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FINITE GROUPS WHOSE COMMUTING GRAPHS ARE INTEGRAL

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Abstract. A finite non-abelian group G is called commuting integral if the commuting graph of G is integral. In this paper, we show that a finite group is commuting integral if its central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m}, where p is any prime integer and D_{2m} is the dihedral group of order $2m$.

1. Introduction

Let G be a non-abelian group with center $Z(G)$. The commuting graph of G, denoted by Γ_G, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two vertices x and y are adjacent if and only if $xy = yx$. In recent years, many mathematicians have considered commuting graphs of different finite groups and studied various graph theoretic aspects (see [4, 7, 11–14]). A finite non-abelian group G is called *commuting integral* if the commuting graph of G is integral. It is natural to ask which finite groups are commuting integral. In this paper, we compute the spectrum of the commuting graphs of finite groups whose central quotients are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, for any prime integer p, or D_{2m}, the dihedral group of order $2m$. Our computation reveals that those groups are commuting integral.

Recall that the spectrum of a graph $\mathcal{G}$, denoted by $\text{Spec}(\mathcal{G})$, is the multiset $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_n^{k_n}\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of $\mathcal{G}$ with multiplicities $k_1, k_2, \ldots, k_n$ respectively. A graph $\mathcal{G}$ is called integral if $\text{Spec}(\mathcal{G})$ contains only integers. It is well known that the complete graph K_n on n vertices is integral and $\text{Spec}(K_n) = \{(-1)^{n-1}, (n-1)^1\}$. Further, if $\mathcal{G} = K_{m_1} \sqcup K_{m_2} \sqcup \cdots \sqcup K_{m_l}$, where K_{m_i}'s are complete graphs on m_i vertices for $1 \leq i \leq l$, then

$$\text{Spec}(\mathcal{G}) = \{(-1)^{\sum_{i=1}^{l} m_i - l}, (m_1 - 1)^1, (m_2 - 1)^1, \ldots, (m_l - 1)^1\}.$$

The notion of integral graph was introduced by Harary and Schwenk [9] in the year 1974. Since then many mathematicians have considered integral graphs, see for example [2, 10, 15]. A very impressive survey on integral graphs can be found in [6].

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Ahmadi et alňoted that integral graphs are of some interest for designing the network topology of perfect state transfer networks, see [3] and the references therein.

For any element x of a group G, the set $C_G(x) = \{y \in G : xy = yx\}$ is called the centralizer of x in G. Let $|\operatorname{Cent}(G)| = |\{C_G(x) : x \in G\}|$, that is the number of distinct centralizers in G. A group G is called an n-centralizer group if $|\operatorname{Cent}(G)| = n$. In [8], Belcastro and Sherman characterized finite n-centralizer groups for $n = 4, 5$. As a consequence of our results, we show that 4-, 5-centralizer finite groups are commuting integral. Further, we show that a finite $(p+2)$-centralizer p-group is commuting integral for any prime p.

2. Main results and consequences

We begin this section with the following theorem.

Theorem 2.1. *Let G be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime integer. Then*

$$\operatorname{Spec}(\Gamma_G) = \{(-1)^{(p^2-1)|Z(G)|-p-1}, ((p-1)|Z(G)| - 1)^{p+1}\}.$$

Proof. Let $|Z(G)| = n$. Then since $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ we have $\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^p, b^p, aba^{-1}b^{-1} \in Z(G)\rangle$, where $a, b \in G$ with $ab \neq ba$. Then for any $z \in Z(G)$, we have

$$C_G(a) = C_G(a^i z) = Z(G) \sqcup aZ(G) \sqcup \cdots \sqcup a^{p-1}Z(G) \text{ for } 1 \leq i \leq p-1,$$
$$C_G(a^j b) = C_G(a^j bz) = Z(G) \sqcup a^j bZ(G) \sqcup \cdots \sqcup a^{(p-1)j} b^{p-1} Z(G) \text{ for } 1 \leq j \leq p.$$

These are the only centralizers of non-central elements of G. Also note that these centralizers are abelian subgroups of G. Therefore

$$\Gamma_G = K_{|C_G(a) \setminus Z(G)|} \sqcup \left(\bigsqcup_{j=1}^{p} K_{|C_G(a^j b) \setminus Z(G)|} \right).$$

Thus $\Gamma_G = K_{(p-1)n} \sqcup (\bigsqcup_{j=1}^{p} K_{(p-1)n})$, since $|C_G(a)| = pn$ and $|C_G(a^j b)| = pn$ for $1 \leq j \leq p$ where as usual K_m denotes the complete graph with m vertices. That is, $\Gamma_G = \bigsqcup_{j=1}^{p+1} K_{(p-1)n}$. Hence the result follows. □

The above theorem shows that G is commuting integral if the central quotient of G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for any prime integer p. Some consequences of Theorem 2.1 are given below.

Corollary 2.2. *Let G be a non-abelian group of order p^3, for any prime p. Then*

$$\operatorname{Spec}(\Gamma_G) = \{(-1)^{p^3-2p-1}, (p^2 - p - 1)^{p+1}\}.$$

Hence, G is commuting integral.

Proof. Note that $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.1. □

Corollary 2.3. *If G is a finite 4-centralizer group then G is commuting integral.*

Proof. If G is a finite 4-centralizer group then by Theorem 2 of [8] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 2.1,

$$\text{Spec}(\Gamma_G) = \{(-1)^{3(|Z(G)|-1)}, (|Z(G)|-1)^3\}.$$

This shows that G is commuting integral. □

Further, we have the following result.

Corollary 2.4. *If G is a finite $(p+2)$-centralizer p-group, for any prime p, then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{(p^2-1)|Z(G)|-p-1}, ((p-1)|Z(G)|-1)^{p+1}\}.$$

Hence, G is commuting integral.

Proof. If G is a finite $(p+2)$-centralizer p-group then by Lemma 2.7 of [5] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Now the result follows from Theorem 2.1. □

The following theorem shows that G is commuting integral if the central quotient of G is isomorphic to the dihedral group $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1}\rangle$.

Theorem 2.5. *Let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, for $m \geq 2$. Then*

$$\text{Spec}(\Gamma_G) = \{(-1)^{(2m-1)|Z(G)|-m-1}, (|Z(G)|-1)^m, ((m-1)|Z(G)|-1)^1\}.$$

Proof. Since $\frac{G}{Z(G)} \cong D_{2m}$ we have $\frac{G}{Z(G)} = \langle xZ(G), yZ(G) : x^2, y^m, xyx^{-1}y \in Z(G)\rangle$, where $x, y \in G$ with $xy \neq yx$. It is not difficult to see that for any $z \in Z(G)$,

$$C_G(y) = C_G(y^i z) = Z(G) \sqcup yZ(G) \sqcup \cdots \sqcup y^{m-1}Z(G), 1 \leq i \leq m-1$$

and

$$C_G(xy^j) = C_G(xy^j z) = Z(G) \sqcup xy^j Z(G), 1 \leq j \leq m$$

are the only centralizers of non-central elements of G. Also note that these centralizers are abelian subgroups of G. Therefore

$$\Gamma_G = K_{|C_G(y)\setminus Z(G)|} \sqcup \left(\bigsqcup_{j=1}^{m} K_{|C_G(xy^j)\setminus Z(G)|}\right).$$

Thus $\Gamma_G = K_{(m-1)n} \sqcup (\bigsqcup_{j=1}^{m} K_n)$, since $|C_G(y)| = mn$ and $|C_G(x^j y)| = 2n$ for $1 \leq j \leq m$, where $|Z(G)| = n$. Hence the result follows. □

Corollary 2.6. *If G is a finite 5-centralizer group then G is commuting integral.*

Proof. If G is a finite 5-centralizer group then by Theorem 4 of [8] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or D_6. Now, if $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then by Theorem 2.1 we have

$$\text{Spec}(\Gamma_G) = \{(-1)^{8|Z(G)|-4}, (2|Z(G)|-1)^4\}.$$

Again, if $\frac{G}{Z(G)} \cong D_6$ then by Theorem 2.5 we have

$$\text{Spec}(\Gamma_G) = \{(-1)^{5|Z(G)|-4}, (|Z(G)|-1)^3, (2|Z(G)|-1)^1\}.$$

In both cases Γ_G is integral. Hence G is commuting integral. □

We also have the following result.

Corollary 2.7. *Let G be a finite non-abelian group and $\{x_1, x_2, \ldots, x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then G is commuting integral if $r = 3, 4$.*

Proof. By Lemma 2.4 of [1], we have that G is a 4-centralizer or a 5-centralizer group according as $r = 3$ or 4. Hence the result follows from Corollaries 2.3 and 2.6. □

We now compute the spectrum of the commuting graphs of some well-known groups, using Theorem 2.5.

Proposition 2.8. *Let $M_{2mn} = \langle a, b : a^m = b^{2n} = 1, bab^{-1} = a^{-1}\rangle$ be a metacyclic group, where $m > 2$. Then*

$$\text{Spec}(\Gamma_{M_{2mn}}) = \begin{cases} \{(-1)^{2mn-m-n-1}, (n-1)^m, (mn-n-1)^1\}, & \textit{if } m \textit{ is odd} \\ \{(-1)^{2mn-2n-\frac{m}{2}-1}, (2n-1)^{\frac{m}{2}}, (mn-2n-1)^1\}, & \textit{if } m \textit{ is even.} \end{cases}$$

Proof. Observe that $Z(M_{2mn}) = \langle b^2\rangle$ or $\langle b^2\rangle \cup a^{\frac{m}{2}}\langle b^2\rangle$ depending whether m is odd or even. Also, it is easy to see that $\frac{M_{2mn}}{Z(M_{2mn})} \cong D_{2m}$ or D_m depending whether m is odd or even. Hence, the result follows from Theorem 2.5. □

The above Proposition 2.8 also gives the spectrum of the commuting graph of the dihedral group D_{2m}, where $m > 2$, as given below:

$$\text{Spec}(\Gamma_{D_{2m}}) = \begin{cases} \{(-1)^{m-2}, 0^m, (m-2)^1\}, & \text{if } m \text{ is odd} \\ \{(-1)^{\frac{3m}{2}-3}, 1^{\frac{m}{2}}, (m-3)^1\}, & \text{if } m \text{ is even.} \end{cases}$$

Proposition 2.9. *The spectrum of the commuting graph of dicyclic group or the generalized quaternion group $Q_{4m} = \langle a, b : a^{2m} = 1, b^2 = a^m, bab^{-1} = a^{-1}\rangle$, where $m \geq 2$, is given by*

$$\text{Spec}(\Gamma_{Q_{4m}}) = \{(-1)^{3m-3}, 1^m, (2m-3)^1\}.$$

Proof. The result follows from Theorem 2.5 noting that $Z(Q_{4m}) = \{1, a^m\}$ and $\frac{Q_{4m}}{Z(Q_{4m})} \cong D_{2m}$. □

Proposition 2.10. *Consider the group $U_{6n} = \langle a, b : a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$. Then* $\text{Spec}(\Gamma_{U_{6n}}) = \{(-1)^{5n-4}, (n-1)^3, (2n-1)^1\}$.

Proof. Note that $Z(U_{6n}) = \langle a^2 \rangle$ and $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$. Hence the result follows from Theorem 2.5. □

We conclude the paper by noting that the groups M_{2mn}, D_{2m}, Q_{4m} and U_{6n} are commuting integral.

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