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# BLENDING TYPE APPROXIMATION BY BERNSTEIN-DURRMEYER TYPE OPERATORS

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**Abstract**. In this note, we introduce the Durrmeyer variant of Stancu operators that preserve the constant functions depending on non-negative parameters. We give a global approximation theorem in terms of the Ditzian-Totik modulus of smoothness, a Voronovskaja type theorem and a local approximation theorem by means of second order modulus of continuity. Also, we obtain the rate of approximation for absolutely continuous functions having a derivative equivalent with a function of bounded variation. Lastly, we compare the rate of approximation of the Stancu-Durrmeyer operators and genuine Bernstein-Durrmeyer operators to certain function by illustrative graphics with the help of the Mathematica software.

## 1. Introduction

The most famous theorem for convergence of linear positive operators is due to the Weierstrass [19] who introduced an important theorem named Weierstrass approximation theorem. This theorem is the first magnificent evolution in approximation theory of one real variable and plays a basic role in the development of approximation theory. The constructive proof of this theorem is given by following Bernstein polynomials $(\mathcal{B}_n)_{n\in\mathbb{N}}$: $\mathcal{B}_n(f;x)=\sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right)$, where $p_{n,\nu}(x)=\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}$, for $f\in C(J)$, with $J=[0,1]$.

Using different methods, many mathematicians generalized Bernstein polynomials with parameters. Stancu [18] proposed the Bernstein type operators based on the two parameters $r,s\in\mathbb{N}\cup\{0\}$, as follows:

$$(S_{n,r,s})\, f(x)=\sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x) f\left(\frac{\mu+\nu r}{n}\right). \tag{1}$$

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For the special case $r = s = 0$, these operators reduce to the classical Bernstein operators.

Gonska and Păltăneă [10] introduced genuine Bernstein-Durrmeyer type operators

$$\mathcal{N}_n^\rho(f;x) = \sum_{\nu=1}^{n-1} p_{n,\nu}(x) \int_0^1 \frac{t^{\nu\rho-1}(1-t)^{(n-\nu)\rho-1}}{B\left(\nu\rho,(n-\nu)\rho\right)} f(t)\,dt + f(0)(1-x)^n + f(1)x^n,$$

where $B\left(\nu\rho,(n-\nu)\rho\right)$ is the beta function defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a,b>0 \text{ and } \rho>0.$$

These operators preserve linear functions. Also, the simultaneous approximation for these operators was obtained.

In 2008, Păltănea [16] considered the generalization of Phillips operators by taking the weights in general form depending on the non-negative parameter $\rho$. Goyal et al. [11] considered a one parameter family of Baskakov-Szàsz type operators and studied quantitative convergence theorems for these operators. In [13], Gupta and Rassias proposed hybrid operators based on Polya distribution and obtained some direct theorems of these operators. Gupta et al. [14] introduced hybrid operators involving inverse Polya-Eggenberger distribution and studied degree of approximation of these operators which include global approximation and uniform convergence. Very recently, Acu and Gupta [6] defined a summation-integral type operators depending on two parameters and discussed some approximation results, e.g., local approximation, Voronovskaja type asymtotic theorem and weighted approximation of these operators. In the literature survey, several authors have studied the approximation behavior of mixed hybrid operators [1–5, 7, 12, 15, 17, 20].

Motivated by the above work, for $f \in C(J)$, we introduce the following Durrmeyer variant of the operators (1) depending on three parameters $r, s$ and $\rho$ as follows:

$$\mathcal{K}_{n,r,s}^\rho(f;x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{\nu=0}^{s} p_{s,\nu}(x) \int_0^1 \Theta_{n,\mu+\nu r}^\rho(t) f(t)\,dt, \tag{2}$$

where $\Theta_{n,\mu+\nu r}^\rho(t) = \dfrac{t^{(\mu+\nu r)\rho}(1-t)^{(n-\mu-\nu r)\rho}}{B\left((\mu+\nu r)\rho+1,(n-\mu-\nu r)\rho+1\right)}$. It can be seen that the operators $\mathcal{K}_{n,r,s}^\rho$ preserve the constant functions.

The aim of this article is to study the approximation properties for the Stancu-Durrmeyer operators based on non-negative parameters of the operators defined in (2). We give a direct approximation theorem with the help of Ditzian-Totik modulus of continuity, a Voronovskaja type theorem and a local approximation theorem by means of second order modulus of continuity. Also, we discuss the rate of convergence for absolutely continuous functions having a derivative equivalent with a function of bounded variation. Furthermore, we show the rate of convergence of these operators and the genuine Bernstein-Durrmeyer operators to certain function by illustrative graphics produced by Mathematica software.

## 2. Auxiliary results

In order to prove the main results, in this section we will show some lemmas.

Let $e_i(x) = x^i, i = \overline{0,4}$.

LEMMA 2.1. *For the operators $\mathcal{K}^{\rho}_{n,r,s}(f;x)$, we have*

*(i)* $\mathcal{K}^{\rho}_{n,r,s}(e_0;x) = 1$;

*(ii)* $\mathcal{K}^{\rho}_{n,r,s}(e_1;x) = \dfrac{n\rho x + 1}{n\rho + 2}$;

*(iii)* $\mathcal{K}^{\rho}_{n,r,s}(e_2;x) = \dfrac{x^2\rho^2\left[n(n-1) - rs(r-1)\right] + x\rho\left[n(3+\rho) + rs\rho(r-1)\right] + 2}{(n\rho+3)(n\rho+2)}$;

*(iv)*
$$\begin{aligned}\mathcal{K}^{\rho}_{n,r,s}(e_3;x) &= \frac{x^3\rho^3\left[n(n-1)(n-2) + r(r-1)(2-3n+2r)s\right]}{(n\rho+4)(n\rho+3)(n\rho+2)}\\ &+ \frac{3x^2\rho^2\left[n^2(2+\rho) - rs(r-1)(2+\rho+r\rho) + n(r^2s\rho - (1+2sr)(2+\rho) + rs(4+\rho))\right]}{(n\rho+4)(n\rho+3)(n\rho+2)}\\ &+ \frac{x\rho\left(rs\rho(r-1)(6+\rho+r\rho) + n(11+\rho(6+\rho))\right)}{(n\rho+4)(n\rho+3)(n\rho+2)}\\ &+ \frac{6}{(n\rho+4)(n\rho+3)(n\rho+2)};\end{aligned}$$

*(v)*
$$\begin{aligned}\mathcal{K}^{\rho}_{n,r,s}(e_4;x) &= \frac{x^4\rho^4}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)}\Big[n(n-1)(n-2)(n-3)\\ &+ n^2(11 + 6r^2s(s-1) - 12rs(1+rs) + 6rs(3+rs)) + 2n(-3 + 2r^3s(s-1)(s-2)\\ &- 11rs - 9r^2s - 2r^3s - 3r^2s(s-1)(1+2rs) + 2rs(2+6rs+3r^2s))\\ &+ (r^4s(-6+11s-6s^2+s^3) - 4r^4s^2(s-1)(s-2) + 6r^3s^2(s-1)(1+rs)\\ &- 4r^2s^2(2+3rs+r^2s) + rs(6+11rs+6r^2s+r^3s))\Big]\\ &+ \frac{x^3\rho^2}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)}\Big[6n^3(1+\rho) + 6n^2(r^2s\rho - 3(1+rs)(1+\rho)\\ &+ rs(3+2\rho)) + 6(r^4s\rho(s-1)(s-2) - rs(2+3rs+r^2s)(1+\rho) + r^2s^2(1+rs)(3+2\rho)\\ &+ r^3s(s-1)(-2+s-2rs\rho) + r^3s^2(3+(2+rs)\rho - s(3+\rho))) + 6n(2r^3s\rho(s-1)\\ &+ (2+6rs+3sr^2)(1+\rho) - rs(1+2rs)(3+2\rho) + r^2s(-3-2(1+rs)\rho + s(3+\rho)))\Big]\\ &+ \frac{x^2\rho}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)}\Big[n^2(\rho+1)(11+7\rho) + n(4r^3s\rho^2 + 6r^2s\rho(\rho+3)\\ &+ 2rs(11+9\rho+2\rho^2) - (1+2rs)(\rho+1)(11+7\rho)) + 7r^4s\rho^2(s-1) - 2r^3s\rho(9-9s+2rs\rho)\\ &- 2r^2s^2(11+9\rho+2\rho^2) + rs(1+rs)(\rho+1)(11+7\rho) + r^2s(-11+11s-6rs\rho(\rho+3))\Big]\\ &+ \frac{x\left[rs(6+\rho(11r+6r^2\rho+r^3\rho^2)) + (\rho+1)(\rho+2)(\rho+3)(n-rs)\right]}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)}\\ &+ \frac{24}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)}.\end{aligned}$$

Lemma 2.2. *For $m = 1, 2$, the $m^{th}$ order central moments of $\mathcal{K}^{\rho}_{n,r,s}$ defined as $\eta^{\rho;m}_{n,r,s}(x) = \mathcal{K}^{\rho}_{n,r,s}((t-x)^m; x)$ satisfy*

*(i)* $\eta^{\rho;1}_{n,r,s}(x) = \dfrac{1-2x}{(n\rho+2)}$;

*(ii)* $\eta^{\rho;2}_{n,r,s}(x) = \dfrac{x^2\left(6-\rho(n+(n+(r-1)rs)\rho)\right)+x\left(-6+\rho(n+(n+(r-1)rs)\rho)\right)+2}{(n\rho+2)(n\rho+3)}$.

Lemma 2.3. *For $n \in \mathbb{N}$, we have*

$$\mathcal{K}^{\rho}_{n,r,s}((t-x)^2; x) \leq \frac{\mathcal{A}^{\rho}_{r,s}\ \ x(1-x)}{(1+n\rho)},$$

*where $\mathcal{A}^{\rho}_{r,s}$ is a positive constant depending on $r, s$ and $\rho$.*

Remark 2.4. For the operators $\mathcal{K}^{\rho}_{n,r,s}$, we get

$$\lim_{n\to\infty} n\ \eta^{\rho;1}_{n,r,s}(x) = \frac{1-2x}{\rho}, \qquad \lim_{n\to\infty} n\ \eta^{\rho,2}_{n,r,s}(x) = \frac{(1+\rho)x(1-x)}{\rho},$$
$$\lim_{n\to\infty} n^2\ \eta^{\rho,4}_{n,r,s}(x) = \frac{12x^2(x+1)^2}{\rho^2}.$$

## 3. Direct estimates

Theorem 3.1. *Let $f \in C(J)$. Then $\lim\limits_{n\to\infty} \mathcal{K}^{\rho}_{n,r,s}(f; x) = f(x)$, uniformly in $J$.*

*Proof.* Since $\mathcal{K}^{\rho}_{n,r,s}(1; x) = 1$, $\mathcal{K}^{\rho}_{n,r,s}(e_1; x) \to x$, $\mathcal{K}^{\rho}_{n,r,s}(e_2; x) \to x^2$ as $n \to \infty$, uniformly in $J$. By Korovkin Theorem, it follows that $\mathcal{K}^{\rho}_{n,r,s}(f; x) \to f(x)$ as $n \to \infty$, uniformly in $J$. □

### 3.1 Voronovskaja type theorem

In this section we establish Voronvoskaja type result for the operators $\mathcal{K}^{\rho}_{n,r,s}$.

Theorem 3.2. *Let $f \in C(J)$. If $f''$ exists at a point $x \in J$, then we have*

$$\lim_{n\to\infty} n\left[\mathcal{K}^{\rho}_{n,r,s}(f; x) - f(x)\right] = \frac{1-2x}{\rho} f'(x) + \frac{(1+\rho)x(1-x)}{2\rho} f''(x).$$

*Proof.* Applying Taylor's expansion of $f$, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \varpi(t,x)(t-x)^2,$$

where $\lim\limits_{t\to x} \varpi(t,x) = 0$. By using the linearity of the operator $\mathcal{K}^{\rho}_{n,r,s}$, we get

$$\begin{aligned}\mathcal{K}^{\rho}_{n,r,s}(f; x) - f(x) =& \mathcal{K}^{\rho}_{n,r,s}((t-x); x) f'(x) + \frac{1}{2}\mathcal{K}^{\rho}_{n,r,s}((t-x)^2; x) f''(x) \\ &+ \mathcal{K}^{\rho}_{n,r,s}(\varpi(t,x)(t-x)^2; x).\end{aligned}$$

Now, using the Cauchy-Schwarz property, we have

$$\mathcal{K}^{\rho}_{n,r,s}(\varpi(t,x)(t-x)^2;x) \leq \sqrt{\mathcal{K}^{\rho}_{n,r,s}(\varpi^2(t,x);x)}\sqrt{n^2\mathcal{K}^{\rho}_{n,r,s}((t-x)^4;x)}.$$

In view of Theorem 3.1, we obtain $\lim\limits_{n\to\infty} \mathcal{K}^{\rho}_{n,r,s}(\varpi^2(t,x);x)= \varpi^2(x,x) = 0$, since $\varpi(t,x) \to 0$ as $t \to x$, and using Remark 2.4 for every $x \in J$, we get

$$\lim_{n\to\infty} n^2\mathcal{K}^{\rho}_{n,r,s}\left((t-x)^4;x\right) = \frac{12x^2(x+1)^2}{\rho^2}.$$

Hence, $n\mathcal{K}^{\rho}_{n,r,s}(\varpi(t,x)(t-x)^2;x) = 0$. From Remark 2.4, we have

$$\lim_{n\to\infty} n\mathcal{K}^{\rho}_{n,r,s}\left(t-x;x\right) = \frac{1-2x}{\rho}, \quad \lim_{n\to\infty} n\mathcal{K}^{\rho}_{n,r,s}\left((t-x)^2;x\right) = \frac{(1+\rho)x(1-x)}{\rho}.$$

Combining the results from the above, the theorem is proved. □

### 3.2 Local approximation

We begin by recalling the following K-functional:

$$K_2(f,\delta) = \inf\{\|f-g\| + \delta\|g''\| : g \in W^2\} \ (\delta > 0),$$

where $W^2 = \{g : g'' \in C(J)\}$ and $\|\cdot\|$ is the uniform norm on $C(J)$. By [8], there exists a positive constant $M > 0$ such that

$$K_2(f,\delta) \leq M\omega_2(f,\sqrt{\delta}),$$

where the modulus of smoothness of second order for $f \in C(J)$ is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \ \sup_{x,x+2h\in J} |f(x+2h) - 2f(x+h) + f(x)|.$$

The modulus of continuity for $f \in C(J)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \ \sup_{x,x+h\in J} |f(x+h) - f(x)|.$$

The Steklov mean is defined as

$$f_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}} [2f(x+u+v) - f(x+2(u+v))]\, du\, dv. \tag{3}$$

By simple computation, it is observed that

a) $\|f_h - f\|_{C(J)} \leq \omega_2(f,h)$;

b) $f_h', f_h'' \in C(J)$ and $\|f_h'\|_{C(J)} \leq \frac{5}{h}\omega(f,h)$, $\|f_h''\|_{C(J)} \leq \frac{9}{h^2}\omega_2(f,h)$.

THEOREM 3.3. *Let $f \in C(J)$. Then for every $x \in J$, the following inequality holds*

$$\left|\mathcal{K}^{\rho}_{n,r,s}(f;x) - f(x)\right| \leq 5\omega\left(f, \sqrt{\eta^{\rho,2}_{n,r,s}(x)}\right) + \frac{13}{2}\omega_2\left(f, \sqrt{\eta^{\rho,2}_{n,r,s}(x)}\right).$$

*Proof.* For $x \in J$, and using the Steklov mean $f_h$ given by (3), we may write

$$\left|\mathcal{K}^{\rho}_{n,r,s}(f;x) - f(x)\right| \leq \mathcal{K}^{\rho}_{n,r,s}\left(|f-f_h|;x\right) + \left|\mathcal{K}^{\rho}_{n,r,s}\left(f_h - f_h(x);x\right)\right| + |f_h(x) - f(x)|. \tag{4}$$

From (2), for every $f \in C(J)$ we get

$$\left|\mathcal{K}^{\rho}_{n,r,s}(f;x)\right| \leq \|f\|. \tag{5}$$

Using assumption $(a)$ of Steklov mean and (5), we have

$$\mathcal{K}^{\rho}_{n,r,s}\left(|f-f_h|;x\right) \leq \|\mathcal{K}^{\rho}_{n,r,s}\left(f-f_h\right)\| \leq \|f-f_h\| \leq \omega_2(f,h).$$

By Taylor's expansion and Cauchy-Schwarz property, we obtain

$$\left|\mathcal{K}^{\rho}_{n,r,s}\left(f_h-f_h(x);x\right)\right| \leq \|f_h'\|\sqrt{\mathcal{K}^{\rho}_{n,r,s}\left((t-x)^2;x\right)}+\frac{1}{2}\|f_h''\|\mathcal{K}^{\rho}_{n,r,s}\left((t-x)^2;x\right).$$

From Lemma 2.2 and inequality $(b)$ of Steklov mean, we have

$$\left|\mathcal{K}^{\rho}_{n,r,s}\left(f_h-f_h(x);x\right)\right| \leq \frac{5}{h}\omega(f,h)\sqrt{\eta^{\rho,2}_{n,r,s}(x)}+\frac{9}{2h^2}\omega_2(f,h)\eta^{\rho,2}_{n,r,s}(x).$$

Choosing $h=\sqrt{\eta^{\rho,2}_{n,r,s}(x)}$, and substituting the values of the above estimates in (4), we get the desired result. □

### 3.3 Global approximation

In this subsection, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the $K$-functional [9]. Let $\phi(x)=\sqrt{x(1-x)}$ and $f\in C(J)$. The first order modulus of smoothness is defined by

$$\omega_\phi(f,t)=\sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|, x\pm\frac{h\phi(x)}{2}\in J\right\},$$

and the Petree's $K$-functional is given by

$$\overline{K}_\phi(f,t)=\inf_{g\in W_\phi}\{\|f-g\|+t\|\phi g'\|+t^2\|g'\|\}\ (t>0),$$

where $W_\phi=\{g:\ g\in AC_{loc}, \|\phi g'\|<\infty, \|g'\|<\infty\}$ and $\|\cdot\|$ is the uniform norm on $C(J)$. It is well known [9, Theorem 3.1.2] that $\overline{K}_\phi(f,t)\sim\omega_\phi(f,t)$ which means that there exists a constant $M>0$ such that

$$M^{-1}\omega_\phi(f,t)\leq\overline{K}_\phi(f,t)\leq M\omega_\phi(f,t).$$

Now we present a global approximation theorem for the operators $\mathcal{K}^{\rho}_{n,r,s}$.

Theorem 3.4. *Let $f$ be in $C(J)$ and $\phi(x)=\sqrt{x(1-x)}$. Then for every $x\in[0,1)$, we have*

$$|\mathcal{K}^{\rho}_{n,r,s}(f;x)-f(x)|\leq C\omega_\phi\left(f,\sqrt{\frac{\mathcal{A}^{\rho}_{r,s}}{(1+n\rho)}}\right),$$

*where $\mathcal{A}^{\rho}_{r,s}$ is defined in Lemma 2.3 and $C>0$ is a constant.*

*Proof.* Applying the relation $g(t)=g(x)+\int_x^t g'(u)\,du$, we may write

$$\left|\mathcal{K}^{\rho}_{n,r,s}(g;x)-g(x)\right|=\left|\mathcal{K}^{\rho}_{n,r,s}\left(\int_x^t g'(u)\,du;x\right)\right|. \tag{6}$$

For any $x,t\in(0,1)$, we have

$$\left|\int_x^t g'(u)\,du\right|\leq\|\phi g'\|\left|\int_x^t\frac{1}{\phi(u)}du\right|. \tag{7}$$

Therefore,

$$\begin{aligned}\left|\int_x^t \frac{1}{\phi(u)}\,du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}\,du\right| \leq \left|\int_x^t \left(\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}\right)du\right| \\ &\leq 2\Big(\,|\sqrt{t}-\sqrt{x}\,|+|\sqrt{1-t}-\sqrt{1-x}\,|\,\Big) \\ &= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}}+\frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right) \\ &< 2|t-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right) \leq \frac{2\sqrt{2}\,|t-x|}{\phi(x)}. \end{aligned} \tag{8}$$

Collecting (6)–(8) and using Cauchy-Schwarz property, we may write

$$\begin{aligned}|\mathcal{K}^{\rho}_{n,r,s}(g;x)-g(x)| &< 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)\mathcal{K}^{\rho}_{n,r,s}(|t-x|;x) \\ &\leq 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)\Big(\mathcal{K}^{\rho}_{n,r,s}((t-x)^2;x)\Big)^{1/2}.\end{aligned}$$

Now applying Lemma 2.3, we obtain

$$|\mathcal{K}^{\rho}_{n,r,s}(g;x)-g(x)| < C\sqrt{\frac{\mathcal{A}^{\rho}_{r,s}}{(1+n\rho)}}\|\phi g'\|. \tag{9}$$

Using Lemma 2.1 and (9), we get

$$\begin{aligned}|\,\mathcal{K}^{\rho}_{n,r,s}(f)-f\,| &\leq |\,\mathcal{K}^{\rho}_{n,r,s}(f-g;x)\,|+|f-g|+|\,\mathcal{K}^{\rho}_{n,r,s}(g;x)-g(x)\,| \\ &\leq C\left(\|f-g\|+\sqrt{\frac{\mathcal{A}^{\rho}_{r,s}}{(1+n\rho)}}\|\phi g'\|\right).\end{aligned}$$

Taking infimum on the right-hand side of the above inequality over all $g \in W_\phi$, we have

$$|\mathcal{K}^{\rho}_{n,r,s}(f;x)-f(x)| \leq C\overline{K}_\phi\left(f;\frac{\mathcal{A}^{\rho}_{r,s}}{(1+n\rho)}\right).$$

Using $\overline{K_\phi}(f,t) \sim \omega_\phi(f,t)$, we get the desired relation. □

Let us consider the Lipschitz-type space with two parameters $\kappa_1 \geq 0, \kappa_2 > 0$. We define

$$Lip_M^{(\kappa_1,\kappa_2)}(\sigma) := \left\{f \in C(J) : |f(t)-f(x)| \leq M\frac{|t-x|^\sigma}{(t+\kappa_1 x^2+\kappa_2 x)^{\frac{\sigma}{2}}}; t \in J, x \in (0,1]\right\},$$

where $0 < \sigma \leq 1$.

THEOREM 3.5. *Let $f \in Lip_M^{(\kappa_1,\kappa_2)}(\sigma)$. Then for all $x \in (0,1]$, we have*

$$\left|\mathcal{K}^{\rho}_{n,r,s}(f;x)-f(x)\right| \leq M\left(\frac{\eta^{\rho;2}_{n,r,s}(x)}{\kappa_1 x^2+\kappa_2 x}\right)^{\sigma/2}.$$

*Proof.* Let us prove the theorem for the case $0 < \sigma \leq 1$, applying Holder's inequality with $p=\frac{2}{\sigma}, q=\frac{2}{2-\sigma}$.

$$\begin{aligned}
\left|\mathcal{K}_{n,r,s}^{\rho}(f;x)-f(x)\right| &\leq \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x)\int_0^1 |f(t)-f(x)|\,\Theta_{n,\mu+\nu r}^{\rho}(t)\,dt\\
&\leq \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x)\left(\int_0^1 |f(t)-f(x)|^{\frac{2}{\sigma}}\,\Theta_{n,\mu+\nu r}^{\rho}(t)\,dt\right)^{\frac{\sigma}{2}}\\
&\leq \left\{\sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x)\int_0^1 |f(t)-f(x)|^{\frac{2}{\sigma}}\,\Theta_{n,\mu+\nu r}^{\rho}(t)\,dt\right\}^{\frac{\sigma}{2}}\\
&\times\left(\sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x)\int_0^1 \Theta_{n,\mu+\nu r}^{\rho}(t)\,dt\right)^{\frac{2-\sigma}{2}}\\
&= \left(\sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x)\int_0^1 |f(t)-f(x)|^{\frac{2}{\sigma}}\,\Theta_{n,\mu+\nu r}^{\rho}(t)\,dt\right)^{\frac{\sigma}{2}}\\
&\leq M\left(\sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x)\int_0^1 \frac{(t-x)^2}{(t+\kappa_1x^2+\kappa_2x)}\Theta_{n,\mu+\nu r}^{\rho}(t)\,dt\right)^{\frac{\sigma}{2}}\\
&\leq \frac{M}{(\kappa_1x^2+\kappa_2x)^{\frac{\sigma}{2}}}\left(\sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x)\sum_{\nu=0}^{s} p_{s,\nu}(x)\int_0^1 (t-x)^2\Theta_{n,\mu+\nu r}^{\rho}(t)\,dt\right)^{\frac{\sigma}{2}}\\
&= \frac{M}{(\kappa_1x^2+\kappa_2x)^{\frac{\sigma}{2}}}\mathcal{K}_{n,r,s}^{\rho}((t-x)^2;x)^{\frac{\sigma}{2}}\\
&= \frac{M}{(\kappa_1x^2+\kappa_2x)^{\frac{\sigma}{2}}}(\eta_{n,r,s}^{\rho,2}(x))^{\frac{\sigma}{2}}. \qquad \square
\end{aligned}$$

THEOREM 3.6. *For $f\in C^1(J)$ and $x\in J$, we have*

$$\left|\mathcal{K}_{n,r,s}^{\rho}(f;x)-f(x)\right| \leq \left|\frac{1-2x}{(n\rho+2)}\right|\left|f'(x)\right|+2\sqrt{\eta_{n,r,s}^{\rho,2}(x)}\,\omega\left(f',\sqrt{\eta_{n,r,s}^{\rho,2}(x)}\right).$$

*Proof.* Let $f\in C^1(J)$. For any $t,x\in J$, we have

$$f(t)-f(x)=f'(x)(t-x)+\int_x^t (f'(u)-f'(x))\,du.$$

Using $\mathcal{K}_{n,r,s}^{\rho}(\cdot;x)$ on both sides of the above relation, we have

$$\mathcal{K}_{n,r,s}^{\rho}(f(t)-f(x);q_n,x)=f'(x)\mathcal{K}_{n,r,s}^{\rho}(t-x;x)+\mathcal{K}_{n,r,s}^{\rho}\left(\int_x^t (f'(u)-f'(x))\,du;x\right)$$

Using the well-known inequality for modulus of continuity $|f(t)-f(x)|\leq\omega(f,\delta)\left(\frac{|t-x|}{\delta}+1\right)$, $\delta>0$, we obtain

$$\left|\int_x^t (f'(u)-f'(x))\,du\right|\leq\omega(f',\delta)\left(\frac{(t-x)^2}{\delta}+|t-x|\right).$$

It follows that

$$\left|\mathcal{K}^{\rho}_{n,r,s}(f;x)-f(x)\right| \leq |f'(x)|\ |\mathcal{K}^{\rho}_{n,r,s}(t-x;x)| + \omega(f',\delta)\left\{\frac{1}{\delta}\mathcal{K}^{\rho}_{n,r,s}((t-x)^2;x)+\mathcal{K}^{\rho}_{n,r,s}(|t-x|;x)\right\}.$$

From Cauchy-Schwarz inequality, we have

$$\left|\mathcal{K}^{\rho}_{n,r,s}(f;x)-f(x)\right| \leq |f'(x)|\ |\mathcal{K}^{\rho}_{n,r,s}(t-x;x)| + \omega(f',\delta)\left\{\frac{1}{\delta}\sqrt{\mathcal{K}^{\rho}_{n,r,s}((t-x)^2;x)}+1\right\}\sqrt{\mathcal{K}^{\rho}_{n,r,s}((t-x)^2;x)}.$$

Now, taking $\delta=\sqrt{\eta^{\rho,2}_{n,r,s}(x)}$, the required result follows. □

### 3.4 Rate of convergence

$DBV_{(J)}$ denotes the class of all absolutely continuous functions $f$ defined on $J$, having on $J$ a derivative $f'$ equivalent with a function of bounded variation on $J$. We observe that the functions $f\in DBV_{(J)}$ possess a representation

$$f(x)=\int_0^x g(t)\,dt+f(0),$$

where $g\in BV_{(J)}$, i.e. $g$ is a function of bounded variation on $J$.

The operators $\mathcal{K}^{\rho}_{n,r,s}(f;x)$ also admit the integral representation

$$\mathcal{K}^{\rho}_{n,r,s}(f;x)=\int_0^1 \mathcal{P}^{\rho}_{n,r,s}(x,t)f(t)\,dt, \tag{10}$$

where the kernel $\mathcal{P}^{\rho}_{n,r,s}(x,t)$ is given by

$$\mathcal{P}^{\rho}_{n,r,s}(x,t)=\sum_{\mu=0}^{n-sr}p_{n-sr,\mu}(x)\sum_{\nu=0}^{s}p_{s,\nu}(x)\Theta^{\rho}_{n,\mu+\nu r}(t).$$

Lemma 3.7. *For a fixed $x\in(0,1)$ and sufficiently large $n$, we have*

*(i)* $\beta^{\rho}_{n,r,s}(x,y)=\int_0^y \mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt \leq \frac{\mathcal{A}^{\rho}_{r,s}}{(1+n\rho)}\frac{x(1-x)}{(x-y)^2},\ 0\leq y<x,$

*(ii)* $1-\beta^{\rho}_{n,r,s}(x,z)=\int_z^1 \mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt \leq \frac{\mathcal{A}^{\rho}_{r,s}}{(1+n\rho)}\frac{x(1-x)}{(z-x)^2},\ x<z<1,$

*where $\mathcal{A}^{\rho}_{r,s}$ is defined in Lemma 2.3.*

*Proof.* $(i)$ In view of Lemma 2.3, we obtain

$$\beta_n(x,y)=\int_0^y \mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt \leq \int_0^y\left(\frac{x-t}{x-y}\right)^2\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt = \mathcal{K}^{\rho}_{n,r,s}((t-x)^2;x)(x-y)^{-2}\leq \frac{\mathcal{A}^{\rho}_{r,s}}{(1+n\rho)}\frac{x(1-x)}{(x-y)^2}.$$

The proof of $(ii)$ is similar, hence we omit the details. □

Theorem 3.8. *Let $f \in DBV(J)$. Then for every $x \in (0,1)$ and sufficiently large $n$, we have*

$$\begin{aligned}|\mathcal{K}^{\rho}_{n,r,s}(f;x) - f(x)| \leq &\frac{(1-2x)}{(n\rho+2)}\frac{|f'(x+)+f'(x-)|}{2}\\ &+\sqrt{\frac{\mathcal{A}^{\rho}_{r,s}x(1-x)}{(1+n\rho)}}\frac{|f'(x+)-f'(x-)|}{2}\\ &+\frac{\mathcal{A}^{\rho}_{r,s}(1-x)}{(1+n\rho)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f'_x) + \frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x)\\ &+\frac{\mathcal{A}^{\rho}_{r,s}x}{(1+n\rho)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+((1-x)/k)}(f'_x) + \frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+((1-x)/\sqrt{n})}(f'_x),\end{aligned}$$

*where $\bigvee_c^d(f'_x)$ denotes the total variation of $f'_x$ on $[c,d]$ and $f'_x$ is defined by*

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x\\ 0, & t = x\\ f'(t) - f'(x+) & x < t < 1. \end{cases} \tag{11}$$

*Proof.* Since $\mathcal{K}^{\rho}_{n,r,s}(1;x) = 1$, by using (10), for every $x \in (0,1)$ we have

$$\begin{aligned}\mathcal{K}^{\rho}_{n,r,s}(f;x) - f(x) &= \int_0^1 \mathcal{P}^{\rho}_{n,r,s}(x,t)(f(t)-f(x))\,dt\\ &= \int_0^1 \mathcal{P}^{\rho}_{n,r,s}(x,t)\left(\int_x^t f'(u)du\right)dt. \end{aligned}\tag{12}$$

For any $f \in DBV(J)$, by (11) we may write

$$\begin{aligned}f'(u) =& f'_x(u) + \frac{1}{2}(f'(x+)+f'(x-)) + \frac{1}{2}(f'(x+)-f'(x-))\,\mathrm{sgn}(u-x)\\ &+\delta_x(u)[f'(u) - \frac{1}{2}(f'(x+)+f'(x-))], \end{aligned}\tag{13}$$

where $\delta_x(u) = \begin{cases} 1, & u = x\\ 0 & u \neq x. \end{cases}$

Obviously,

$$\int_0^1\left(\int_x^t\left(f'(u) - \frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)\,du\right)\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt = 0.$$

By (10) and simple calculations we have

$$\begin{aligned}&\int_0^1\left(\int_x^t \frac{1}{2}(f'(x+)+f'(x-))\,du\right)\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt\\ &\quad= \frac{1}{2}(f'(x+)+f'(x-))\int_0^1 (t-x)\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt\\ &\quad= \frac{1}{2}(f'(x+)+f'(x-))\mathcal{K}^{\rho}_{n,r,s}((t-x);x)\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_0^1 \mathcal{P}^{\rho}_{n,r,s}(x,t)\left(\int_x^t \frac{1}{2}(f'(x+)-f'(x-))\operatorname{sgn}(u-x)\,du\right)dt\right|\\
&\quad\le \frac{1}{2}\mid f'(x+)-f'(x-)\mid \int_0^1 |t-x|\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt\\
&\quad\le \frac{1}{2}\mid f'(x+)-f'(x-)\mid \mathcal{K}^{\rho}_{n,r,s}(|t-x|;x)\\
&\quad\le \frac{1}{2}\mid f'(x+)-f'(x-)\mid \left(\mathcal{K}^{\rho}_{n,r,s}((t-x)^2;x)\right)^{1/2}.
\end{aligned}$$

In view of Lemmas 2.2 and 2.3, applying (12)–(13) we find the following estimate

$$\begin{aligned}
|\mathcal{K}^{\rho}_{n,r,s}(f;x)-f(x)| \le &\frac{1}{2}|f'(x+)-f'(x-)|\sqrt{\frac{\mathcal{A}^{\rho}_{r,s}x(1-x)}{(1+n\rho)}}\\
&+\left|\int_0^x\left(\int_x^t f'_x(u)\,du\right)\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt\right.\\
&\left.+\int_x^1\left(\int_x^t f'_x(u)\,du\right)\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt\right|. \qquad (14)
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{G}^{\rho}_{n,r,s}(f'_x,x) &= \int_0^x\left(\int_x^t f'_x(u)\,du\right)\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt,\\
\mathcal{F}^{\rho}_{n,r,s}(f'_x,x) &= \int_x^1\left(\int_x^t f'_x(u)\,du\right)\mathcal{P}^{\rho}_{n,r,s}(x,t)\,dt.
\end{aligned}$$

To complete the proof, it is sufficient to determine the terms $\mathcal{G}^{\rho}_{n,r,s}(f'_x,x)$ and $\mathcal{F}^{\rho}_{n,r,s}(f'_x,x)$. Since $\int_c^d d_t\beta^{\rho}_{n,r,s}(x,t) \le 1$ for all $[c,d]\subseteq J$, applying integration by parts and using Lemma 3.7, with $y = x-(x/\sqrt{n})$, we may write

$$\begin{aligned}
|\mathcal{G}^{\rho}_{n,r,s}(f'_x,x)| &= \left|\int_0^x\left(\int_x^t f'_x(u)\,du\right)d_t\beta^{\rho}_{n,r,s}(x,t)\right| = \left|\int_0^x \beta^{\rho}_{n,r,s}(x,t)f'_x(t)\,dt\right|\\
&\le \left(\int_0^y+\int_y^x\right)|f'_x(t)|\;|\beta^{\rho}_{n,r,s}(x,t)|\,dt\\
&\le \frac{\mathcal{A}^{\rho}_{r,s}x(1-x)}{(1+n\rho)}\int_0^y \bigvee_t^x(f'_x)(x-t)^{-2}\,dt+\int_y^x\bigvee_t^x(f'_x)\,dt\\
&\le \frac{\mathcal{A}^{\rho}_{r,s}x(1-x)}{(1+n\rho)}\int_0^{x-(x/\sqrt{n})}\bigvee_t^x(f'_x)(x-t)^{-2}\,dt+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x).
\end{aligned}$$

By the substitution of $u = x/(x-t)$, we obtain

$$\frac{\mathcal{A}_{r,s}^{\rho} x(1-x)}{(1+n\rho)} \int_0^{x-(x/\sqrt{n})} (x-t)^{-2} \bigvee_t^x (f_x')\, dt = \frac{\mathcal{A}_{r,s}^{\rho}(1-x)}{(1+n\rho)} \int_1^{\sqrt{n}} \bigvee_{x-(x/u)}^{x} (f_x')\, du$$

$$\leq \frac{\mathcal{A}_{r,s}^{\rho}(1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-(x/u)}^{x} (f_x')\, du \leq \frac{\mathcal{A}_{r,s}^{\rho}(1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^{x} (f_x').$$

Thus,

$$|\mathcal{G}_{n,r,s}^{\rho}(f_x', x)| \leq \frac{\mathcal{A}_{r,s}^{\rho}(1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^{x} (f_x') + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^{x} (f_x'). \tag{15}$$

Using integration by parts and applying Lemma 3.7 with $z = x + ((1-x)/\sqrt{n})$, we have

$$\begin{aligned}
|\mathcal{F}_{n,r,s}^{\rho}(f_x', x)| &= \left| \int_x^1 \left( \int_x^t f_x'(u)\, du \right) \mathcal{P}_{n,r,s}^{\rho}(x,t)\, dt \right| \\
&= \left| \int_x^z \left( \int_x^t f_x'(u)\, du \right) d_t(1-\beta_{n,r,s}^{\rho}(x,t)) \right. \\
&\quad \left. + \int_z^1 \left( \int_x^t f_x'(u)\, du \right) d_t(1-\beta_{n,r,s}^{\rho}(x,t)) \right| \\
&= \left| \left[ \int_x^t f_x'(u)(1-\beta_{n,r,s}^{\rho}(x,t))\, du \right]_x^z - \int_x^z f_x'(t)(1-\beta_{n,r,s}^{\rho}(x,t))\, dt \right. \\
&\quad \left. + \int_z^1 \left( \int_x^t f_x'(u)\, du \right) d_t(1-\beta_{n,r,s}^{\rho}(x,t)) \right| \\
&= \left| \int_x^z f_x'(u)\, du(1-\beta_{n,r,s}^{\rho}(x,z)) - \int_x^z f_x'(t)(1-\beta_{n,r,s}^{\rho}(x,t))\, dt \right. \\
&\quad \left. + \left[ \int_x^t f_x'(u)\, du(1-\beta_{n,r,s}^{\rho}(x,t)) \right]_z^1 - \int_z^1 f_x'(t)(1-\beta_{n,r,s}^{\rho}(x,t))\, dt \right| \\
&= \left| \int_x^z f_x'(t)(1-\beta_{n,r,s}^{\rho}(x,t))\, dt + \int_z^1 f_x'(t)(1-\beta_{n,r,s}^{\rho}(x,t))\, dt \right| \\
&\leq \frac{\mathcal{A}_{r,s}^{\rho} x(1-x)}{(1+n\rho)} \int_z^1 \bigvee_x^t (f_x')(t-x)^{-2}\, dt + \int_x^z \bigvee_x^t (f_x')\, dt \\
&= \frac{\mathcal{A}_{r,s}^{\rho} x(1-x)}{(1+n\rho)} \int_{x+((1-x)/\sqrt{n})}^1 \bigvee_x^t (f_x')(t-x)^{-2}\, dt + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f_x').
\end{aligned}$$

By the substitution of $v=(1-x)/(t-x)$, we get

$$\begin{aligned}|\mathcal{F}^{\rho}_{n,r,s}(f'_x,x)| &\leq \frac{\mathcal{A}^{\rho}_{r,s}x(1-x)}{(1+n\rho)}\int_1^{\sqrt{n}}\bigvee_x^{x+((1-x)/v)}(f'_x)(1-x)^{-1}dv+\frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+((1-x)/\sqrt{n})}(f'_x)\\ &\leq \frac{\mathcal{A}^{\rho}_{r,s}x}{(1+n\rho)}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_x^{x+((1-x)/v)}(f'_x)dv+\frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+((1-x)/\sqrt{n})}(f'_x)\\ &= \frac{\mathcal{A}^{\rho}_{r,s}x}{(1+n\rho)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+((1-x)/k)}(f'_x)+\frac{(1-x)}{\sqrt{n}}\bigvee_x^{x+((1-x))/\sqrt{n}}(f'_x). \qquad (16)\end{aligned}$$

Combining the estimates (14)–(16), we get the desired relation. □

## 4. Numerical examples

In this section, we show the comparison of the convergence of the Stancu Durrmeyer type operators $\mathcal{K}^{\rho}_{n,r,s}(f;x)$ and the genuine Bernstein-Durrmeyer operators $\mathcal{N}^{\rho}_n(f;x)$ to the certain function for different values of parameters $r,s$ and $\rho$ using Mathematica algorithms.

Example 4.1. In Figure 1, for $n=20,\ r=2,\ s=3,\ \rho=3$, the comparison of convergence of $\mathcal{K}^{\rho}_{n,r,s}(f;x)$ (orange) and the genuine Bernstein-Durrmeyer [10] (thick) operators to $f(x)=x^2-2x+3$ (blue) is illustrated. It is seen that the genuine Bernstein-Durrmeyer $\mathcal{N}^{\rho}_n(f;x)$ operators give a better convergence to $f(x)$ than $\mathcal{K}^{\rho}_{n,r,s}(f;x)$ for $n=20,\ r=2,\ s=3,\ \rho=3$.

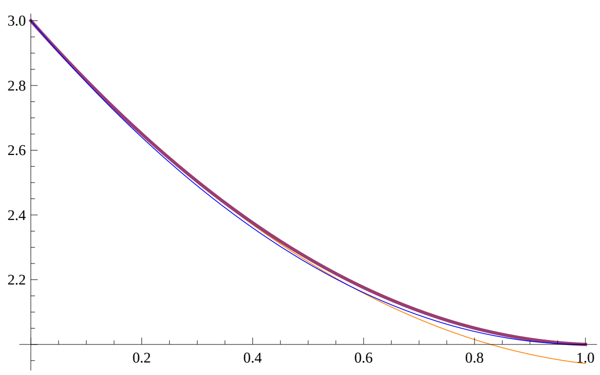


Figure 1: The convergence of $\mathcal{K}^{\rho}_{n,r,s}(f;x)$ and $\mathcal{N}^{\rho}_n(f;x)$ to $f(x)$

Example 4.2. In Figure 2, for $n=60,\ r=2,\ s=6,\ \rho=7$ the comparison of convergence of $\mathcal{K}^{\rho}_{n,r,s}(f;x)$ (orange) and the genuine Bernstein-Durrmeyer [10] (thick) operators to $f(x)=x^2-2x+3$ (blue) is illustrated. It is observed that the genuine

Bernstein-Durrmeyer $\mathcal{N}_n^\rho(f;x)$ operators give a better approximation to $f(x)$ than $\mathcal{K}_{n,r,s}^\rho(f;x)$ for $n = 60$, $r = 2$, $s = 6$, $\rho = 7$.

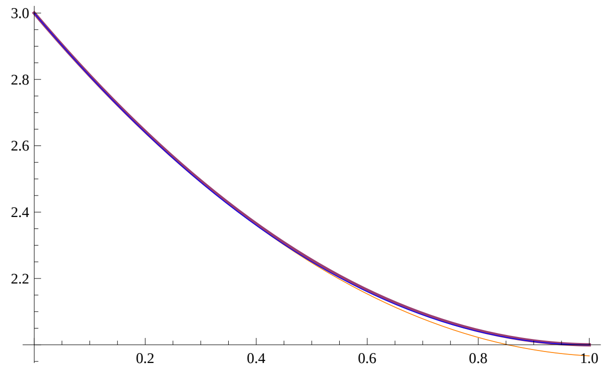


Figure 2: The convergence of $\mathcal{K}_{n,r,s}^\rho(f;x)$ and $\mathcal{N}_n^\rho(f;x)$ to $f(x)$

Example 4.3. In Figure 3, for $n = 100$, $r = 1$, $s = 5$, $\rho = 5$ the comparison of convergence of $\mathcal{K}_{n,r,s}^\rho(f;x)$ (orange) and the genuine Bernstein-Durrmeyer [10] (thick) operators to $f(x) = x^2 - 2x + 3$ (blue) is illustrated. It is seen that both the operators $\mathcal{N}_n^\rho(f;x)$ and $\mathcal{K}_{n,r,s}^\rho(f;x)$ give a good convergence to $f(x)$ for $n = 100$, $r = 1$, $s = 5$, $\rho = 5$.

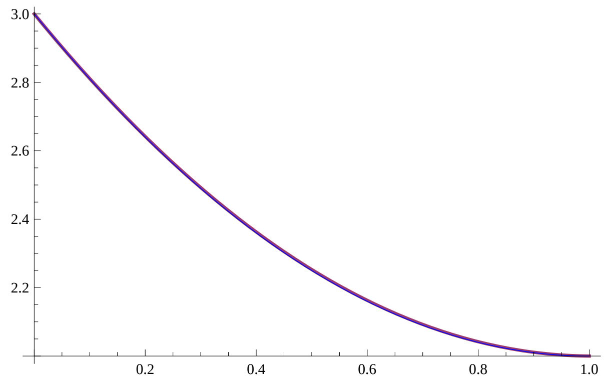


Figure 3: The convergence of $\mathcal{K}_{n,r,s}(f;x)$ and $\mathcal{N}_n^\rho(f;x)$ to $f(x)$

Remark 4.4. From the above examples, we conclude that the operators $\mathcal{K}_{n,r,s}^\rho(f;x)$ converge to $f(x)$ for large $n$.

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