bochner integral inequalities of various types.

1. Introduction

Here we use the Bochner integral for Banach space valued functions, which is a direct generalization of Lebesgue integral to this case. The reader may read about Bochner integral and its properties from [2, 6, 7, 9–12].

Using Bochner integral properties and the great article [12], we develop a right and left generalized Canavati type, [8], strong fractional theory for the first time in the literature, which is the direct analog of the real one, but now dealing with Banach space valued functions.

In the literature there are very few articles only about the left weak fractional theory of Banach space valued functions with one of the best [1].

However we found the left weak theory, using Pettis integral and functionals, complicated, less clear, difficult and unnecessary.

With this article and [4, 5] earlier, we try to simplify matters and put the related theory on its natural grounds and resemble the theory on real numbers.

2. Main results

Here $C([a, b], X)$ stands for the space of continuous functions from $[a, b]$ into X , where X is a Banach space.

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All integrals here are of Bochner type. By [4], we have that: if $f \in C([a, b], X)$, then $f \in L_\infty([a, b], X)$ and $f \in L_1([a, b], X)$. Derivatives for vector valued functions are defined according to [11, p. 83], similar to numerical ones.

We need the following reverse Taylor's formula:

THEOREM 2.1. *Let $n \in \mathbb{N}$ and $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and X is a Banach space. Then*

$$f(a) = \sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b) + \frac{1}{(n-1)!} \int_b^a (a-t)^{n-1} f^{(n)}(t) dt. \tag{2}$$

Proof. We consider

$$F(x) := \sum_{i=0}^{n-1} \frac{(a-x)^i}{i!} f^{(i)}(x), \quad x \in [a, b].$$

Clearly $F \in C([a, b], X)$ and $F(a) = f(a)$, with $F(b) = \sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b)$. Furthermore it holds

$$F'(x) = \frac{(a-x)^{n-1}}{(n-1)!} f^{(n)}(x), \quad \forall x \in [a, b],$$

and $F' \in C([a, b], X)$.

By the Fundamental Theorem of Calculus for Banach space valued functions and Bochner integral, see [12], we get $F(b) - F(a) = \int_a^b F'(t) dt$. That is we have

$$\begin{aligned} \sum_{i=0}^{n-1} \frac{(a-b)^i}{i!} f^{(i)}(b) - f(a) &= \int_a^b \frac{(a-t)^{n-1}}{(n-1)!} f^{(n)}(t) dt \\ &= - \int_b^a \frac{(a-t)^{n-1}}{(n-1)!} f^{(n)}(t) dt, \end{aligned} \tag{3}$$

proving (2). □

THEOREM 2.2. *Let $f \in C([a, b], X)$. Then the function*

$$F(t) = \int_b^t f(\tau) d\tau = - \int_t^b f(\tau) d\tau, \quad \forall t \in [a, b]$$

is continuous and $F'(t) = f(t)$, $\forall t \in [a, b]$, i.e. $F \in C^1([a, b], X)$.

Proof. Let $a \leq t \leq t_0 \leq b$. We have that

$$\begin{aligned} \frac{F(t) - F(t_0)}{t - t_0} &\stackrel{(2)}{=} \frac{1}{t_0 - t} \int_t^{t_0} f(\tau) d\tau \text{ (see [2, p. 426, Theorem 11.43])} \\ &= \frac{1}{t_0 - t} \int_t^{t_0} f(t_0) dt + \frac{1}{t_0 - t} \int_t^{t_0} [f(\tau) - f(t_0)] d\tau \\ &= f(t_0) + \frac{1}{t_0 - t} \int_t^{t_0} [f(\tau) - f(t_0)] d\tau, \end{aligned}$$

where

$$\begin{aligned} \left\| \frac{1}{t_0 - t} \int_t^{t_0} [f(\tau) - f(t_0)] d\tau \right\| &\leq \frac{1}{t_0 - t} \int_t^{t_0} \|f(\tau) - f(t_0)\| d\tau \\ &\leq \max_{t \leq \tau \leq t_0} \|f(\tau) - f(t_0)\| \rightarrow 0, \end{aligned}$$

as $t \rightarrow t_0$, since $f(t)$ is continuous at t_0 .

Thus there exists $F'(t_0) = f(t_0)$, for any $t_0 \in [a, b]$. By [11, p. 83], now F is continuous. \square

THEOREM 2.3. *Let $g \in C([a, b], X)$, where X is a Banach space, $n \in \mathbb{N}$. Then*

$$\begin{aligned} g(t) &= \frac{1}{(n-1)!} \left(\int_b^t (t-z)^{n-1} g(z) dz \right)^{(n)} \\ &= \frac{(-1)^n}{(n-1)!} \left(\int_t^b (z-t)^{n-1} g(z) dz \right)^{(n)}, \quad \forall t \in [a, b]. \end{aligned}$$

Proof. We apply Theorem 2.2 repeatedly. We consider the function

$$\begin{aligned} f(t) &= \int_b^t \left(\int_b^{t_1} \left(\int_b^{t_2} \left(\dots \left(\int_b^{t_{n-1}} g(t_n) dt_n \right) dt_{n-1} \right) \dots \right) dt_2 \right) dt_1 \\ &= (-1)^n \left(\int_t^b \left(\int_{t_1}^b \left(\int_{t_2}^b \left(\dots \left(\int_{t_{n-1}}^b g(t_n) dt_n \right) dt_{n-1} \right) \dots \right) dt_2 \right) dt_1 \right), \end{aligned}$$

$\forall t \in [a, b]$. Hence it holds

$$\begin{aligned} f'(t) &= \int_b^t \left(\int_b^{t_1} \left(\dots \left(\int_b^{t_{n-2}} g(t_{n-1}) dt_{n-1} \right) dt_{n-2} \right) \dots \right) dt_1 \\ &= (-1)^{n-1} \left(\int_t^b \left(\int_{t_1}^b \left(\dots \left(\int_{t_{n-2}}^b g(t_{n-1}) dt_{n-1} \right) dt_{n-2} \right) \dots \right) dt_1 \right), \end{aligned}$$

$\forall t \in [a, b]$, etc. Continuing, similarly, we get

$$f^{(n-2)}(t) = \int_b^t \left(\int_b^{t_1} g(t_2) dt_2 \right) dt_1 = (-1)^2 \left(\int_t^b \left(\int_{t_1}^b g(t_2) dt_2 \right) dt_1 \right),$$

$\forall t \in [a, b]$, and

$$f^{(n-1)}(t) = \int_b^t g(t_1) dt_1 = - \int_t^b g(t_1) dt_1, \quad \forall t \in [a, b].$$

Finally, we have that $f^{(n)}(t) = g(t)$, $\forall t \in [a, b]$. Clearly $f \in C^n([a, b], X)$ with $f^{(i)}(b) = 0$, for $i = 0, 1, \dots, n-1$.

By Theorem 2.1 now we obtain

$$f(t) = \frac{1}{(n-1)!} \int_b^t (t-z)^{n-1} f^{(n)}(z) dz,$$

and finally $f^{(n)}(t) = \frac{1}{(n-1)!} \left(\int_b^t (t-z)^{n-1} f^{(n)}(z) dz \right)^{(n)}$, $\forall t \in [a, b]$,

proving the claim. \square

DEFINITION 2.4. Let $f \in C([a, b], X)$, where X is a Banach space. Let $\nu > 0$, we define the right Riemann-Liouville fractional Bochner integral operator

$$(J_{b-}^{\nu} f)(x) := \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} f(z) dz, \quad \forall x \in [a, b],$$

where Γ is the gamma function.

In [5], we have proved that $(J_{b-}^{\nu} f) \in C([a, b], X)$. Furthermore in [5], we have proved that

$$J_{b-}^{\nu} J_{b-}^{\mu} f = J_{b-}^{\nu+\mu} f = J_{b-}^{\mu} J_{b-}^{\nu} f,$$

for any $\mu, \nu > 0$; any $f \in C([a, b], X)$.

LEMMA 2.5. Let $f \in C([a, b], X)$, $\nu \geq 1$, $n = [\nu]$ ($[\cdot]$ integral part), $\alpha = \nu - n$. Then

$$((J_{b-}^{\nu} f)(x))^{(k)} = (-1)^k J_{b-}^{\nu-k} f(x),$$

$k = 0, 1, \dots, n - 1$. Also

$$((J_{b-}^{\nu} f)(x))^{(n)} = (-1)^n J_{b-}^{\alpha} f(x), \quad \text{if } \alpha > 0,$$

and

$$(J_{b-}^{\nu} f)^{(n)} = (-1)^n f, \quad \text{if } \alpha = 0. \tag{4}$$

Proof. We notice that

$$\begin{aligned} ((J_{b-}^{\nu} f)(x))^{(k)} &= (D^k J_{b-}^{\nu} f)(x) = (D^k J_{b-}^k J_{b-}^{\nu-k} f)(x) \\ &= (-1)^k (I J_{b-}^{\nu-k} f)(x) = (-1)^k (J_{b-}^{\nu-k} f)(x), \end{aligned}$$

$k = 0, 1, \dots, n - 1$, where I is the identity operator. If $\alpha > 0$, we get

$$\begin{aligned} ((J_{b-}^{\nu} f)(x))^{(n)} &= (D^n J_{b-}^{\nu} f)(x) = (D^n J_{b-}^{n+\alpha} f)(x) \\ &= (D^n J_{b-}^n J_{b-}^{\alpha} f)(x) = (-1)^n (I J_{b-}^{\alpha} f)(x) = (-1)^n (J_{b-}^{\alpha} f)(x). \end{aligned}$$

Equality (4) is obvious by Theorem 2.3. □

THEOREM 2.6. $J_{b-}^{\nu} : C([a, b], X) \rightarrow C([a, b], X)$, $\nu > 0$, is 1-1.

Proof. Let $f \in C([a, b], X)$ such that $J_{b-}^{\nu} f = 0$. If $0 < \nu < 1$, then $J_{b-}^1 f = J_{b-}^{1-\nu} J_{b-}^{\nu} f = 0$, hence $J_{b-}^1 f = 0$. That is by Theorem 2.3, $(-1) f = (J_{b-}^1 f)' = 0$, and $f = 0$.

If now $\nu \geq 1$, then $\nu = n + \alpha$, (where $n = [\nu]$, $\alpha := \nu - n$, $n \geq 1$, and $0 \leq \alpha < 1$). If $\alpha = 0$, then $J_{b-}^n f = 0$, hence by Theorem 2.3, $(-1)^n f = (J_{b-}^n f)^{(n)} = 0$, so that $f = 0$.

If $\alpha > 0$, then

$$J_{b-}^{\alpha} (J_{b-}^n f) = J_{b-}^{n+\alpha} f = J_{b-}^{\nu} f = 0.$$

Hence by the first case of this proof we get $J_{b-}^n f = 0$. And as in the second case of this proof we get $f = 0$. The proof now is complete. □

DEFINITION 2.7. Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $\nu \notin \mathbb{N}$. Define the subspace of functions

$$C_{b-}^{\nu}([a, b], X) := \left\{ f \in C^n([a, b], X) : J_{b-}^{1-\alpha} f^{(n)} \in C^1([a, b], X) \right\}.$$

Define the Banach space valued right generalized ν -fractional derivative of f over $[a, b]$ as

$$D_{b-}^{\nu} f := (-1)^{n-1} \left(J_{b-}^{1-\alpha} f^{(n)} \right)'.$$

Notice that

$$J_{b-}^{1-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b (z-x)^{-\alpha} f^{(n)}(z) dz$$

exists for $f \in C_{b-}^{\nu}([a, b], X)$, and

$$(D_{b-}^{\nu} f)(x) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (z-x)^{-\alpha} f^{(n)}(z) dz,$$

i.e.

$$(D_{b-}^{\nu} f)(x) = \frac{(-1)^{n-1}}{\Gamma(n-\nu+1)} \frac{d}{dx} \int_x^b (z-x)^{n-\nu} f^{(n)}(z) dz.$$

If $\nu \in \mathbb{N}$, then $\alpha = 0$, $n = \nu$, and

$$(D_{b-}^{\nu} f)(x) = (D_{b-}^n f)(x) = (-1)^n f^{(n)}(x).$$

Notice that $D_{b-}^{\nu} f \in C([a, b], X)$.

We give the following right fractional Taylor's formula.

THEOREM 2.8. Let $f \in C_{b-}^{\nu}([a, b], X)$, $\nu > 0$, $n := [\nu]$. Then

1) If $\nu \geq 1$, we get

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + (J_{b-}^{\nu} D_{b-}^{\nu} f)(x), \quad \forall x \in [a, b]. \quad (5)$$

2) If $0 < \nu < 1$, we get

$$f(x) = J_{b-}^{\nu} D_{b-}^{\nu} f(x), \quad \forall x \in [a, b]. \quad (6)$$

We have that

$$J_{b-}^{\nu} D_{b-}^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} (D_{b-}^{\nu} f)(z) dz, \quad \forall x \in [a, b]. \quad (7)$$

Proof. Let $f \in C_{b-}^{\nu}([a, b], X)$. We see that

$$\begin{aligned} J_{b-}^1 (D_{b-}^{\nu} f)(x) &= \int_x^b (D_{b-}^{\nu} f)(z) dz = (-1)^{n-1} \int_x^b \frac{d}{dz} \left(J_{b-}^{1-\alpha} f^{(n)} \right)(z) dz \\ &= (-1)^{n-1} \left[\left(J_{b-}^{1-\alpha} f^{(n)} \right)(b) - \left(J_{b-}^{1-\alpha} f^{(n)} \right)(x) \right] = (-1)^n J_{b-}^{1-\alpha} f^{(n)}(x). \end{aligned}$$

That is

$$J_{b-}^{1-\alpha} f^{(n)}(x) = (-1)^n J_{b-}^1 (D_{b-}^{\nu} f)(x) = (-1)^n J_{b-}^{1-\alpha} (J_{b-}^{\alpha} (D_{b-}^{\nu} f))(x).$$

Hence since $J_{b-}^{1-\alpha}$ is 1-1 we get $f^{(n)}(x) = (-1)^n J_{b-}^\alpha (D_{b-}^\nu f)(x)$. Consequently

$$\begin{aligned} J_{b-}^n f^{(n)}(x) &= (-1)^n J_{b-}^n J_{b-}^\alpha (D_{b-}^\nu f)(x) \\ &= (-1)^n J_{b-}^{n+\alpha} (D_{b-}^\nu f)(x) = (-1)^n J_{b-}^\nu (D_{b-}^\nu f)(x). \end{aligned}$$

That is

$$J_{b-}^n f^{(n)}(x) = (-1)^n J_{b-}^\nu (D_{b-}^\nu f)(x). \quad (8)$$

Let now $\nu \geq 1$, hence $n \geq 1$, $n \in \mathbb{N}$. By Theorem 2.1 we have that

$$\begin{aligned} f(x) - \sum_{i=0}^{n-1} \frac{(x-b)^i}{i!} f^{(i)}(b) &= \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} f^{(n)}(t) dt \\ &= \frac{(-1)^n}{\Gamma(n)} \int_x^b (t-x)^{n-1} f^{(n)}(t) dt = (-1)^n J_{b-}^n f^{(n)}(x) \\ &\stackrel{(8)}{=} (-1)^{2n} J_{b-}^\nu (D_{b-}^\nu f)(x) = J_{b-}^\nu (D_{b-}^\nu f)(x). \end{aligned}$$

That is

$$f(x) - \sum_{i=0}^{n-1} \frac{(x-b)^i}{i!} f^{(i)}(b) = J_{b-}^\nu (D_{b-}^\nu f)(x), \quad \forall x \in [a, b],$$

proving (5).

If $0 < \nu < 1$, then $n = 0$. Then by (8) we get

$$f(x) = J_{b-}^\nu (D_{b-}^\nu f)(x), \quad \forall x \in [a, b],$$

proving (6). The theorem is proved. \square

COROLLARY 2.9. *Let $f \in C_{b-}^\nu([a, b], X)$, $\nu \geq 1$, and $f^{(i)}(b) = 0$, $i = 0, 1, \dots, n-1$. Then*

$$f(x) = (J_{b-}^\nu D_{b-}^\nu f)(x), \quad \forall x \in [a, b].$$

We give the following Taylor's formula:

THEOREM 2.10. *Let $n \in \mathbb{N}$ and $f \in C^n([a, b], X)$. Then*

$$f(b) = \sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) dx.$$

Proof. We consider

$$F(x) = \sum_{i=0}^{n-1} \frac{(b-x)^i}{i!} f^{(i)}(x), \quad x \in [a, b].$$

Clearly $F \in C([a, b], X)$ and $F(b) = f(b)$, and $F(a) = \sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a)$. Furthermore it holds

$$F'(x) = \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x), \quad \forall x \in [a, b],$$

and $F' \in C([a, b], X)$.

By the Fundamental Theorem of Calculus for Banach space valued functions and Bochner integral, see [12], we get

$$F(b) - F(a) = \int_a^b F'(t) dt.$$

That is we have

$$f(b) - \left(\sum_{i=0}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) \right) = \int_a^b \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) dx,$$

proving the claim. \square

THEOREM 2.11. *Let $f \in C([a, b], X)$. Then the function*

$$F(t) = \int_a^t f(\tau) d\tau, \quad \forall t \in [a, b]$$

is continuous and $F'(t) = f(t)$, $\forall t \in [a, b]$, i.e. $F \in C^1([a, b], X)$.

Proof. Let $a \leq t_0 \leq t \leq b$. We have that

$$\begin{aligned} \frac{F(t) - F(t_0)}{t - t_0} &\stackrel{(3)}{=} \frac{1}{t - t_0} \int_{t_0}^t f(\tau) d\tau \text{ (see [2, p 426, Theorem 11.43])} \\ &= \frac{1}{t - t_0} \int_{t_0}^t f(t_0) d\tau + \frac{1}{t - t_0} \int_{t_0}^t [f(\tau) - f(t_0)] d\tau \\ &= f(t_0) + \frac{1}{t - t_0} \int_{t_0}^t [f(\tau) - f(t_0)] d\tau, \end{aligned}$$

where

$$\begin{aligned} \left\| \frac{1}{t - t_0} \int_{t_0}^t [f(\tau) - f(t_0)] d\tau \right\| &\leq \frac{1}{t - t_0} \int_{t_0}^t \|f(\tau) - f(t_0)\| d\tau \\ &\leq \max_{t_0 \leq \tau \leq t} \|f(\tau) - f(t_0)\| \rightarrow 0, \end{aligned}$$

as $t \rightarrow t_0$, since $f(t)$ is continuous at t_0 .

Thus, there exists $F'(t_0) = f(t_0)$, for any $t_0 \in [a, b]$. By [11, p. 83], now F is continuous. \square

THEOREM 2.12. *Let $g \in C([a, b], X)$, $n \in \mathbb{N}$. Then*

$$g(t) = \frac{1}{(n-1)!} \left(\int_a^t (t-z)^{n-1} g(z) dz \right)^{(n)}, \quad \forall t \in [a, b].$$

Proof. We apply Theorem 2.11 repeatedly. We consider the function

$$f(t) = \int_a^t \left(\int_a^{t_1} \left(\int_a^{t_2} \left(\dots \left(\int_a^{t_{n-1}} g(t_n) dt_n \right) dt_{n-1} \right) \dots \right) dt_2 \right) dt_1, \quad \forall t \in [a, b].$$

Hence it holds

$$f'(t) = \int_a^t \left(\int_a^{t_1} \left(\dots \left(\int_a^{t_{n-2}} g(t_{n-1}) dt_{n-1} \right) dt_{n-2} \right) \dots \right) dt_1, \quad \forall t \in [a, b],$$

etc. Continuing, similarly, we get

$$f^{(n-2)}(t) = \int_a^t \left(\int_a^{t_1} g(t_2) dt_2 \right) dt_1, \quad \forall t \in [a, b],$$

and

$$f^{(n-1)}(t) = \int_a^t g(t_1) dt_1, \quad \forall t \in [a, b].$$

Finally, we have that $f^{(n)}(t) = g(t)$, $\forall t \in [a, b]$. Clearly $f \in C^n([a, b], X)$ with $f^{(i)}(a) = 0$, for $i = 0, 1, \dots, n-1$.

By Theorem 2.10 now we obtain

$$f(t) = \frac{1}{(n-1)!} \int_a^t (t-x)^{n-1} f^{(n)}(x) dx,$$

and finally

$$f^{(n)}(t) = \frac{1}{(n-1)!} \left(\int_a^t (t-x)^{n-1} f^{(n)}(x) dx \right)^{(n)}, \quad \forall t \in [a, b],$$

proving the claim. \square

DEFINITION 2.13. Let $f \in C([a, b], X)$. Let $\nu > 0$, we define the left Riemann-Liouville fractional Bochner integral operator

$$(J_a^\nu f)(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} f(z) dz, \quad \forall x \in [a, b].$$

In [4], we have proved that $(J_a^\nu f) \in C([a, b], X)$. Furthermore in [4], we have proved that

$$J_a^\nu J_a^\mu f = J_a^{\nu+\mu} f = J_a^\mu J_a^\nu f,$$

$\forall \mu, \nu > 0, \forall f \in C([a, b], X)$.

LEMMA 2.14. Let $f \in C([a, b], X)$, $\nu \geq 1$, $n = [\nu]$, $\alpha = \nu - n$. Then

$$((J_a^\nu f)(x))^{(k)} = J_a^{\nu-k} f(x),$$

$k = 0, 1, \dots, n-1$. Also

$$((J_a^\nu f)(x))^{(n)} = J_a^\alpha f(x), \quad \text{if } \alpha > 0,$$

and

$$(J_a^\nu f)^{(n)} = f, \quad \text{if } \alpha = 0. \tag{9}$$

Proof. We notice that

$$\begin{aligned} ((J_a^\nu f)(x))^{(k)} &= (D^k J_a^\nu f)(x) = (D^k J_a^k J_a^{\nu-k} f)(x) \\ &= (I J_a^{\nu-k} f)(x) = (J_a^{\nu-k} f)(x), \end{aligned}$$

$k = 0, 1, \dots, n-1$. If $\alpha > 0$, we get

$$\begin{aligned} ((J_a^\nu f)(x))^{(n)} &= (D^n J_a^\nu f)(x) = (D^n J_a^{n+\alpha} f)(x) \\ &= (D^n J_a^n J_a^\alpha f)(x) = (I J_a^\alpha f)(x) = (J_a^\alpha f)(x). \end{aligned}$$

Equality (9) is obvious by Theorem 2.12. \square

THEOREM 2.15. $J_a^\nu : C([a, b], X) \rightarrow C([a, b], X)$, $\nu > 0$, is 1-1.

Proof. Let $f \in C([a, b], X)$ such that $J_a^\nu f = 0$. If $0 < \nu < 1$, then $J_a^1 f = J_a^{1-\nu} J_a^\nu f = 0$, hence $J_a^1 f = 0$. That is by Theorem 12, $f = (J_a^1 f)' = 0$, and $f = 0$.

If now $\nu \geq 1$, then $\nu = n + \alpha$, $0 \leq \alpha < 1$. If $\alpha = 0$, then $J_a^n f = 0$, hence by Theorem 12, $f = (J_a^n f)^{(n)} = 0$, so that $f = 0$.

If $\alpha > 0$, then

$$J_a^\alpha (J_a^n f) = J_a^{n+\alpha} f = J_a^\nu f = 0.$$

Hence by the first case of this proof we get $J_a^n f = 0$. And as in the second case of this proof we get $f = 0$. The theorem is proved. \square

DEFINITION 2.16. Let $\nu > 0$, $n := [\nu]$, $\alpha = \nu - n$, $0 < \alpha < 1$, $\nu \notin \mathbb{N}$. Define the subspace of functions

$$C_a^\nu([a, b], X) := \left\{ f \in C^n([a, b], X) : J_a^{1-\alpha} f^{(n)} \in C^1([a, b], X) \right\}.$$

Define the Banach space valued left generalized ν -fractional derivative of f over $[a, b]$ as

$$(D_a^\nu f) := \left(J_a^{1-\alpha} f^{(n)} \right)'.$$

Notice that

$$J_a^{1-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-z)^{-\alpha} f^{(n)}(z) dz$$

exists for $f \in C_a^\nu([a, b], X)$, and

$$(D_a^\nu f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-z)^{-\alpha} f^{(n)}(z) dz,$$

i.e.

$$(D_a^\nu f)(x) = \frac{1}{\Gamma(n-\nu+1)} \frac{d}{dx} \int_a^x (x-z)^{n-\nu} f^{(n)}(z) dz.$$

If $\nu \in \mathbb{N}$, then $\alpha = 0$, $n = \nu$, and

$$(D_a^\nu f)(x) = (D_a^n f)(x) = f^{(n)}(x).$$

Notice that $D_a^\nu f \in C([a, b], X)$.

We give the following left fractional Taylor's formula.

THEOREM 2.17. Let $f \in C_a^\nu([a, b], X)$, $\nu > 0$, $n := [\nu]$. Then

1) If $\nu \geq 1$, we get

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + (J_a^\nu D_a^\nu f)(x), \quad \forall x \in [a, b]. \quad (10)$$

2) If $0 < \nu < 1$, we get

$$f(x) = J_a^\nu D_a^\nu f(x), \quad \forall x \in [a, b]. \quad (11)$$

We have that

$$J_a^\nu D_a^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_a^\nu f)(z) dz, \quad \forall x \in [a, b]. \quad (12)$$

Proof. Let $f \in C_a^\nu([a, b], X)$. We see that

$$\begin{aligned} J_a^1(D_a^\nu f)(x) &= \int_a^x (D_a^\nu f)(z) dz = \int_a^x \frac{d}{dz} \left(J_a^{1-\alpha} f^{(n)} \right) (z) dz \\ &= \left[\left(J_a^{1-\alpha} f^{(n)} \right) (x) - \left(J_a^{1-\alpha} f^{(n)} \right) (a) \right] = \left(J_a^{1-\alpha} f^{(n)} \right) (x). \end{aligned}$$

That is $(J_a^{1-\alpha} f^{(n)})(x) = J_a^1(D_a^\nu f)(x) = J_a^{1-\alpha}(J_a^\alpha(D_a^\nu f))(x)$. Since $J_a^{1-\alpha}$ is 1-1 we get $f^{(n)}(x) = (J_a^\alpha(D_a^\nu f))(x)$. Consequently

$$J_a^n f^{(n)}(x) = (J_a^n J_a^\alpha(D_a^\nu f))(x) = (J_a^{n+\alpha}(D_a^\nu f))(x) = (J_a^\nu(D_a^\nu f))(x).$$

That is

$$J_a^n f^{(n)}(x) = (J_a^\nu D_a^\nu f)(x), \quad \forall x \in [a, b]. \tag{13}$$

Let now $\nu \geq 1$, hence $n \geq 1, n \in \mathbb{N}$. By Theorem 2.10 we have that

$$\begin{aligned} f(x) - \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} f^{(i)}(a) &= \frac{1}{(n-1)!} \int_a^x (x-z)^{n-1} f^{(n)}(z) dz \\ &= \left(J_a^n f^{(n)} \right) (x) \stackrel{(13)}{=} (J_a^\nu D_a^\nu f)(x), \quad \forall x \in [a, b], \end{aligned}$$

proving (10).

If $0 < \nu < 1$, then $n = 0$. Then by (13) we get $f(x) = (J_a^\nu D_a^\nu f)(x)$, proving (11). \square

COROLLARY 2.18. *Let $f \in C_a^\nu([a, b], X)$, $\nu > 0$, $n = [\nu]$, and $f^{(i)}(a) = 0, i = 0, 1, \dots, n - 1$. Then $f(x) = (J_a^\nu D_a^\nu f)(x), \forall x \in [a, b]$.*

We give the following fractional Polya type integral inequality without any boundary conditions, see also [3, p. 4].

THEOREM 2.19. *Let $0 < \nu < 1, f \in C([a, b], X)$. Assume that $f \in C_a^\nu([a, \frac{a+b}{2}], X)$ and $f \in C_{b-}^\nu([\frac{a+b}{2}, b], X)$. Set*

$$M(f) = \max \left\{ \left\| \| D_a^\nu f \| \right\|_{\infty, [a, \frac{a+b}{2}]}, \left\| \| D_{b-}^\nu f \| \right\|_{\infty, [\frac{a+b}{2}, b]} \right\}.$$

Then

$$\left\| \int_a^b f(x) dx \right\| \leq \int_a^b \| f(x) \| dx \leq M(f) \frac{(b-a)^{\nu+1}}{\Gamma(\nu+2) 2^\nu}. \tag{14}$$

Inequality (14) is sharp, namely it is attained by

$$f_*(x) = \begin{cases} (x-a)^\nu \vec{i}, & x \in [a, \frac{a+b}{2}], \\ (b-x)^\nu \vec{i}, & x \in [\frac{a+b}{2}, b] \end{cases}, \quad 0 < \nu < 1,$$

$\vec{i} \in X : \| \vec{i} \| = 1$. Clearly here non zero constant vector function f are excluded.

Proof. By (6) and (7) we get that

$$\begin{aligned} \| f(x) \| &= \frac{1}{\Gamma(\nu)} \left\| \int_x^b (z-x)^{\nu-1} (D_{b-}^\nu f)(z) dz \right\| \\ &\leq \frac{1}{\Gamma(\nu)} \int_x^b (z-x)^{\nu-1} \| (D_{b-}^\nu f)(z) \| dz \leq \left\| \| D_{b-}^\nu f \| \right\|_{\infty, [\frac{a+b}{2}, b]} \frac{(b-x)^\nu}{\Gamma(\nu+1)}. \end{aligned}$$

That is

$$\|f(x)\| \leq \left\| \|D_{b-}^{\nu} f\| \right\|_{\infty, [\frac{a+b}{2}, b]} \frac{(b-x)^{\nu}}{\Gamma(\nu+1)}, \quad \forall x \in \left[\frac{a+b}{2}, b \right]. \quad (15)$$

Similarly by (11) and (12) we get:

$$\|f(x)\| \leq \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \|(D_a^{\nu} f)(z)\| dz \leq \frac{\| \|D_a^{\nu} f\| \|_{\infty, [a, \frac{a+b}{2}]}}{\Gamma(\nu+1)} (x-a)^{\nu}, \quad (16)$$

$\forall x \in [a, \frac{a+b}{2}]$. Hence we find

$$\begin{aligned} \int_a^b \|f(x)\| dx &= \int_a^{\frac{a+b}{2}} \|f(x)\| dx + \int_{\frac{a+b}{2}}^b \|f(x)\| dx \quad (\text{by (15) and (16)}) \\ &\leq \frac{1}{\Gamma(\nu+1)} \left\{ \left(\int_a^{\frac{a+b}{2}} (x-a)^{\nu} dx \right) \| \|D_a^{\nu} f\| \|_{\infty, [a, \frac{a+b}{2}]} \right. \\ &\quad \left. + \left(\int_{\frac{a+b}{2}}^b (b-x)^{\nu} dx \right) \| \|D_{b-}^{\nu} f\| \|_{\infty, [\frac{a+b}{2}, b]} \right\} \\ &= \frac{1}{(\Gamma(\nu+1))(\nu+1)} \left\{ \left(\frac{b-a}{2} \right)^{\nu+1} \| \|D_a^{\nu} f\| \|_{\infty, [a, \frac{a+b}{2}]} \right. \\ &\quad \left. + \left(\frac{b-a}{2} \right)^{\nu+1} \| \|D_{b-}^{\nu} f\| \|_{\infty, [\frac{a+b}{2}, b]} \right\} \\ &= \frac{1}{\Gamma(\nu+2)} \left(\frac{b-a}{2} \right)^{\nu+1} \left\{ \| \|D_a^{\nu} f\| \|_{\infty, [a, \frac{a+b}{2}]} + \| \|D_{b-}^{\nu} f\| \|_{\infty, [\frac{a+b}{2}, b]} \right\}. \end{aligned}$$

So we have proved that

$$\int_a^b \|f(x)\| dx \leq \max \left\{ \| \|D_a^{\nu} f\| \|_{\infty, [a, \frac{a+b}{2}]} , \| \|D_{b-}^{\nu} f\| \|_{\infty, [\frac{a+b}{2}, b]} \right\} \frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)2^{\nu}},$$

proving (14).

Notice that

$$f_* \left(\left(\frac{a+b}{2} \right)_- \right) = f_* \left(\left(\frac{a+b}{2} \right)_+ \right) = \left(\frac{b-a}{2} \right)^{\nu} \vec{i},$$

so that $f_* \in C([a, b], X)$.

Here, very similarly, as in [3, p. 5-6], we get that

$$D_a^{\nu} \left((x-a)^{\nu} \vec{i} \right) = \Gamma(\nu+1) \vec{i}, \quad \text{for all } x \in \left[a, \frac{a+b}{2} \right].$$

Therefore it holds

$$\left\| \|D_a^{\nu} \left((\cdot - a)^{\nu} \vec{i} \right) \| \|_{\infty, [a, \frac{a+b}{2}]} = \Gamma(\nu+1).$$

Similarly, it holds

$$\left\| \|D_{b-}^{\nu} \left((b - \cdot)^{\nu} \vec{i} \right) \| \|_{\infty, [\frac{a+b}{2}, b]} = \Gamma(\nu+1).$$

Consequently we find that $M(f_*) = \Gamma(\nu + 1)$. Applying f_* into (13) we obtain:

$$R.H.S. (14) = \frac{(b-a)^{\nu+1}}{(\nu+1)2^\nu},$$

$$\begin{aligned} \text{and } L.H.S. (14) &= \left\| \int_a^b f_*(x) dx \right\| = \int_a^b \|f_*(x)\| dx \\ &= \int_a^{\frac{a+b}{2}} (x-a)^\nu dx + \int_{\frac{a+b}{2}}^b (b-x)^\nu dx = \frac{(b-a)^{\nu+1}}{(\nu+1)2^\nu}, \end{aligned}$$

proving optimality of (14). \square

We present the following fractional Ostrowski type inequality, see also [3, p. 379–381].

THEOREM 2.20. *Let $\nu \geq 1$, $n = [\nu]$, $f \in C([a, b], X)$, $x_0 \in [a, b]$. Assume that $f|_{[a, x_0]} \in C_{x_0-}^\nu([a, x_0], X)$, $f|_{[x_0, b]} \in C_{x_0}^\nu([x_0, b], X)$, and $f^{(i)}(x_0) = 0$, for $i = 1, \dots, n-1$, which is void when $1 \leq \nu < 2$. Then*

$$\begin{aligned} \left\| \frac{1}{b-a} \int_a^b f(x) dx - f(x_0) \right\| &\leq \frac{1}{(b-a)\Gamma(\nu+2)} \\ &\times \left\{ \left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]} (x_0-a)^{\nu+1} + \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} (b-x_0)^{\nu+1} \right\} \\ &\leq \frac{1}{(b-a)\Gamma(\nu+2)} \max \left(\left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]}, \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} \right) \\ &\times \left[(b-x_0)^{\nu+1} + (x_0-a)^{\nu+1} \right] \\ &\leq \max \left(\left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]}, \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} \right) \frac{(b-a)^\nu}{\Gamma(\nu+2)}. \end{aligned} \quad (17)$$

Proof. By (5) we get that

$$f(x) - f(x_0) = \frac{1}{\Gamma(\nu)} \int_x^{x_0} (z-x)^{\nu-1} (D_{x_0-}^\nu f)(z) dz, \quad \forall x \in [a, x_0]. \quad (18)$$

And from (10) we get that

$$f(x) - f(x_0) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-z)^{\nu-1} (D_{x_0}^\nu f)(z) dz, \quad \forall x \in [x_0, b]. \quad (19)$$

Hence, by (18),

$$\begin{aligned} \|f(x) - f(x_0)\| &\leq \frac{1}{\Gamma(\nu)} \int_x^{x_0} (z-x)^{\nu-1} \|(D_{x_0-}^\nu f)(z)\| dz \\ &\leq \left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]} \frac{(x_0-x)^\nu}{\Gamma(\nu+1)}, \quad \forall x \in [a, x_0]. \end{aligned}$$

Furthermore it holds

$$\begin{aligned} \|f(x) - f(x_0)\| &\stackrel{(19)}{\leq} \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-z)^{\nu-1} \|(D_{x_0}^\nu f)(z)\| dz \\ &\leq \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} \frac{(x-x_0)^\nu}{\Gamma(\nu+1)}, \quad \forall x \in [x_0, b]. \end{aligned}$$

Next we observe that

$$\begin{aligned}
& \left\| \frac{1}{b-a} \int_a^b f(x) dx - f(x_0) \right\| = \frac{1}{b-a} \left\| \int_a^b (f(x) - f(x_0)) dx \right\| \\
& \leq \frac{1}{b-a} \int_a^b \|f(x) - f(x_0)\| dx \\
& = \frac{1}{b-a} \left\{ \int_a^{x_0} \|f(x) - f(x_0)\| dx + \int_{x_0}^b \|f(x) - f(x_0)\| dx \right\} \\
& \leq \frac{1}{(b-a)\Gamma(\nu+1)} \left\{ \left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]} \int_a^{x_0} (x_0-x)^\nu dx \right. \\
& \quad \left. + \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} \int_{x_0}^b (x-x_0)^\nu dx \right\} \\
& = \frac{1}{(b-a)\Gamma(\nu+2)} \left\{ \left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]} (x_0-a)^{\nu+1} \right. \\
& \quad \left. + \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} (b-x_0)^{\nu+1} \right\} \\
& \leq \frac{1}{(b-a)\Gamma(\nu+2)} \max \left(\left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]}, \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} \right) \\
& \quad \times \left[(x_0-a)^{\nu+1} + (b-x_0)^{\nu+1} \right] \\
& \leq \frac{1}{\Gamma(\nu+2)} \max \left(\left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]}, \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} \right) (b-a)^\nu.
\end{aligned}$$

Notice here that

$$\left\| \|D_{x_0-}^\nu f\| \right\|_{\infty, [a, x_0]}, \left\| \|D_{x_0}^\nu f\| \right\|_{\infty, [x_0, b]} < \infty.$$

The theorem is proved. \square

Inequalities (17) are optimal.

THEOREM 2.21. *All as in Theorem 2.20. Inequalities (17) are sharp, namely are attained by*

$$\bar{f}(z) = \begin{cases} (x_0-z)^\nu \vec{i}, & z \in [a, x_0], \\ (z-x_0)^\nu \vec{i}, & z \in [x_0, b], \end{cases}$$

where $\vec{i} \in X : \|\vec{i}\| = 1, \nu \geq 1, x_0 \in [a, b]$ is fixed.

Proof. See that

$$\bar{f}^{(k)}(x_{0-}) = \bar{f}^{(k)}(x_{0+}) = 0, \quad k = 0, 1, \dots, n-1.$$

We have that

$$\left\| \|D_{x_0-}^\nu \bar{f}\| \right\|_{\infty, [a, x_0]} = \left\| \|D_{x_0}^\nu \bar{f}\| \right\|_{\infty, [x_0, b]} = \Gamma(\nu+1).$$

The

$$\begin{aligned}
 \text{R.H.S. of (17)} &= \frac{1}{(b-a)\Gamma(\nu+2)} \max \left(\left\| \|D_{x_0-}^\nu \bar{f}\| \right\|_{\infty, [a, x_0]}, \left\| \|D_{x_0}^\nu \bar{f}\| \right\|_{\infty, [x_0, b]} \right) \\
 &\quad \times \left[(b-x_0)^{\nu+1} + (x_0-a)^{\nu+1} \right] \\
 &= \frac{1}{(b-a)\Gamma(\nu+2)} \Gamma(\nu+1) \left[(b-x_0)^{\nu+1} + (x_0-a)^{\nu+1} \right] \\
 &= \frac{\left[(b-x_0)^{\nu+1} + (x_0-a)^{\nu+1} \right]}{(b-a)(\nu+1)}. \tag{20}
 \end{aligned}$$

The

$$\begin{aligned}
 \text{L.H.S. of (17)} &= \left\| \frac{1}{b-a} \int_a^b \bar{f}(x) dx - \bar{f}(x_0) \right\| = \left\| \frac{1}{b-a} \int_a^b \bar{f}(x) dx \right\| \\
 &= \frac{1}{b-a} \left[\int_a^{x_0} (x_0-z)^\nu dz + \int_{x_0}^b (z-x_0)^\nu dz \right] \\
 &= \frac{\left((x_0-a)^{\nu+1} + (b-x_0)^{\nu+1} \right)}{(b-a)(\nu+1)}. \tag{21}
 \end{aligned}$$

By (20) and (21) we get optimality of (17). □

We continue with a right fractional Poincaré type inequality.

THEOREM 2.22. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = [\alpha]$. Let $f \in C_{b-}^\alpha([a, b], X)$. Assume that $f^{(k)}(b) = 0$, $k = 0, 1, \dots, m-1$, when $\alpha \geq 1$. Then*

$$\|f\|_{L_q([a,b],X)} \leq \frac{(b-a)^\alpha \|D_{b-}^\alpha f\|_{L_q([a,b],X)}}{\Gamma(\alpha) (p(\alpha-1) + 1)^{\frac{1}{p}} (q\alpha)^{\frac{1}{q}}}.$$

Proof. We have that (by (5)–(7))

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} (D_{b-}^\alpha f)(z) dz, \quad \forall x \in [a, b].$$

Hence

$$\begin{aligned}
 \|f(x)\| &= \frac{1}{\Gamma(\alpha)} \left\| \int_x^b (z-x)^{\alpha-1} (D_{b-}^\alpha f)(z) dz \right\| \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \|D_{b-}^\alpha f(z)\| dz \\
 &\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (z-x)^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \left(\int_x^b \|(D_{b-}^\alpha f)(z)\|^q dz \right)^{\frac{1}{q}} \\
 &\leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1) + 1)^{\frac{1}{p}}} \|D_{b-}^\alpha f\|_{L_q([a,b],X)}.
 \end{aligned}$$

We have proved that

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \|D_{b-}^{\alpha} f\|_{L_q([a,b],X)}, \quad \forall x \in [a,b].$$

Then

$$\|f(x)\|^q \leq \frac{(b-x)^{(p(\alpha-1)+1)\frac{q}{p}}}{(\Gamma(\alpha))^q (p(\alpha-1)+1)^{\frac{q}{p}}} \|D_{b-}^{\alpha} f\|_{L_q([a,b],X)}^q, \quad \forall x \in [a,b].$$

Hence it holds

$$\int_a^b \|f(x)\|^q dx \leq \frac{(b-a)^{q\alpha}}{(\Gamma(\alpha))^q (p(\alpha-1)+1)^{\frac{q}{p}} q\alpha} \|D_{b-}^{\alpha} f\|_{L_q([a,b],X)}^q.$$

The last inequality implies

$$\left(\int_a^b \|f(x)\|^q dx \right)^{\frac{1}{q}} \leq \frac{(b-a)^{\alpha} \|D_{b-}^{\alpha} f\|_{L_q([a,b],X)}}{\Gamma(\alpha) (p(\alpha-1)+1)^{\frac{1}{p}} (q\alpha)^{\frac{1}{q}}},$$

proving the claim. \square

We finish with a Poincaré like left fractional inequality:

THEOREM 2.23. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = [\nu]$. Let $f \in C_a^{\nu}([a,b], X)$. Assume that $f^{(k)}(a) = 0$, $k = 0, 1, \dots, n-1$, if $\nu \geq 1$. Then*

$$\|f\|_{L_q([a,b],X)} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}} \|D_a^{\nu} f\|_{L_q([a,b],X)}.$$

Proof. We have that (by (9)–(11))

$$f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} (D_a^{\nu} f)(z) dz, \quad \forall x \in [a,b].$$

Thus

$$\begin{aligned} \|f(x)\| &= \frac{1}{\Gamma(\nu)} \left\| \int_a^x (x-z)^{\nu-1} (D_a^{\nu} f)(z) dz \right\| \\ &\leq \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \|D_a^{\nu} f(z)\| dz \\ &\leq \left(\frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{p(\nu-1)} dz \right)^{\frac{1}{p}} \left(\int_a^x \|D_a^{\nu} f(z)\|^q dz \right)^{\frac{1}{q}} \\ &\leq \frac{1}{\Gamma(\nu)} \frac{(x-a)^{\frac{p(\nu-1)+1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}} \|D_a^{\nu} f\|_{L_q([a,b],X)}. \end{aligned}$$

We have proved that

$$\|f(x)\| \leq \frac{(x-a)^{\nu-\frac{1}{q}}}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}}} \|D_a^{\nu} f\|_{L_q([a,b],X)}, \quad \forall x \in [a,b].$$

Then

$$\|f(x)\|^q \leq \frac{(x-a)^{q\nu-1}}{(\Gamma(\nu))^q (p(\nu-1)+1)^{\frac{q}{p}}} \|D_a^\nu f\|_{L_q([a,b],X)}^q,$$

and

$$\int_a^b \|f(x)\|^q dx \leq \frac{(b-a)^{q\nu} \|D_a^\nu f\|_{L_q([a,b],X)}^q}{(\Gamma(\nu))^q (p(\nu-1)+1)^{\frac{q}{p}} q\nu}.$$

This last results into

$$\left(\int_a^b \|f(x)\|^q dx \right)^{\frac{1}{q}} \leq \frac{(b-a)^\nu}{\Gamma(\nu) (p(\nu-1)+1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}} \|D_a^\nu f\|_{L_q([a,b],X)}.$$

proving the claim. \square

One can prove, with the above built fractional machinery, all kinds of inequalities but we choose to stop here.

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