MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
70, 1 (2018), 55–63
March 2018

research paper
оригинални научни рад

NEW CONGRUENCES FOR OVERCUBIC PARTITION FUNCTION

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Abstract. In 2010, Byungchan Kim introduced a new class of partition function $\overline{a}(n)$, the number of overcubic partitions of n and established $\overline{a}(3n+2) \equiv 0 \pmod 3$. Our goal is to consider this function from an arithmetic point of view in the spirit of Ramanujan's congruences for the unrestricted partition function $p(n)$. We prove a number of results for $\overline{a}(n)$, for example, for $\alpha \geq 0$ and $n \geq 0$, $\overline{a}(8n+5) \equiv 0 \pmod{16}$, $\overline{a}(8n+7) \equiv 0 \pmod{32}$, $\overline{a}(8 \cdot 3^{2\alpha+2} n + 3^{2\alpha+2}) \equiv 3^{\alpha} \overline{a}(8n+1) \pmod 8$.

1. Introduction

In 2010, Byungchan Kim [6] introduced the overcubic partitions of the number n, partitions of n in which odd parts come in two colours, one of which can occur at most once and in which the even parts come in four colours, two of which can occur at most once each. Let the number of overcubic partitions of n be $\overline{a}(n)$. For example, there are 6 such partitions of 2: 2_1, 2_2, 2_3, 2_4, $1_1 + 1_1$, $1_1 + 1_2$.

The generating function for overcubic partition function $\overline{a}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{a}(n) q^n = \frac{(-q;q)_\infty (-q^2;q^2)_\infty}{(q;q)_\infty (q^2;q^2)_\infty} = \frac{f_4}{f_1^2 f_2}. \tag{1}$$

Here and throughout this paper $f_k := \prod_{i=1}^{\infty} (1 - q^{ki}) = (q^k;q^k)_\infty$, for any positive k.

Using theory of modular forms, Kim proved

$$\sum_{n=0}^{\infty} \overline{a}(3n+2) q^n = 6 \frac{(q^3;q^3)_\infty^6 (q^4;q^4)_\infty^3}{(q;q)_\infty^8 (q^2;q^2)_\infty^3}.$$

This implies $\overline{a}(3n+2) \equiv 0 \pmod 6$. Using elementary generating function methods, Hirschhorn [5] proved Kim's generating function result. Clearly, this generating function result implies that $\overline{a}(3n+2) \equiv 0 \pmod 6$ for all $n \geq 0$.

Recently, Sellers [10] proved number of arithmetic properties satisfied by $\overline{a}(n)$ by employing elementary generating function methods. For example, he showed that

2010 Mathematics Subject Classification: 11P83, 05A15, 05A17
Keywords and phrases: Overcubic partitions; congruences; theta function.

for $n > 1$, $\overline{a}(n) \equiv 0 \pmod 2$ and for $n \geq 0$, $j \geq 0$, $\overline{a}(2^j(8n+5)) \equiv 0 \pmod 8$ and $\overline{a}(2^j(8n+7)) \equiv 0 \pmod 8$. For more details on cubic and overcubic partition functions, one can see [2,3,7,8].

Motivated by works of the above authors, our aim in this paper is to establish some new congruences modulo powers of 2 for overcubic partition function $\overline{a}(n)$ by using some elementary generating function and dissection formulas due to some authors. Our main results can be stated as follows.

Theorem 1.1. *For all $n \geq 0$,*

$$\overline{a}(8n+5) \equiv 0 \pmod{16}, \tag{2}$$

$$\overline{a}(8n+7) \equiv 0 \pmod{32}. \tag{3}$$

Theorem 1.2. *For any prime $p \geq 5$, $(\frac{-2}{p}) = -1$, $\alpha \geq 0$, $i = 1, 2, ..., p-1$ and $n \geq 0$,*

$$\overline{a}(8p^{2\alpha+2}n + 8p^{2\alpha+1}i + 4p^{2\alpha+2}) \equiv 0 \pmod 8, \tag{4}$$

$$\overline{a}(8p^{2\alpha+2}n + 8p^{2\alpha+1}i + p^{2\alpha+2}) \equiv 0 \pmod{16}, \tag{5}$$

$$\overline{a}(4p^{2\alpha+2}n + 4p^{2\alpha+1}i + 2p^{2\alpha+2}) \equiv 0 \pmod{16}, \tag{6}$$

$$\overline{a}(8p^{2\alpha+2}n + 8p^{2\alpha+1}i + 3p^{2\alpha+2}) \equiv 0 \pmod{16}. \tag{7}$$

Theorem 1.3. *For any $\alpha \geq 0$, $n \geq 0$,*

$$\overline{a}(8 \cdot 3^{2\alpha+2}n + 3^{2\alpha+2}) \equiv 3^{\alpha}\overline{a}(8n+1) \pmod 8, \tag{8}$$

$$\overline{a}(8 \cdot 3^{2\alpha+2}n + 33 \cdot 3^{2\alpha}) \equiv 0 \pmod 8, \tag{9}$$

$$\overline{a}(8 \cdot 3^{2\alpha+2}n + 57 \cdot 3^{2\alpha}) \equiv 0 \pmod 8. \tag{10}$$

2. Preliminary results

To prove our main results, we need the following dissection formulas.

We define Ramanujan's general theta function $f(a,b)$ for $|ab| < 1$ as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

The special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{11}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{12}$$

and $$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1,$$

where the product representations arise from Jacobi's triple product identity [1, p. 35, Entry 19] $f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$.

Lemma 2.1. *The following 2-dissections hold*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{13}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{14}$$

Lemma 2.1 is a consequence of dissection formulas of Ramanujan, collected in Berndt's book [1].

Lemma 2.2. *[1, Entry 25(i) and (ii), p. 40]*

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \tag{15}$$

Lemma 2.3. *[1, p. 49]) For any prime p,*

$$\varphi(q) = \varphi(q^{p^2}) + \sum_{r=0}^{p-1} q^{r^2} f\left(q^{p(p-2r)}, q^{p(p+2r)}\right). \tag{16}$$

Lemma 2.4. *[4, Theorem 2.1.] For any odd prime p,*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{17}$$

Furthermore, $\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod p$ *for* $0 \le k \le \frac{p-3}{2}$.

Lemma 2.5. *[9, Theorem 1.] If* $\sum_{n=0}^{\infty} b_1(n)q^n = \varphi(q)\psi(q)$ *then*

$$\sum_{n=0}^{\infty} b_1(3n+1)q^n = 4\psi(q)\psi(q^2) - \psi(q^3)\varphi(q^3). \tag{18}$$

3. Proof of main results

Lemma 3.1. *We have*

$$\sum_{n=0}^{\infty} \overline{a}(4n)q^n = \frac{f_2^{19}}{f_1^{18} f_4 f_8^2} + 8q\frac{f_2^9 f_4 f_8^2}{f_1^{14}}, \tag{19}$$

$$\sum_{n=0}^{\infty} \overline{a}(4n+2)q^n = 2\frac{f_2^{21} f_8^2}{f_1^{18} f_4^7} + 4\frac{f_2^7 f_4^7}{f_1^{14} f_8^2}, \tag{20}$$

$$\sum_{n=0}^{\infty} \overline{a}(4n+1)q^n = 2\frac{f_2^{13} f_4^3}{f_1^{16} f_8^2} + 16q\frac{f_2^3 f_4^5 f_8^2}{f_1^{12}}, \tag{21}$$

$$\sum_{n=0}^{\infty} \overline{a}(4n+3)q^n = 4\frac{f_2^{15} f_8^2}{f_1^{16} f_4^3} + 8\frac{f_2 f_4^{11}}{f_1^{12} f_8^2}. \tag{22}$$

Proof. Substituting (13) into (1),

$$\sum_{n=0}^{\infty}\overline{a}(n)q^n=\frac{f_4}{f_2}\frac{1}{f_1^2}=\frac{f_4f_8^5}{f_2^6f_{16}^2}+2q\frac{f_4^3f_{16}^2}{f_2^6f_8},$$

which yields

$$\sum_{n=0}^{\infty}\overline{a}(2n)q^n=\frac{f_2f_4^5}{f_1^6f_8^2} \tag{23}$$

and

$$\sum_{n=0}^{\infty}\overline{a}(2n+1)q^n=2\frac{f_2^3f_8^2}{f_1^6f_4}. \tag{24}$$

Combining (13) and (14), we see that

$$\frac{1}{f_1^2}\frac{1}{f_1^4}=\frac{1}{f_1^6}=\frac{f_4^{14}f_8}{f_2^{19}f_{16}^2}+4q\frac{f_4^2f_8^9}{f_2^{15}f_{16}^2}+2q\frac{f_4^{16}f_{16}^2}{f_2^{19}f_8^5}+8q^2\frac{f_4^4f_8^3f_{16}^2}{f_2^{15}}. \tag{25}$$

Substituting (25) into (23) and (24), we find that

$$\sum_{n=0}^{\infty}\overline{a}(2n)q^n=\frac{f_4^{19}}{f_2^{18}f_8f_{16}^2}+4q\frac{f_4^7f_8^7}{f_2^{14}f_{16}^2}+2q\frac{f_4^{21}f_{16}^2}{f_2^{18}f_8^7}+8q^2\frac{f_4^9f_8f_{16}^2}{f_2^{14}} \tag{26}$$

and

$$\sum_{n=0}^{\infty}\overline{a}(2n+1)q^n=2\frac{f_4^{13}f_8^3}{f_2^{16}f_{16}^2}+8q\frac{f_4f_8^{11}}{f_2^{12}f_{16}^2}+4q\frac{f_4^{15}f_{16}^2}{f_2^{16}f_8^3}+16q^2\frac{f_4^3f_8^5f_{16}^2}{f_2^{12}}. \tag{27}$$

Lemma 3.1 follows from (26) and (27). This completes the proof. □

THEOREM 3.2. *For $\alpha\geq 0$, $n\geq 0$, we have*

$$\overline{a}(2^{4+\alpha}n)\equiv\overline{a}(2^3n)\pmod 8, \tag{28}$$

$$\sum_{n=0}^{\infty}\overline{a}(8n+4)q^n\equiv 2\varphi(q)\psi(q^4)\pmod 8. \tag{29}$$

Proof. Following (19), we have

$$\sum_{n=0}^{\infty}\overline{a}(4n)q^n\equiv\frac{f_2^{19}}{f_1^{18}f_4f_8^2}\pmod 8. \tag{30}$$

By binomial theorem, it is easy to see that

$$f_{2m}\equiv f_m^2\pmod 2,\quad f_{2m}^2\equiv f_m^4\pmod{2^2},\quad f_{2m}^4\equiv f_m^8\pmod{2^3}. \tag{31}$$

Utilizing (31) in (30), we deduce that

$$\sum_{n=0}^{\infty}\overline{a}(4n)q^n\equiv\frac{f_2^{19}}{f_1^{18}f_4f_8^2}\equiv\frac{f_2^{11}}{f_1^2f_4f_8^2}\pmod 8. \tag{32}$$

But

$$\frac{f_2^{11}}{f_1^2f_4f_8^2}\equiv\frac{f_2^5}{f_1^2f_4^2}\frac{f_4^5}{f_2^2f_8^2}\equiv\varphi(q)\varphi(q^2)\pmod 8. \tag{33}$$

In view of (32) and (33), we have

$$\sum_{n=0}^{\infty}\overline{a}(4n)q^n\equiv\varphi(q)\varphi(q^2)\pmod 8. \tag{34}$$

Employing (15) into the above equation, we see that

$$\sum_{n=0}^{\infty} \overline{a}(4n)q^n \equiv \varphi(q^2)\varphi(q^4) + 2q\varphi(q^2)\psi(q^8) \pmod{8}, \tag{35}$$

which implies

$$\sum_{n=0}^{\infty} \overline{a}(8n)q^n \equiv \varphi(q)\varphi(q^2) \pmod{8}. \tag{36}$$

Congruence (28) follows from (34), (36) and by induction on α. Equating odd powers of q from both sides of (35), we arrive at (29). □

Proof (of Theorem 1.1). In the view of (21), we deduce that

$$\sum_{n=0}^{\infty} \overline{a}(4n+1)q^n \equiv 2\frac{f_2^{13}f_4^3}{f_1^{16}f_8^2} \equiv 2\frac{f_2^5f_4^3}{f_8^2} \pmod{16}. \tag{37}$$

Congruence (2) follows from (37). Extracting the terms involving q^{2n} from both sides of (37), we obtain

$$\sum_{n=0}^{\infty} \overline{a}(8n+1)q^n \equiv 2\frac{f_1^5f_2^3}{f_4^2} \pmod{16}. \tag{38}$$

But

$$\frac{f_1^5f_2^3}{f_4^2} \equiv \frac{f_2^5}{f_1^2f_4^2}\frac{f_2^2}{f_1} \equiv \varphi(q)\psi(q) \pmod{8}. \tag{39}$$

Combining (38) and (39), we see that

$$\sum_{n=0}^{\infty} \overline{a}(8n+1)q^n \equiv 2\varphi(q)\psi(q) \pmod{16}. \tag{40}$$

Applying (31) into (22), we obtain

$$\sum_{n=0}^{\infty} \overline{a}(4n+3)q^n \equiv 4\frac{f_2^7f_8^2}{f_4^3} + 8\frac{f_4^7}{f_2^5} \pmod{32}. \tag{41}$$

Congruence (3) follows from (41). Again, equating even powers of q from both sides of (41), we get

$$\sum_{n=0}^{\infty} \overline{a}(8n+3)q^n \equiv 4\frac{f_1^7f_4^2}{f_2^3} + 8\frac{f_2^7}{f_1^5} \pmod{32}. \tag{42}$$

But

$$\frac{f_1^7f_4^2}{f_2^3} \equiv \frac{f_2^2}{f_1}\frac{f_4^2}{f_2} \equiv \psi(q)\psi(q^2) \pmod{4} \tag{43}$$

and

$$\frac{f_2^7}{f_1^5} \equiv \frac{f_2^2}{f_1}\frac{f_4^2}{f_2} \equiv \psi(q)\psi(q^2) \pmod{2}. \tag{44}$$

In view of (42), (43) and (44), we deduce that

$$\sum_{n=0}^{\infty} \overline{a}(8n+3)q^n \equiv 12\psi(q)\psi(q^2) \pmod{16}. \tag{45}$$

□

Proof (of Theorem 1.2). Define

$$\sum_{n=0}^{\infty} a(n)q^n = \varphi(q)\psi(q^4). \tag{46}$$

In view of (29) and (46), for $n \geq 0$,

$$\overline{a}(8n+4) \equiv 2a(n) \pmod{8}. \tag{47}$$

Now, we consider the congruence equation

$$r^2 + 4 \cdot \frac{k^2+k}{2} \equiv \frac{4p^2-4}{8} \pmod{p}, \tag{48}$$

which is equivalent to $(2r)^2 + 2 \cdot (2k+1)^2 \equiv 0 \pmod{p}$, where $0 \leq k, r \leq p-1$ and p is a prime such that $(\frac{-2}{p}) = -1$. Since $(\frac{-2}{p}) = -1$ for $p \equiv 5$ or $7 \pmod{8}$, the congruence relation (48) holds if and only if $r = 0$ and $k = \frac{p-1}{2}$. Therefore, if we substitute (16) and (17) into (47) and then extracting the terms in which the powers of q are $pn + \frac{p^2-1}{2}$, we arrive at

$$\sum_{n=0}^{\infty} a\left(pn + \frac{p^2-1}{2}\right) q^{pn+\frac{p^2-1}{2}} = q^{\frac{p^2-1}{2}} \varphi(q^{p^2})\psi(q^{4p^2}). \tag{49}$$

Dividing by $q^{\frac{p^2-1}{2}}$ both sides of (49) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} a\left(pn + \frac{p^2-1}{2}\right) q^n = \varphi(q^p)\psi(q^{4p}),$$

implying

$$\sum_{n=0}^{\infty} a\left(p^2n + \frac{p^2-1}{2}\right) q^n = \varphi(q)\psi(q^4) \tag{50}$$

and for $n \geq 0$,

$$a\left(p^2n + pi + \frac{p^2-1}{2}\right) = 0, \tag{51}$$

where i is an integer and $1 \leq i \leq p-1$. Combining (46) and (50), for $n \geq 0$, we have $a\left(p^2n + \frac{p^2-1}{2}\right) = a(n)$. From here, by mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$ $a\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{2}\right) = a(n)$. Replacing n by $p^2n + pi + \frac{p^2-1}{2}$ and using (51), we deduce that for $n \geq 0$, $\alpha \geq 0$, $a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{2}\right) = 0$. Congruence (4) follows from here and from (47).

Now, we prove (5). Define

$$\sum_{n=0}^{\infty} b(n)q^n = \varphi(q)\psi(q). \tag{52}$$

Combining (40) and (52), for $n \geq 0$,

$$\overline{a}(8n+1) \equiv 2b(n) \pmod{16}. \tag{53}$$

Now, we consider the congruence equation

$$r^2 + \frac{k^2+k}{2} \equiv \frac{p^2-1}{8} \pmod{p}, \tag{54}$$

which is equivalent to $2\cdot(2r)^2+(2k+1)^2 \equiv 0 \pmod{p}$, where $0 \leq k, r \leq p-1$ and p is a prime such that $(\frac{-2}{p}) = -1$. Since $(\frac{-2}{p}) = -1$ for $p \equiv 5\, or\, 7 \pmod{8}$, the congruence relation (54) holds if and only if $r = 0$ and $k = \frac{p-1}{2}$. Therefore, if we substitute (16) and (17) into (52) and then extracting the terms in which the powers of q are $pn + \frac{p^2-1}{8}$, we arrive at

$$\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{8}\right) q^{pn+\frac{p^2-1}{8}} = q^{\frac{p^2-1}{8}} \varphi(q^{p^2})\psi(q^{p^2}). \tag{55}$$

Dividing by $q^{\frac{p^2-1}{8}}$ both sides of (55) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{8}\right) q^{n} = \varphi(q^{p})\psi(q^{p}),$$

which implies that

$$\sum_{n=0}^{\infty} b\left(p^2n + \frac{p^2-1}{8}\right) q^{n} = \varphi(q)\psi(q) \tag{56}$$

and for $n \geq 0$,

$$b\left(p^2n + pi + \frac{p^2-1}{8}\right) = 0, \tag{57}$$

where i is an integer and $1 \leq i \leq p-1$. Combining (52) and (56), for $n \geq 0$, we have $b\left(p^2n + \frac{p^2-1}{8}\right) = b(n)$. From here, by mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$, $b\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{8}\right) = b(n)$. Replacing n by $p^2n+pi+\frac{p^2-1}{8}$ and using (57), we deduce that for $n \geq 0$, $\alpha \geq 0$,

$$b\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{8}\right) = 0. \tag{58}$$

Congruence (5) follows from (53) and (58).

Next, we prove (6). By (11), (12) and (31), it is easy to see that

$$2\frac{f_2^{21}f_8^2}{f_1^{18}f_4^7} \equiv 2\varphi(q)\psi(q^4) \pmod{16} \tag{59}$$

and

$$4\frac{f_2^{7}f_4^7}{f_1^{14}f_8^2} \equiv 4\varphi(q)\psi(q^4) \pmod{16}. \tag{60}$$

Let $c(n)$ be defined by

$$\sum_{n=0}^{\infty} c(n)q^n = \varphi(q)\psi(q^4). \tag{61}$$

Combining (20), (59), (60) and (61), for $n \geq 0$, $\overline{a}(4n+2) \equiv 6c(n) \pmod{16}$ holds.

The remaining part of the proof is exactly similar to the proof of congruence (4), hence we omit the details.

Finally, to conclude this section, we give a proof of (7). Define

$$\sum_{n=0}^{\infty} d(n)q^n = \psi(q)\psi(q^2). \tag{62}$$

Combining (45) and (62), for $n \geq 0$,

$$\overline{a}(8n+3) \equiv 12d(n) \pmod{16}. \tag{63}$$

Now, we consider the congruence equation

$$\frac{k^2+k}{2} + 2 \cdot \frac{m^2+m}{2} \equiv \frac{3p^2-3}{8} \pmod{p}, \tag{64}$$

which is equivalent to $2 \cdot (2k+1)^2 + (2m+1)^2 \equiv 0 \pmod{p}$, where $0 \leq k, m \leq p-1$ and p is a prime such that $(\frac{-2}{p}) = -1$. Since $(\frac{-2}{p}) = -1$ for $p \equiv 5\,or\,7 \pmod{8}$, the congruence relation (64) holds if and only if both $k = m = \frac{p-1}{2}$. Therefore, if we substitute (17) into (62) and then extract the terms in which the powers of q are $pn + \frac{3p^2-3}{8}$, we arrive at

$$\sum_{n=0}^{\infty} d\left(pn + \frac{3p^2-3}{8}\right) q^{pn+\frac{3p^2-3}{8}} = q^{\frac{3p^2-3}{8}} \psi(q^{p^2})\psi(q^{2p^2}). \tag{65}$$

Dividing by $q^{\frac{3p^2-3}{8}}$ both sides of (65) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} d\left(pn + \frac{3p^2-3}{8}\right) q^n = \psi(q^p)\psi(q^{2p}),$$

which implies

$$\sum_{n=0}^{\infty} d\left(p^2n + \frac{3p^2-3}{8}\right) q^n = \psi(q)\psi(q^2) \tag{66}$$

and for $n \geq 0$,

$$d\left(p^2n + pi + \frac{3p^2-3}{8}\right) = 0, \tag{67}$$

where i is an integer and $1 \leq i \leq p-1$. Combining (62) and (66), for $n \geq 0$, we get

$$d\left(p^2n + \frac{3p^2-3}{8}\right) = d(n). \tag{68}$$

By (68) and mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$d\left(p^{2\alpha}n + \frac{3p^{2\alpha}-3}{8}\right) = d(n). \tag{69}$$

Replacing n by $p^2n+pi+\frac{3p^2-3}{8}$ in (69) and using (67), we deduce that for $n \geq 0$, $\alpha \geq 0$,

$$d\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{3p^{2\alpha+2}-3}{8}\right) = 0. \tag{70}$$

Congruence (7) follows from (63) and (70). This completes the proof. □

Proof (of Theorem 1.3). If $\sum\limits_{n=0}^{\infty} b_1(n)q^n = \varphi(q)\psi(q)$, then (40) can be expressed as

$$\sum_{n=0}^{\infty} \overline{a}(8n+1)q^n \equiv 2\sum_{n=0}^{\infty} b_1(n)q^n \pmod{16}, \tag{71}$$

which yields

$$\sum_{n=0}^{\infty} \overline{a}(24n+9)q^n \equiv 2\sum_{n=0}^{\infty} b_1(3n+1)q^n \pmod{16}. \tag{72}$$

Invoking (18) and (72), we find that

$$\sum_{n=0}^{\infty} \overline{a}(24n+9)q^n \equiv 8\psi(q)\psi(q^2) - 2\psi(q^3)\varphi(q^3) \pmod{16}, \tag{73}$$

which implies

$$\sum_{n=0}^{\infty} \overline{a}(24n+9)q^n \equiv 6\psi(q^3)\varphi(q^3) \pmod{8}, \tag{74}$$

which yields

$$\overline{a}(72n+9) \equiv 3\overline{a}(8n+1) \pmod{8}, \tag{75}$$

$$\overline{a}(72n+33) \equiv 0 \pmod{8} \tag{76}$$

and

$$\overline{a}(72n+57) \equiv 0 \pmod{8}. \tag{77}$$

From (75) and by induction on α, we arrive at (8). Substituting (76) and (77) into (8), we arrive at (9) and (10). □

Acknowledgement. The authors would like to thank anonymous referees for their helpful suggestions and comments which improved the original version of the manuscript.

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(received 23.03.2017; in revised form 12.07.2017; available online 03.08.2017)

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