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SYMMETRIC TILINGS IN THE SQUARE LATTICE

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Abstract. We apply the method of Gröbner bases to polyomino tilings, following and developing the ideas of Bodini and Nouvel. The emphasis is, in the spirit of the paper M. Muzika Dizdarević, R. T. Živaljević, *Symmetric polyomino tilings, tribones, ideals and Gröbner bases*, Publ. Inst. Math. **98** (112) (2015), 1–23., on tiling problems with added symmetry conditions. The main problem studied in the paper covers case of tiling by three-in-line polyominoes, centrally symmetric with respect to the origin.

1. Introduction and a summary of main results

The art of designing tiling is very old and well developed. Every human civilization has used some form of tiling, which consisted of tiles made of stone, ceramic, wood or similar material, to cover the plane or some other surface with no gaps and no overlaps. By contrast, the study of mathematical properties of tiling is quite new and many parts of the subject are still unexplored.

Tiling problems have become popular in mathematics since 1954 when Solomon Golomb published his famous book "Checker Board and Polyominoes". Golomb points out that tiling of the plane represent *a subject which is accessible to amateurs but lies close to the very heart of mathematics and continues to provide a seemingly inexhaustible supply of intriguing and provocative questions.* He first introduced polyomino as a plane figure with connected interior which can be divided into n congruent squares, of which any two have a side, a vertex or nothing in common. In his book [6] and papers Golomb considered a special class of the tiling problems and focused primarily on the question: Which polyominoes have the property that a finite number of copies of the basic shape, allowing all rotations and reflections, can be assembled to a rectangle?

J.H. Conway and J.C. Lagarias [3] developed techniques of *tile homology groups* and applied it to another type of tiling problems - the $\mathbb{Z}$-tiling problems. They give necessary conditions for the existence of such tilings using boundary words which are

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combinatorial-group invariants associated to the boundaries of the tile shapes and the regions to be tiled.

M. Reid [10] studied a particular refinement of Conway and Lagariases method and continued the development of the theory of tiling homotopy groups. In some cases where the set of tiles consists of some simple polyominoes he gave a complete description of corresponding homotopy groups.

O. Bodini and B. Nouvel [2] used algebraic method based on the theory of Gröbner basis to $\mathbb{Z}$-tiling problems.

In earlier papers [9] and [8] the authors explored polyomino tilings with tribones in hexagonal lattice and generalized well-known result of Conway and Lagarias [3] about tilings of the triangular regions by tribones. In the paper [9] we explored tilings by tribones which are symmetric with respect to a group generated by the 120°-rotation. We reduced the symmetric tiling problem to *submodule membership problem* and applied the theory of Gröbner basis for the polynomial rings with integer coefficients. In that paper we developed techniques which enable us to consider not only ordinary $\mathbb{Z}$-tiling problems in a lattice but the problems of tilings which are invariant under some subgroups of the symmetry group of the given lattice.

In this paper we continue the development of this method and apply it to the problem of $\mathbb{Z}$-tilings in the square lattice which are invariant under the central symmetry.

The paper is organized as follows.

In Section 2 we recall definitions of the lattice and introduce the module $P(A)$. In the third and fourth section we consider a general $\mathbb{Z}$-tiling problem in the lattice and symmetric tilings with respect to some subgroup G of the group of all isometric transformations of the lattice. In Section 5 we look more closely at the structure of the ring of invariants P^G and determine its generators and relations among them. In Section 6 we investigate the structure of the module $P(B(\mathcal{T}))^G$ of all regions in the lattice Λ which is possible to tile by tribones symmetric with respect to the origin. Section 7 establishes the relations between modules and rings which allows us to reduce *the submodule membership problem* to *the ideal membership problem.* In Section 8 we form ideal $J^G_{B(\mathcal{T})}$ and find its Gröbner basis. In the last section we apply the general theory to a *diamond shaped* region $\diamondsuit_n$ and prove a result (Theorem 9.7) which determines when they admit a centrally symmetric tiling by three-in-line polyominoes.

We work with Gröbner bases with integer coefficients. Standard references are [1] and [5]. See also [7] for an overview and some applications.

2. Lattices in the Euclidean plane

We want to study $\mathbb{Z}$-tiling problems in the square lattice in the Euclidean plane, so let us first mention basic definitions and concepts that will be needed in the paper.

Definition 2.1. A two-dimensional lattice Λ in the Euclidean plane is a group of the form $\{n_1\vec{v_1} + n_2\vec{v_2} : n_1, n_2 \in \mathbb{Z}\}$.

If $|\vec{v_1}| = |\vec{v_2}|$ and $\vec{v_1} \perp \vec{v_2}$, lattice Λ is called square lattice.

We refer to $\vec{v_1}$ and $\vec{v_2}$ as a basis of the lattice Λ.

DEFINITION 2.2. Let L be the matrix whose columns are vectors $\vec{v_1}$ and $\vec{v_2}$.

$$a_{k,l} = \left\{ L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \in [k, k+1), x_2 \in [l, l+1), k, l \in \mathbb{Z} \right\}$$

is the elementary cell of the lattice Λ. The cell $a = a_{0,0} = \{Lx : x \in [0,1)^2\}$ is the fundamental elementary cell.

Let $\vec{u_1}$ and $\vec{u_2}$ be radius vectors of barycenters of the elementary cells $a_{0,0}$ and $a_{-1,0}$. We denote by $\overline{\Lambda}$ the square lattice generated by the vectors $\vec{u_1}$ and $\vec{u_2}$. It is obvious that $\vec{v_1} = \vec{u_1} - \vec{u_2}, \quad \vec{v_2} = \vec{u_1} + \vec{u_2}$ so the lattice Λ can be viewed as a sublattice of the lattice $\overline{\Lambda}$.

The lattice Λ is the sublattice of $\overline{\Lambda}$ of index 2 and $\overline{\Lambda}/\Lambda \cong \mathbb{Z}_2$.

If we denote elements of the lattice Λ with white dots and the remaining elements of the lattice $\overline{\Lambda}$ with black dots, then each black dot represents one elementary cell in the lattice Λ. The set of all black dots of the lattice $\overline{\Lambda}$ is invariant under translation for the vectors of the form $n_1\vec{v_1} + n_2\vec{v_2}$, for $n_1, n_2 \in \mathbb{Z}$ which means that white dots act on the set of black dots.

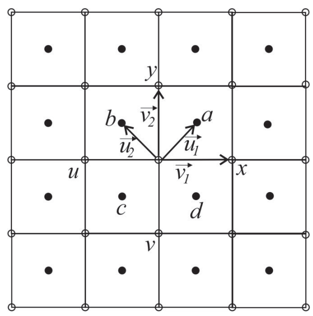

Figure 1: The Lattice Λ and the Lattice $\overline{\Lambda}$

There is a natural identification between the lattice Λ and the group Γ of all translations of the set of all elementary cells.

The group Γ is a free abelian group with two generators $\vec{v_1}$ and $\vec{v_2}$. Let H be a multiplicative group with four generators, x, y, u, v and defining relations $xu = 1$ and $yv = 1$. Then there is an isomorphism $\phi : H \longrightarrow \Gamma$ defined on the basis as follows

$$\phi(x) = \vec{v_1}, \phi(u) = -\vec{v_1},$$
$$\phi(y) = \vec{v_2}, \phi(v) = -\vec{v_2}.$$

If we form group rings $\mathbb{Z}[\Gamma]$ and $\mathbb{Z}[H]$ then there exists isomorphism between group rings induced by the isomorphism ϕ.

We call $\mathbb{Z}[\Gamma] \cong \mathbb{Z}[H] \cong \mathbb{Z}[x, y, u, v]/\langle xu - 1, yv - 1\rangle$, the ring of all translations that keep the lattice Λ invariant and denoted it by P.

Let A be a free Abelian group generated by all elementary cells a_{lk} of the lattice Λ, or what is the same, with all black dots of the lattice $\overline{\Lambda}$. If we define the map $\psi : P \times A \to A$ by $\psi(p, a_{k,l}) = a_{k+n_1,l+n_2}$ where $\phi(p) = n_1\vec{v_1} + n_2\vec{v_2}$, then ψ defines group action of P on the set A. Under this action the group A can be seen as a module over the ring P generated by only one elementary cell, for example by fundamental elementary cell a. We denote that module by $P(A)$. For our purpose it is convenient to regard module $P(A)$ as a module with four generators a, b, c, d, where relations among generators are given by $ua = b, \quad va = d, \quad uva = c.$

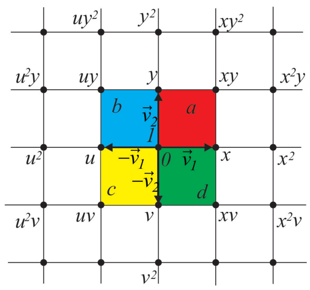

Figure 2: The lattice Λ and the module $P(A)$

3. Tilings in the square lattice

A polyomino T is a finite region consisting of elementary cells in the lattice Λ which are not necessarily connected. We will use a slightly more general definition of a polyomino as a multiset of elementary cells of lattice Λ with multiplicity, that can be negative.

Definition 3.1. A polyomino T is a finite weighted subset of Λ (a multiset) which contains each elementary cell of the lattice Λ with some (positive or negative) multiplicity. In other words $T = \sum_{k,l} w_{k,l} a_{k,l} = \sum_i p_i a$, $w_{k,l} \in \mathbb{Z}$, $p_i \in P$ is an element of the module $P(A)$.

For example polyomino T in Figure 3 is an element of module $P(A)$ and can be represented as a sum of the form $T = a + b + ub + d + xd$, where a, b, d are elements of A with the coefficients from the ring P.

Translation of the polyomino T for vector $\vec{v} = n_1\vec{v_1} + n_2\vec{v_2}$, $n_1, n_2 \geq 0$, is algebraically described as multiplying polynomial T by the monomial $x^{n_1}y^{n_2}$.

Let us now consider $\mathbb{Z}$-tiling problems in the lattice Λ.

Let $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ be the collection of basic tiles which we want to use for tiling a region T in lattice Λ. We have already seen that $T_1, T_2, \ldots, T_n$ and T are elements of the module $P(A)$. A region T can be $\mathbb{Z}$-tiled by elements of the set $\mathcal{T}$ if and only if T can be written as a sum of $T_1, T_2, \ldots, T_n$ with the coefficients in P, as follows $T = p_1T_1 + p_2T_2 + \cdots + p_nT_n$.

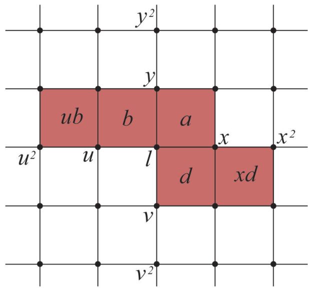

Figure 3: Polyomino T in the lattice Λ

Therefore, T can be tiled by the elements of the set $\mathcal{T}$ if and only if T belongs to a P-submodule of the module $P(A)$ which is generated by $T_1, T_2, \ldots, T_n$. Thus, we translated the $\mathbb{Z}$-tiling problem in the lattice Λ to the submodule membership problem.

Proposition 3.2. *A polyomino T admits a $\mathbb{Z}$-tiling by translates of elements of the set $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$ if and only if $T \in P(\mathcal{T})$, where $P(\mathcal{T})$ is the P-submodule of $P(A)$ generated by elements of $\mathcal{T}$.*

4. Symmetric tilings

We want to investigate the conditions of tilings of the region T in the square lattice Λ symmetrical with respect to some subgroup G of the group of all isometric transformations of the lattice Λ.

Let S_Λ be the group of all isometric transformations that keep the lattice Λ invariant. The group of all translations Γ is the subgroup of the group S_Λ.

Let G be a finite subgroup of the group S_Λ. The group G acts on the set of all elementary cells of the lattice Λ and on the module $P(A)$, preserving its P-module structure. This means that G acts on the elements of the ring

$$P = \mathbb{Z}[x, y, u, v]/\langle xu - 1, yv - 1\rangle$$

and that action may be non-trivial.

Let P^G be the set of all elements of the ring P which are invariant under the group G. The P^G is a subring of the ring P. The elements of the ring P^G act on the elements of the set A, and in the same way as before, A can be regarded as a module over the ring P^G. We will denote that P^G-module as $(P(A))^G$. An element of the module $(P(A))^G$ is referred as a symmetric polyomino.

Assume that the set $\mathcal{T}$ of basic tiles is invariant under the group G. The main problem is to decide when a given G-invariant polyomino $T \in \mathcal{T}$ admits a G-symmetric $\mathbb{Z}$-tiling by translates of a G-invariant set of basic tiles $\mathcal{T}$. Let $(P(\mathcal{T}))^G$ be the P^G-module of G-symmetric polyominoes generated by the elements of the set $\mathcal{T}$. The following criterion is a symmetric analogue of Proposition 4.

Proposition 4.1. *Let $\mathcal{T}$ be a G-invariant set of basic tiles. A G-symmetric polyomino $T \in (P(A))^G$ has a G-symmetric $\mathbb{Z}$-tiling by translates of the elements of the set $\mathcal{T}$ if and only if $T \in (P(\mathcal{T}))^G$.*

From now on we will denote by G the subgroup of S_Λ generated by the symmetry with respect to the origin. Clearly, $G \cong C_2$ because symmetry to the origin is an involution.

A three-in-line polyomino or a *tribone* is a translate of one of the following two types $V_a = a(1+y+y^2), \quad H_a = a(1+x+x^2)$. Let

$$\mathcal{T} = \{a(1+x+x^2),\, a(1+y+y^2),\, b(1+u+u^2),\, b(1+y+y^2)$$
$$c(1+u+u^2),\, c(1+v+v^2),\, d(1+x+x^2),\, d(1+v+v^2)\}$$

be the set of basic prototiles that we want to use to tile regions in the lattice Λ. The set $\mathcal{T}$ is invariant under the group G.

We apply the criterion from Proposition 4.1 in the case of tiling by tribones of a region in the lattice Λ which is invariant under the group G.

5. Ring of invariants

Now we want to determine generators and relations among them in the ring P^G.

The main tools of the classical theory of invariants such as the theorem of Emmy Noether, Moliens theorem and other tools, deal with polynomial rings $k[x_1, x_2, \dots, x_n]$, where k is the field of characteristic zero. The nature of our tiling problem requires dealing with polynomial rings with the coefficients in $\mathbb{Z}$. Therefore we have to use other techniques which include calculating Gröbner bases for the rings with the coefficients in the ring of integers.

Definition 5.1. Let G be the subgroup of the symmetric group S_n and $\mathbb{X} = \{x_1, x_2, \dots, x_n\}$ set of variables. The group G acts on the set $\mathbb{X}$ by $g \star x_i = x_{g(i)}$ for all $g \in G$. A polynomial $f(x_1, x_2, \dots, x_n) \in \mathbb{Z}$ is said to be invariant under G if

$$f(x_1, x_2, \dots, x_n) = f(x_{g(1)}, x_{g(2)}, \dots, x_{g(n)})$$

for all $g \in G$. The set of all invariant polynomials is denoted $\mathbb{Z}[x_1, x_2, \dots, x_n]^G$.

The set $\mathbb{Z}[x_1, x_2, \dots, x_n]^G$ is closed under addition and multiplication and contains the constant polynomials, so $\mathbb{Z}[x_1, x_2, \dots, x_n]^G$ is a subring of $\mathbb{Z}[x_1, x_2, \dots, x_n]$.

Let G be the subgroup of S_4, generated by the product of two transpositions $g = (13)(24)$. $|G| = 2$ and $G \cong C_2$.

Let s_1, s_2 and t be the following binomials in $\mathbb{Z}[x, y, u, v]/\langle xu-1, yv-1\rangle$

$$s_1 = x + u, \quad s_2 = y + v, \quad t = xy + uv.$$

Binomials s_1, s_2, and t are $G-$invariant, and we claim that they, together with 1, generate the ring $(\mathbb{Z}[x, y, u, v]/\langle xu-1, yv-1\rangle)^G$.

Lemma 5.2. *For every $m, n \in \mathbb{N}$, polynomial of the form $g_{m,n} = x^m y^n + u^m v^n \in \mathbb{Z}[x, y, u, v]^G$ can be written as a polynomial in the s_1, s_2, t with the coefficients in $\mathbb{Z}$.*

Proof. Let us initially handle the case $n = 1$.

For $m = 1$ we have $g_{1,1} = xy + uv = t$ and, since $s_1 g_{1,1} = (x+u)(xy+uv) = x^2y + u^2v + y + v$, we get $g_{2,1} = s_1 g_{1,1} - s_2 = s_1 t - s_2$. Thus, we see that for $m = 1$ and $m = 2$, $g_{m,n}$ can be written as a polynomial in s_1, s_2 and t.

Assume that for every $k < m$ polynomial $g_{k,1}$ can be represented as a polynomial in s_1, s_2 and t. Since

$$\begin{aligned} s_1 g_{m-1,1} &= (x+u)(x^{m-1}y + u^{m-1}v) \\ &= x^m y + u^m v + x^{m-2}y + u^{m-2}v = g_{m,1} + g_{m-2,1}, \end{aligned}$$

we have $g_{m,1} = s_1 g_{m-1,1} - g_{m-2,1}$. Because of the inductive hypothesis, the polynomial $g_{m,1}$ can also be obtained as a polynomial in s_1, s_2 and t.

The property is true for all $m \in \mathbb{N}$.

Let m be an arbitrary but fixed natural number. We have already shown that $g_{m,1} \in \mathbb{Z}[s_1, s_2, T]$. For $n = 2$, we get $g_{m,2} = s_2 g_{m,1} - (x^n - u^n)$. Since $x^n + u^n$ is a symmetric polynomial in $\mathbb{Z}[x, u]$, it can be written as a polynomial in the elementary symmetric functions $x + u$ and xu. This means that in the ring $\mathbb{Z}[x, y, u, v]/\langle xu - 1, yv - 1\rangle$, $x^n + u^n$ can be written as a polynomial in $x + u = s_1$, and we conclude that $g_{m,2} \in \mathbb{Z}[s_1, s_2, t]$.

Assume that for every $k < n$ polynomial $g_{m,k}$ can be represented as a polynomial in s_1, s_2 and t. Since

$$\begin{aligned} s_2 g_{m,n-1} &= (y+v)(x^m y^{n-1} + u^m v^{n-1}) \\ &= x^m y^n + u^m v^n + x^m y^{n-1} + u^m v^{n-1} = g_{m,n} + g_{m,n-2}, \end{aligned}$$

we have $g_{m,n} = s_2 g_{m,n-1} - g_{m,n-2}$. Due to inductive hypothesis, the polynomial $g_{m,n}$ can be written as a polynomial in s_1, s_2 and t.

We conclude by induction that $g_{m,n} \in \mathbb{Z}[s_1, s_2, t]$ for each $m, n \in \mathbb{N}$. □

The following lemma can be proved in the same way.

LEMMA 5.3. *Polynomial of the form* $h_{m,n} = x^m v^n + u^m y^n \in \mathbb{Z}[x, y, u, v]^G$ *for every* $m, n \in \mathbb{N}$ *can be written as a polynomial in the* s_1, s_2, t *with the coefficients in* $\mathbb{Z}$.

THEOREM 5.4. *The ring* $(\mathbb{Z}[x, y, u, v]/\langle xu - 1, yv - 1\rangle)^G$ *is generated by the* 1, s_1,s_2 *and* t. *Moreover,*

$$(\mathbb{Z}[x, y, u, v]/\langle xu - 1, yv - 1\rangle)^G \cong \mathbb{Z}[s_1, s_2, t]/\langle \Theta \rangle$$

where $\Theta := s_1^2 + s_2^2 - s_1 s_2 t + t^2 - 4 = 0$.

Proof. Every polynomial can be written uniquely as a sum of homogeneous components. A polynomial $f \in \mathbb{Z}[x, y, u, v]$ is invariant under G if and only if its homogeneous components are. For arbitrary polynomial $f \in (\mathbb{Z}[x, y, u, v]/\langle xu - 1, yv - 1\rangle)^G$ its homogeneous components can have one of the following forms:

$$g_{m,n} = x^m v^n + u^m y^n, \quad h_{m,n} = x^m v^n + u^m y^n \,.$$

From the Lemma 5.2 and Lemma 5.3 it follows that f can be written as a polynomial in s_1, s_2, and t.

Now we want to determine the ideal I generated by all algebraic relations among s_1,s_2 and t. The ideal I is said to be *ideal of relations among generators* or *syzygy* ideal. We will use the following proposition:

PROPOSITION 5.5. *[4, Chapter 7, §4, Proposition 3] If* $k[x_1,\dots,x_n]^G = k[f_1,\dots,f_n]$, *consider the ideal* $J_F = \langle f_1 - y_1,\dots,f_m - y_m\rangle \subset k[x_1,\dots,x_n,y_1,\dots,y_m]$.

(i) I_F *is the* $n-th$ *elimination ideal of* J_F. *Thus,* $I_F = J_f \cap k[y_1,\dots,y_m]$

(ii) Fix a monomial order in $k[x_1,\dots,x_n,y_1,\dots,y_m]$ *where any monomial involving one of* $x_1,\dots,x_n$ *is greater than all monomials in* $k[y_1,\dots,y_m]$ *and let* G *be a Gröbner bases for* J_F. *Then* $G \cap k[y_1,\dots,y_m]$ *is a Gröbner bases for* I_F *in the monomial order induced on* $k[y_1,\dots,y_m]$.

REMARK 5.6. It is not difficult to see that Proposition 5.5 is valid (with a similar proof) for Gröbner bases for the polynomial rings with integer coefficients.

If we form the ideal $J = \langle x+u-s_1, y+v-s_2, xy+uv-t, xu-1, yv-1\rangle$ and use the lexicographic order with $x > y > u > v > s_1 > s_2 > t$ then the Gröbner basis for the ideal I consists of the polynomials

$$-4+t^2+s_1^2-ts_1s_2+s_2^2,\ 1+v^2-vs_2, -4u-2tv+2s_1+ts_2+vs_1s_2+us_2^2-s_1s_2^2,$$
$$-2u-tv+s_1+uvs_2,\ -4+t^2+2us_1+tvs_1-tus_2-2vs_2-ts_1s_2+2s_2^2,$$
$$-tu-2v+uvs_1+s_2,$$
$$-2tu-t^2v+ts_1-2s_2+t^2s_2+us_1s_2+tvs_1s_2-vs_2^2-ts_1s_2^2+s_2^3,$$
$$u+x-s_1,\ -2+tuv+us_1-tus_2-vs_2+s_2^2,$$
$$v+y-s_2,\ -3+t^2+u^2+us_1+tvs_1-tus_2-2vs_2-ts_1s_2+2s_2^2,$$
$$-t+2uv-vs_1-us_2+s_1s_2.$$

From Proposition 5.5 follows that the ideal of relations among generators is given by $I = \langle -4+t^2+s_1^2-ts_1s_2+s_2^2\rangle$ and we conclude that

$$(\mathbb{Z}[x,y,u,v]/\langle xu-1, yv-1\rangle)^G \cong \mathbb{Z}[s_1,s_2,t]/\langle s_1^2+s_2^2-s_1s_2t+t^2-4\rangle. \qquad \square$$

6. Submodule $(P(\mathcal{T}))^G$ generated by tribones

Our goal in this section is to determine a set of generators of the module $(P(\mathcal{T}))^G$, where $\mathcal{T}$ is the set of tribones

$$\mathcal{T} = \{a(1+x+x^2),\ a(1+y+y^2),\ b(1+u+u^2),\ b(1+y+y^2)$$
$$c(1+u+u^2),\ c(1+v+v^2),\ d(1+x+x^2),\ d(1+v+v^2)\}$$

Recall that in the module $(P(A))^G$ we have following relations $b = ua,\ d = va\ c = uva$. Let

$$V_1 = a+d+dv+c+b+by, \quad H_1 = b+a+ax+cu+c+d,$$
$$V_2 = ay+a+d+b+c+cv, \quad H_2 = bu+b+a+c+d+dx.$$

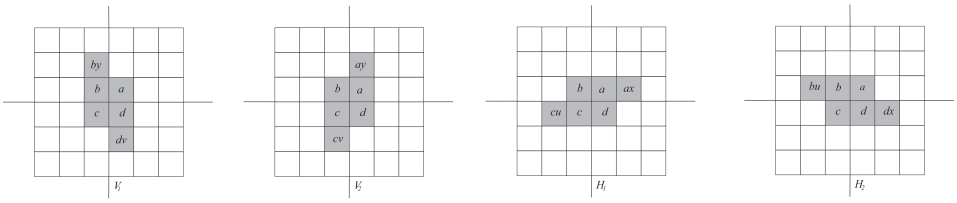

Figure 4: Symmetric Pairs of Polyominoes V_1,V_2, H_1 and H_2

LEMMA 6.1. *For every $m \in \mathbb{N}$ polynomials*

$$V_{1x^m} = x^m(a + d + dv) + u^m(c + b + by)$$
$$V_{2x^m} = x^m(a + b + by) + u^m(cv + c + d)$$

can be written as a sum of V_1 and V_2 with the coefficients in the ring P^G.

Proof. For $m = 1$ we have $V_{1x} = x(a + d + dv) + u(c + b + by)$. Since,

$$\begin{aligned} s_1V_1 &= (x + u)[(a + d + dv) + (c + b + by)] \\ &= [x(a + d + dv) + u(c + b + by)] + [u(a + d + dv) + x(c + b + by)] \\ &= V_{1x} + [(b + c + cv) + (d + a + ay)] = V_{1x} + V_2, \end{aligned}$$

we get $V_{1x} = s_1V_1 - V_2 = s_1V_{1x^0} - V_{2x^0}$.

Assume that for every $k \le m$ polynomial V_{1x^m} can be represented as a sum of V_1 and V_2 with the coefficients in the ring P^G. Then,

$$\begin{aligned} s_1V_{1x^m} &= (x + u)[x^m(a + d + dv) + u^m(c + b + by)] \\ &= [x^{m+1}(a + d + dv) + u^{m+1}(c + b + by)] + [x^{m-1}(a + d + dv) + u^{m-1}(c + b + by)] \\ &= V_{1x^{m+1}} + V_{1x^{m-1}} \end{aligned}$$

and we have $V_{1x^{m+1}} = s_1V_{1x^m} - V_{1x^{m-1}}$. Because of the inductive hypothesis, the polynomial $V_{1x^{m+1}}$ can also be obtained as a sum of V_1 and V_2 with the coefficients in the ring P^G. In the same manner we can see that for V_{2x^m} following relations apply $V_{2x^m} = s_1V_{2x^{m-1}} - V_{2x^{m-2}}$, which completes the proof. □

LEMMA 6.2. *For every $m, n \in \mathbb{N} \cup \{0\}$ polynomials*

$$V_{ax^my^n} = ax^my^n(1 + y + y^2) + cu^mv^n(1 + v + v^2)$$
$$V_{bu^my^n} = bu^my^n(1 + y + y^2) + dx^mv^n(1 + v + v^2)$$

can be written as a sum of V_1 and V_2 with the coefficients in the ring P^G.

Proof. For $m = 0$ and $n = 0$

$$\begin{aligned} s_2V_2 - V_1 &= (y + v)(d + a + ay + b + c + cv) - (dv + d + a + c + b + by) \\ &= a(1 + y + y^2) + c(1 + v + v^2) = V_{ax^0y^0} = V_a \end{aligned}$$

and $s_2V_1 - V_2 = V_{bu^0y^0} = V_b$.

An easy computations show that $V_{ax} = s_1V_a - V_b,\ V_{ay} = s_2V_a - V_2$. Since,

$$\begin{aligned} tV_a - V_1 &= (xy + uv)[a(1 + y + y^2) + c(1 + v + v^2)] - [(c + b + by) + (a + d + dv)] \\ &= [axy(1 + y + y^2) + cuv(1 + v + v^2)] = V_{axy} \end{aligned}$$

for $m = 1$ and $n = 1$, we have $V_{axy} = tV_a - V_1$.

Now proceed by induction. Let first fix $n = 1$. For $m = 2$

$$\begin{aligned} s_1V_{axy} - V_a &= (x + u)[axy(1 + y + y^2) + cuv(1 + v + v^2)] - [a(1 + y + y^2) + c(1 + v + v^2)] \\ &= ax^2y(1 + y + y^2) + cu^2v(1 + v + v^2) = V_{ax^2y} \end{aligned}$$

and we have $V_{ax^2y} = s_1V_{axy} - V_a$. Thus we see that for $n = 1$ and $m = 2$ $V_{ax^my^n}$ can be written as a sum of V_1 and V_2 with the coefficients in the ring P^G.

Assume that the same is true for every polynomial V_{ax^ky} for $2 \le k \le m$. Since,

$$\begin{aligned} s_1V_{ax^my} &= (x + u)[ax^my(1 + y + y^2) + cu^mv(1 + v + v^2)] \\ &= [ax^{m+1}y(1 + y + y^2) + cu^{m+1}v(1 + v + v^2)] \\ &\quad + [ax^{m-1}y(1 + y + y^2) + cu^{m-1}v(1 + v + v^2)] \\ &= V_{ax^{m+1}y} + V_{ax^{m-1}y} \end{aligned}$$

we have $V_{ax^{m+1}y} = s_1V_{ax^my} - V_{ax^{m-1}y}$.

Because of the inductive hypothesis, the polynomial $V_{ax^{m+1}y}$ can also be obtained as a sum of V_1 and V_2 with the coefficients in the ring P^G, so the property is true for all $m \in \mathbb{N} \cup \{0\}$.

Let m be an arbitrary but fixed natural number. We have already shown that for $n = 1$ polynomial V_{ax^my} has the desired characteristic. Assume that the same is true for $V_{ax^my^k}$ for $1 \le k \le n$. Since,

$$\begin{aligned} s_2V_{ax^my^n} &= (y + v)[ax^my^n(1 + y + y^2) + cu^mv^n(1 + v + v^2)] \\ &= [ax^my^{n+1}(1 + y + y^2) + cu^mv^{n+1}(1 + v + v^2)] \\ &\quad + [ax^my^{n-1}(1 + y + y^2) + cu^mv^{n-1}(1 + v + v^2)] \\ &= V_{ax^my^{n+1}} + V_{ax^my^{n-1}} \end{aligned}$$

we have $V_{ax^my^{n+1}} = s_2V_{ax^my^n} - V_{ax^my^{n-1}}$.

By the principles of the mathematical induction we conclude that for all $m, n \in \mathbb{N} \cup \{0\}$ polynomial $V_{ax^my^n}$ can be written as a sum of V_1 and V_2 with the coefficients in the ring P^G.

The proof for the polynomial $V_{bu^mv^n}$ can be handled in much the same way. □

The following lemmas can be proved in the similar way as the previous two.

Lemma 6.3. *For every $m \in \mathbb{N}$ polynomials*

$$\begin{aligned} H_{1y^m} &= y^m(ax + a + b) + v^m(cu + c + d) \\ H_{2y^m} &= y^m(a + b + bu) + v^m(c + d + dx) \end{aligned}$$

can be written as a sum of H_1 and H_2 with the coefficients in the ring P^G.

Lemma 6.4. *For every $m, n \in \mathbb{N} \cup \{0\}$ polynomials*

$$H_{ax^m y^n} = ax^m y^n (1 + x + x^2) + cu^m v^n (1 + u + u^2)$$
$$H_{bu^m y^n} = bu^m y^n (1 + u + u^2) + dx^m v^n (1 + x + x^2)$$

can be written as a sum of H_1 and H_2 with the coefficients in the ring P^G.

Theorem 6.5. *The module $(P(\mathcal{T}))^G \subset (P(A))^G$ of $G-$invariant polyominoes which admit a signed , symmetric tiling by tribones is generated, as a module over P^G by the $G-$symmetric pairs of tribones V_1, V_2, H_1, and H_2.*

Proof. It is obvious that an any vertical pair of tribones which is symmetric to the origin can have one of the following four forms

$$V_{1x^m} = x^m (a + d + dv) + u^m (c + b + by)$$
$$V_{2x^m} = x^m (a + b + by) + u^m (cv + c + d)$$
$$V_{ax^m y^n} = ax^m y^n (1 + y + y^2) + cu^m v^n (1 + v + v^2)$$
$$V_{bu^m y^n} = bu^m y^n (1 + y + y^2) + dx^m v^n (1 + v + v^2)$$

and an arbitrary horizontal pair of tribones symmetric to the origin can have one of the following forms

$$H_{1y^m} = y^m (ax + a + b) + v^m (cu + c + d)$$
$$H_{2y^m} = y^m (a + b + bu) + v^m (c + d + dx)$$
$$H_{ax^m y^n} = ax^m y^n (1 + x + x^2) + cu^m v^n (1 + u + u^2)$$
$$H_{bu^m y^n} = bu^m y^n (1 + u + u^2) + dx^m v^n (1 + x + x^2).$$

From the above four lemmas follows that each of the observed polynomial can be written as a sum of the V_1, V_2, H_1 and H_2 with the coefficients in the ring P^G which means that they are elements of the module generated by V_1, V_2, H_1 and H_2 over the ring P^G. Reverse Inclusion is obvious, so we conclude that the module of all central symmetric tribones is the module generated by the V_1, V_2, H_1 and H_2 . □

7. The ring $\overline{P}$ and the ring P

We have already seen that the lattice Λ can be seen as a sublattice of the lattice $\overline{\Lambda}$. On the same way that we formed group ring $P = \mathbb{Z}[x, y, u, v]/\langle xu - 1, yv - 1\rangle$ of the lattice Λ, we can form group ring of the lattice $\overline{\Lambda}$. Let us denote group ring of the lattice $\overline{\Lambda}$ by $\overline{P} = \mathbb{Z}[a, b, c, d]/\langle ac - 1, bd - 1\rangle$. All structures results that apply to the ring P apply to the ring $\overline{P}$ as well. In particular there is an isomorphism

$$(\mathbb{Z}[a, b, c, d]/\langle ac - 1, bd - 1\rangle)^G \cong \mathbb{Z}[s_1, s_2, t]/\langle s_1^2 + s_2^2 - s_1 s_2 t + t^2 - 4\rangle \qquad (2)$$

where $s_1 = a + c$, $s_2 = b + d$ and $t = ab + cd$.

The fact that Λ is sublattice of the lattice $\overline{\Lambda}$ of index 2 allows us to define a $\mathbb{Z}_2$-grading in the rings $\overline{P}$ and P by 'degree mod 2'.

The rings $\overline{P}^G$ and P^G inherits the $\mathbb{Z}_2-$gradation from the ring $\overline{P}$.

$\overline{P}^G$ is $\mathbb{Z}$-generated by 1 and by binomials $a^m b^n + c^m d^n$, $a^m d^n + c^m b^n$ for $m, n \in \mathbb{N} \cup \{0\}$. Similarly, the ring P^G is $\mathbb{Z}$-generated by 1 and by $x^m y^n + u^m v^n$ and $x^m v^n + u^m y^n$. Since, $x = ad$, $y = ab$, $u = bc$ and $v = cd$ the ring P^G is obviously a subring of the ring $\overline{P}^G$ of all elements graded by $0 \in \mathbb{Z}_2$. The subset of $\overline{P}^G$ of all elements graded by $1 \in \mathbb{Z}_2$ is precisely the set of elements of the module $(P(A))^G$. This characterization make us able to give the following proposition:

PROPOSITION 7.1. *Let $(P(\mathcal{T}))^G \subseteq (P(A))^G$ be a P^G- submodule of $(P(A))^G$ generated by the set $\mathcal{T}$. Let $I_{(P(\mathcal{T}))^G}$ be the ideal in $\overline{P}^G$ generated by $\mathcal{T}$. Suppose that $p \in (P(A))^G$. Then, $p \in (P(\mathcal{T}))^G \Longleftrightarrow p \in I_{(P(\mathcal{T}))^G}$.*

Proof. The implication $p \in (P(\mathcal{T}))^G \Longrightarrow p \in I_{(P(\mathcal{T}))^G}$ is clear. If $p \in I_{(P(\mathcal{T}))^G}$ and $p = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k$ for some elements $p_i \in \mathcal{T}$ and homogeneous (in the sense of the $\mathbb{Z}_2$-graduation) elements $\alpha_i \in \overline{P}^G$ then we can assume that all $\alpha_i \in P^G$ (the other terms cancel out). □

The above proposition allows us to reduce *submodule membership problem* to *ideal membership problem* in the ring $\overline{P}^G$. Because of the isomorphism (2) *ideal membership problem* for the ideal generated by the set $\mathcal{T}$ becomes ideal membership problem for the ideal $J_{(P(\mathcal{T}))^G} = I_{(P(\mathcal{T}))^G} + \langle s_1^2 + s_2^2 - s_1 s_2 t + t^2 - 4\rangle \subset \mathbb{Z}[s_1, s_2, t]$.

8. Ideal $J_{(P(\mathcal{T}))^G}$ and its Gröbner bases

In this section we express the generating elements for the ideal $I_{(P(\mathcal{T}))^G}$ in terms of variables s_1, s_2 and t which appear in the description of the ambient ring and find its Gröbner basis.

PROPOSITION 8.1. *In the ring $\overline{P}^G$ polynomials V_1, V_2, H_1 and H_2 have following forms*

$$V_1 = s_2(1+t), \quad H_1 = s_1(1 + s_1 s_2 - t),$$
$$V_2 = s_1(1+t), \quad H_2 = s_2(1 + s_1 s_2 - t).$$

Proof. The proof is by direct calculations. For example,

$$\begin{aligned} H_1 &= b + a + ax + cu + c + d = (a+c) + (b+d) + a^2 d + bc^2 \\ &= s_1 + s_2 + (a+c)(ad+bc) - abc - adc \\ &= s_1 + s_2 + s_1(s_1 s_2 - t) - s_2 = s_1(1 + s_1 s_2 - t). \end{aligned}$$
□

In light of Proposition 7.1 we can form the ideal $J_{(P(\mathcal{T}))^G}$

$$\begin{aligned} J_{(B(\mathcal{T}))^G} = \langle & s_2(1+t),\ s_1(1+t), s_1(1+s_1 s_2 - t), \\ & s_2(1+s_1 s_2 - t),\ s_1^2 + s_2^2 - s_1 s_2 t + t^2 - 4\rangle. \end{aligned}$$

With the aid of *Wolfram Mathematica 9.0* we determine the Gröbner bases for the ideal $J_{(P(\mathcal{T}))^G}$.

Proposition 8.2. *The Gröbner bases $GJ_{(P(\mathcal{T}))^G}$ of the ideal $J_{(P(\mathcal{T}))^G} \subset \overline{P}^G$ with respect to the lexicographic order of variables s_1, s_2 and t is given by the following list of polynomials:*

$$-4-4t+t^2+t^3, \quad s_2+s_2t, \quad 16-15s_2^2+3s_2^4-4t^2, \quad 4s_2-5s_2^3+s_2^5,$$
$$2s_1+5s_2-s_2^3, \quad s_1+s_1t, 8+s_1s_2-5s_2^2+s_2^4-2t^2, \quad -12+s_1^2+6s_2^2-s_2^4+3t^2.$$

9. The diamond region $\diamondsuit_n$

In the previous chapters we have determined generators and the Gröbner bases for the ideal $J_{(P(\mathcal{T}))^G}$. Now we will use these results to investigate wether it is possible to tile the region $\diamondsuit_n$ symmetric with respect to the origin. Region $\diamondsuit_n$ is depicted in Figure 5 and has hole in his center. n is the number of squares on each side of the region $\diamondsuit_n$. The given diamond region is symmetric with respect to the origin and it is an element of the P^G-module $(P(A))^G$.

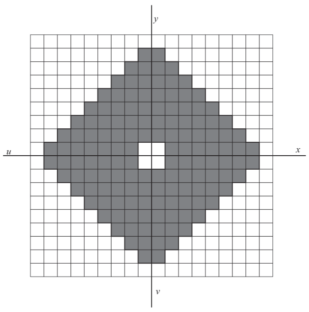

Figure 5: The diamond region $\diamondsuit_8$

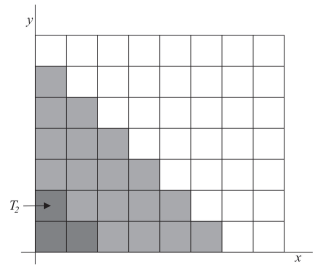

Figure 6: Regions T_6 and T_2

Denote by $\Box$ an operation on the module A which adds to each element another elements that are symmetric to the initial one with respect to xu, yv axes and with respect to the origin. For example, $\Box(ax^2y) = ax^2y + bu^2y + cu^2v + dx^2y$.

If I is the Ideal $I = \langle a(1+x+x^2), a(1+y+y^2)\rangle$, then the next proposition easily follows.

Proposition 9.1. *For each $f \in P(A)$*

a) If $f \in I$, then $\Box f \in I_{(P(\mathcal{T}))^G}$.

b) If $f_1 \equiv_I f_2$, then $\Box f_1 \equiv_{I_{(P(\mathcal{T}))^G}} \Box f_2$.

We will now focus on the first quadrant and a triangular region T_n depicted on the Figure 6.

Let $Hl_n = a(1+x+x^2+\cdots+x^{n-1})$. Then we have

$$T_n = Hl_n + yHl_{n-1} + y^2Hl_{n-2} + \cdots + ay^{n-1},$$
$$T_2 = a(1+x+y).$$

Lemma 9.2. *If $m \equiv n (mod\ 3)$, then $ax^m \equiv_I ax^n$ and $ay^m \equiv_I ay^n$.*

Lemma 9.3. *For arbitrarily $n \in \mathbb{N}$ we have $Hl_n + yHl_{n-1} + y^2 Hl_{n-2} \equiv_I x^{n-2}T_2$.*

Proof.

$$\begin{aligned} &Hl_n + yHl_{n-1} + y^2 Hl_{n-2} \\ &\quad = a(1 + x \cdots + x^{n-1}) + ay(1 + x \cdots + x^{n-2}) + ay^2(1 + x \cdots + x^{n-3}) \\ &\quad = a(1 + x + x^2 \cdots + x^{n-3})(1 + y + y^2) + ax^{n-2}(1 + x + y). \end{aligned}$$ □

The following proposition is a consequence of Lemmas 9.2 and 9.3

Proposition 9.4.

$$T_n \equiv_I \begin{cases} kT_2, & \text{if} \quad n = 3k - 1 \\ kxT_2, & \text{if} \quad n = 3k \\ kx^2T_2 + a, & \text{if} \quad n = 3k + 1. \end{cases}$$

From Proposition 9.1 and Theorem 9.4 we obtain

Proposition 9.5. *The polynomial of the region $\diamondsuit_n$ is congruent mod $J_{(P(\mathcal{T}))^G}$ to*

$$k[a(1 + x + y) + b(1 + u + y) + c(1 + u + v) + d(1 + x + v)] - (a + b + c + d) \qquad (n = 3k - 1)$$
$$k[ax(1 + x + y) + bu(1 + u + y) + cu(1 + u + v) + dx(1 + x + v)] - (a + b + c + d) \qquad (n = 3k)$$
$$k[ax^2(1 + x + y) + bu^2(1 + u + y) + cu^2(1 + u + v) + dx^2(1 + x + v)] \qquad (n = 3k + 1)$$

The proof of the next proposition follows from Proposition 9.5 by direct calculation or preferably by a computer algebra system.

Proposition 9.6. *The polynomial of the region $\diamondsuit_n$ in the ring $\overline{P}^G$ is congruent mod $J_{(P(\mathcal{T}))^G}$ to $kP - Q$ where:*

– *If $n = 3k - 1$:*

$$P = -s_1 - s_2 + s_1^2 s_2 + s_1 s_2, \quad Q = s_1 + s_2.$$

– *If $n = 3k$:*

$$P = -s_1 - s_2 + s_1^2 s_2 + s_1 s_2, \qquad Q = s_1 + s_2.$$

– *If $n = 3k + 1$:*

$$\begin{aligned} P &= s_1 - 7s_2 - 10s_1 s_2^2 + s_1^3 s_2^2 + 2s_2^3 - s_1^2 s_2^3 + s_1^4 s_2^3 + 2s_1 s_2^4 + s_1^3 s_2^4 + s_1 t + 13 s_2 t \\ &\quad - 2s_1^2 s_2 t - s_1 s_2^2 t - 2s_1^3 s_2^2 t - 3s_2^3 t - 4s_1^2 s_2^3 t + 2s_2 t^2 + s_1^2 s_2 t^2 + 6s_1 s_2^2 t^2 - 3s_2 t^3, \\ Q &= 0. \end{aligned}$$

Theorem 9.7. *Let $\diamondsuit_n$ be the diamond region in the square lattice depicted in Figure 5 where n is the number of squares on the edge of the region. Then $\diamondsuit_n$ admits a symmetric, signed tiling by tribones if and only if $n = 3k + 1$ for some integer k.*

Proof. By Proposition 9.6 the polynomial which is congruent to the polynomial of the region $\diamondsuit_n$ can be expressed (in variables s_1, s_2 and t) as the sum $kP - Q$.

With the aid of *Wolfram Mathematica 9.0* we can calculate the remainders $\overline{P}$ and $\overline{Q}$ of the polynomials P and Q on division by the Gröbner bases $GJ_{(P(\mathcal{T}))^G}$ of the ideal $J_{(P(\mathcal{T}))^G}$. We obtain:

If $n = 3k - 1$: $\overline{P} = s_1 + 7s_2 - 2s_2^3$, $\overline{Q} = s_1 + s_2$.
If $n = 3k$: $\overline{P} = s_1 + 2s_2 + s_2^3$, $\overline{Q} = s_1 + s_2$.
If $n = 3k + 1$: $\overline{P} = 0$, $\overline{Q} = 0$.

We see that the polynomial $kP - Q$ is reduced to zero in the case $n = 3k + 1$. In the other two cases the polynomial $kP - Q$ can not be reduced to zero because of the structure of the Gröbner bases we always get remainder equal to the sum of monomials s_1, s_2 and s_2^3 which cannot be reduced. □

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