MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
70, 2 (2018), 97–109
June 2018

research paper
оригинални научни рад

STRONGLY I AND I^*-STATISTICALLY PRE-CAUCHY DOUBLE SEQUENCES IN PROBABILISTIC METRIC SPACES

Prasanta Malik, Argha Ghosh and Manojit Maity

Abstract. In this paper we consider the notion of strongly I-statistically pre-Cauchy double sequences in probabilistic metric spaces in line of Das et. al. (*On I-statistically pre-Cauchy sequences*, Taiwanese J. Math 18 (1) (2014), 115–126) and introduce the new concept of strongly I^*-statistically pre-Cauchy double sequences in probabilistic metric spaces. We mainly study interrelationship among strong I-statistical convergence, strong I-statistical pre-Cauchy condition and strong I^*-statistical pre-Cauchy condition for double sequences in probabilistic metric spaces and examine some basic properties of these notions.

1. Introduction

The notion of probabilistic metric (PM) was introduced by K. Menger [12] under the name of "statistical metric spaces" by considering the distance between two points x and y as a distribution function F_{xy} instead of a non-negative real number and the value of the function F_{xy} at any $t > 0$, i.e. $F_{xy}(t)$ is interpreted as the probability that the distance between the points x and y is $\leq t$. After Menger, the theory of probabilistic metric was developed by Schwiezer and Sklar [19–22], Tardiff [26], Thorp [27] and many others. A thorough development of probabilistic metric spaces can be seen from the famous book of Schwiezer and Sklar [23]. Many different topologies may be defined on a PM space, but strong topology is one of them, which has received most attention and it is the main tool of our paper.

The idea of usual notion of convergence of real sequences was extended to statistical convergence by Fast [10] and Schoenberg [18] independently. For the last few years a lot of work has been done on this convergence (see [2, 6, 10, 11, 25]). In [2], the notion of statistically pre-Cauchy sequences of real numbers was introduced and it was shown that statistically convergent sequences are always statistically pre-Cauchy and

2010 Mathematics Subject Classification: 54E70, 40B05
Keywords and phrases: Probabilistic metric space; strong I-statistical convergence; strong I-statistical pre-Cauchy condition; strong I^*-statistical pre-Cauchy condition.

the converse statement holds under certain conditions. The notion of statistical convergence was further extended to I-convergence in [15] using the ideals of $\mathbb{N}$ and also to I-statistical convergence in [6]. The notion of strong I-statistical convergence for double sequences of real numbers had been introduced by Belen et. al. in [1]. Recently, in [7], Das and Savas have introduced the notion of I-statistically pre-Cauchy sequences of real numbers as a generalization of I-statistical convergence. They proved that every I-statistically convergent sequence is I-statistically pre-Cauchy and the converse is true under certain sufficient conditions.

Following the line of Das and Savas [7], we introduce in this paper the notion of strongly I-statistically pre-Cauchy double sequences in probabilistic metric spaces. We also introduce the new notion of strongly I^*-statistically pre-Cauchy double sequences in probabilistic metric spaces. We show that every strongly I-statistically convergent double sequence is strongly I-statistically pre-Cauchy and every strongly I^*-statistically pre-Cauchy double sequence is strongly I-statistically pre-Cauchy and the converse of each of the results holds under certain conditions.

2. Basic definitions and notations

First we recall some basic concepts related to the probabilistic metric spaces (or PM spaces) (see [19–23] for more details).

Definition 2.1. A non decreasing function $F : \mathbb{R} \to [0,1]$ with $F(-\infty) = 0$ and $F(\infty) = 1$, where $\mathbb{R} = [-\infty, \infty]$, is called a distribution function.

We denote the set of all left continuous distribution function over $(-\infty, \infty)$ by Δ. We consider the relation $\leq$ on Δ defined by $G \leq F$ if and only if $G(x) \leq F(x)$ for all $x \in \mathbb{R}$. Clearly the relation '$\leq$' is a partial order on Δ.

Definition 2.2. For any $p \in [-\infty, \infty]$ the unit step at p is denoted by ϵ_p and is defined to be a function in Δ given by

$$\epsilon_p(x) = \begin{cases} 0, & -\infty \leq x \leq p \\ 1, & p < x \leq \infty. \end{cases}$$

Definition 2.3. A sequence $\{F_n\}_{n\in\mathbb{N}}$ of distribution functions is said to converge weakly to a distribution function F, if the sequence $\{F_n(x)\}_{n\in\mathbb{N}}$ converges to $F(x)$ at each continuity point x of F. We write $F_n \xrightarrow{w} F$

Definition 2.4. The distance between F and G in Δ is denoted by $d_L(F, G)$ and is defined by the infimum of all numbers $s \in (0, 1]$ such that the inequalities

$$F(x-s) - s \leq G(x) \leq F(x+s) + s \quad \text{and} \quad G(x-s) - s \leq F(x) \leq G(x+s) + s$$

hold for every $x \in (-\frac{1}{s}, \frac{1}{s})$.

It is known that d_L is a metric on Δ and for any sequence $\{F_n\}_{n\in\mathbb{N}}$ in Δ and $F \in \Delta$, we have $F_n \xrightarrow{w} F$ if and only if $d_L(F_n, F) \to 0$. Here we will be interested in the subset of Δ consisting of those elements G for which $G(0) = 0$.

Definition 2.5. A non-decreasing function G defined on $\mathbb{R}^+ = [0, \infty]$ that satisfies $G(0) = 0$ and $G(\infty) = 1$ and is left continuous on $(0, \infty)$ is called a distance distribution function.

The set of all distance distribution functions is denoted by Δ^+. The function d_L is clearly a metric on Δ^+. The metric space (Δ^+, d_L) is compact and hence complete.

Theorem 2.6. *Let $F \in \Delta^+$ be given. Then for any $t > 0$, $F(t) > 1 - t$ if and only if $d_L(F, \epsilon_0) < t$.*

Definition 2.7. A triangle function is a binary operation τ on Δ^+ which is commutative, nondecreasing, associative in each place, and ϵ_0 is the identity.

Definition 2.8. A probabilistic metric space, briefly PM space, is a triplet $(S, \mathfrak{F}, \tau)$ where S is a nonempty set whose elements are the points of the space; $\mathfrak{F}$ is a function from $S \times S$ into Δ^+, τ is a triangle function, and the following conditions are satisfied for all $x, y, z \in S$:

1. $\mathfrak{F}(x, x) = \epsilon_0$
2. $\mathfrak{F}(x, y) \neq \epsilon_0$ if $x \neq y$
3. $\mathfrak{F}(x, y) = \mathfrak{F}(y, x)$
4. $\mathfrak{F}(x, z) \geq \tau(\mathfrak{F}(x, y), \mathfrak{F}(y, z))$.

From now on we will denote $\mathfrak{F}(x, y)$ by F_{xy} and its value at a by $F_{xy}(a)$.

Definition 2.9. Let $(S, \mathfrak{F}, \tau)$ be a PM space. For $x \in S$ and $t > 0$, the strong t-neighborhood of x is denoted by $\mathcal{N}_x(t)$ and is defined by $\mathcal{N}_x(t) = \{y \in S : F_{xy}(t) > 1 - t\}$. The collection $\mathfrak{N}_x = \{\mathcal{N}_x(t) : t > 0\}$ is called the strong neighborhood system at x, and the union $\mathfrak{N} = \bigcup_{x \in S} \mathfrak{N}_x$ is called the strong neighborhood system for S.

From Theorem 2.6, we can write $\mathcal{N}_x(t) = \{y \in S : d_L(F_{xy}, \epsilon_0) < t\}$. If τ is continuous, then the strong neighborhood system $\mathfrak{N}$ determines a Hausdorff topology for S. This topology is called the strong topology for S.

Definition 2.10. Let $(S, \mathfrak{F}, \tau)$ be a PM space. Then for any $t > 0$, the subset $\mathfrak{U}(t)$ of $S \times S$ given by $\mathfrak{U}(t) = \{(x, y) : F_{xy}(t) > 1 - t\}$ is called the strong t-vicinity.

Theorem 2.11. *Let $(S, \mathfrak{F}, \tau)$ be a PM space and τ be continuous. Then for any $t > 0$, there is an $\eta > 0$ such that $\mathfrak{U}(\eta) \circ \mathfrak{U}(\eta) \subset \mathfrak{U}(t)$, where $\mathfrak{U}(\eta) \circ \mathfrak{U}(\eta) = \{(x, z) :$ for some $y, (x, y)$ and $(y, z) \in \mathfrak{U}(t)\}$.*

Remark 2.12. From the hypothesis of Theorem 2.11 we can say that for any $t > 0$, there is an $\eta > 0$ such that $F_{ab}(t) > 1 - t$ whenever $F_{ac}(\eta) > 1 - \eta$ and $F_{cb}(\eta) > 1 - \eta$. Equivalently it can be written as: for any $t > 0$, there is an $\eta > 0$ such that $d_L(F_{ab}, \epsilon_0) < t$ whenever $d_L(F_{ac}, \epsilon_0) < \eta$ and $d_L(F_{cb}, \epsilon_0) < \eta$.

Throughout the rest of the paper, in a PM space $(S, \mathfrak{F}, \tau)$, we always assume that τ is continuous and S is endowed with the strong topology.

DEFINITION 2.13. Let $(S, \mathfrak{F}, \tau)$ be a PM space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in S is said to be strongly convergent to a point $x \in S$ if for any $t > 0$, there exists a natural number N such that $x_n \in \mathcal{N}_x(t)$ whenever $n \geq N$ and we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

Similarly, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in S is called strongly Cauchy sequence if for any $t > 0$, there exists a natural number N such that $(x_m, x_n) \in \mathfrak{U}(t)$ whenever $m, n \geq N$.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [14]. A real double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ is said to converge to a real number a if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{jk} - a| < \epsilon$ whenever $j, k \geq N$.

A real double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ is said to be a Cauchy sequence if for every $\epsilon > 0$, there exist $N, M \in \mathbb{N}$ such that for all $j, p \geq N$; $k, q \geq M$, $|x_{jk} - x_{pq}| < \epsilon$.

DEFINITION 2.14. ([8]) Let $(S, \mathfrak{F}, \tau)$ be a PM space. A double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ in S is said to be strongly convergent to a point $\xi \in S$ if for any $t > 0$, there exists a natural number K such that $x_{jk} \in \mathcal{N}_\xi(t)$ whenever $j, k \geq K$.

In this case we write $x_{jk} \to \xi$ or $\lim\limits_{\substack{j\to\infty\\k\to\infty}} x_{jk} = \xi$.

Similarly, a double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ in S is called strongly Cauchy if for any $t > 0$, there exist natural numbers N, M such that for all $j, p \geq N$; $k, q \geq M$, $(x_{jk}, x_{pq}) \in \mathfrak{U}(t)$.

We now recall some basic concepts related to statistical convergence, $I-$convergence and I-statistical convergence for double sequences and also the concept of I-statistical pre-Cauchy condition for single sequences (see [1,4,7,8,13,15–17,24] for more details).

Let $K \subset \mathbb{N} \times \mathbb{N}$ and $K(n, m)$ be the number of $(j, k) \in K$ such that $j \leq n, k \leq m$. If the sequence $\{\frac{K(n,m)}{nm}\}_{n,m\in\mathbb{N}}$ has a limit in Pringsheim's sense, then we say that K has the double natural density and it is denoted by

$$d_2(K) = \lim_{\substack{m\to\infty\\n\to\infty}} \frac{K(n, m)}{nm}.$$

DEFINITION 2.15. ([13]) A double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$ if for every $\epsilon > 0$, we have $d_2(A(\epsilon)) = 0$ where $A(\epsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : |x_{jk} - \xi| \geq \epsilon\}$. In this case we write $st - \lim\limits_{\substack{j\to\infty\\k\to\infty}} x_{jk} = \xi$.

DEFINITION 2.16. ([13]) A double sequence $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ of real numbers is said to be statistically Cauchy if for every $\epsilon \geq 0$, there exist natural numbers $N = N(\epsilon)$ and $M = M(\epsilon)$ such that for all $j, p \geq N$ and $k, q \geq M$, $d_2(\{(j, k) \in \mathbb{N} \times \mathbb{N} : |x_{jk} - x_{pq}| \geq \epsilon\}) = 0$.

The notions of strong statistical convergence and strong statistical Cauchy condition for double sequences in a PM space are defined similarly using the concept of strong neighborhood.

DEFINITION 2.17. ([4]) Let $X \neq \phi$. A class I of subsets of X is said to be an ideal in X provided I satisfies the conditions:

(i) $\phi \in I$, (ii) $A, B \in I \Rightarrow A \cup B \in I$, (iii) $A \in I, B \subset A \Rightarrow B \in I$.

An ideal I in a non-empty set X is called non-trivial if X $\notin I$.

Definition 2.18. ([4]) Let $X \neq \phi$. A non-empty class $\mathbb{F}$ of subsets of X is said to be a filter in X provided that:

(i) $\phi \notin \mathbb{F}$, (ii) $A, B \in \mathbb{F} \Rightarrow A \cap B \in \mathbb{F}$, (iii) $A \in \mathbb{F}, B \supset A \Rightarrow B \in \mathbb{F}$.

Definition 2.19. Let I be a non-trivial ideal in a non-empty set X. Then the class $\mathbb{F}(I) = \{M \subset X : \exists A \in I \text{ such that } M = X \setminus A\}$ is a filter on X. This filter $\mathbb{F}(I)$ is called the filter associated with I.

A non-trivial ideal I in $X(\neq \phi)$ is called admissible if $\{x\} \in I$ for each $x \in X$. Throughout the paper we take I as a non-trivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.20. ([4]) A non-trivial ideal I of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to I for each $i \in \mathbb{N}$.

Clearly every strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$ is admissible.

Definition 2.21. ([15]) A double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is said to converge to $\eta \in \mathbb{R}$ with respect to the ideal I, if for every $\epsilon > 0$ the set $A(\epsilon) = \{(m, n) : |x_{mn} - \eta| \geq \epsilon\} \in I$.

The notion of strong ideal convergence for double sequences in a PM space can be defined in the same way using the concept of strong neighborhood.

Definition 2.22. ([1]) We say that a double sequence $(x_{jk})_{j,k \in \mathbb{N}}$ of real numbers is I-statistically convergent to a real number L, and we write $x_{jk} \xrightarrow{I^s} L$, provided for any $\epsilon > 0$ and $\delta > 0$ the set $\left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : |x_{jk} - L| \geq \epsilon, j \leq m, k \leq n\}| \geq \delta\right\} \in I$.

Definition 2.23. ([7]) A sequence $(x_j)_{j \in \mathbb{N}}$ of real numbers is said to be I-statistically pre-Cauchy if for any $\epsilon > 0$ and $\delta > 0$:

$$\left\{n \in \mathbb{N} : \frac{1}{n^2} |\{(j, k) : |x_j - x_k| \geq \epsilon; j, k \leq n\}| \geq \delta\right\} \in I.$$

3. Main results

In this section, we introduce strong I-statistical pre-Cauchy condition and the notion of strong I-statistical convergence for double sequences in a PM space $(S, \mathfrak{F}, \tau)$ and investigate some basic properties of these concepts.

Definition 3.1. Let $(S, \mathfrak{F}, \tau)$ be a PM space. A double sequence $(x_{jk})_{j,k \in \mathbb{N}}$ in S is said to be strongly I-statistically convergent to $p \in S$, and write $x_{jk} \xrightarrow{str-I^s} p$, provided that for $t, \delta > 0$: $\left\{(m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |\{(j, k) : x_{jk} \notin \mathcal{N}_p(t), j \leq m, k \leq n\}| \geq \delta\right\} \in I$.

Now for fixed $p_1, q_1, p_2, q_2 \in \mathbb{N}$, we consider the ordered pairs $(j,k)_{p_1q_1}$ and $(j,k)_{p_2q_2}$ as different elements of $\mathbb{N}\times\mathbb{N}$ provided $(p_1,q_1) \neq (p_2,q_2)$.

Definition 3.2. Let $(S,\mathfrak{F},\tau)$ be a PM space. A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ in S is said to be strongly I-statistically pre-Cauchy if for any $t>0$ and $\delta>0$: $\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2}\left|\{(j,k)_{pq} : (x_{jk},x_{pq}) \notin \mathfrak{U}(t); j,p\leq m; k,q\leq n\}\right| \geq \delta\right\} \in I$.

Remark 3.3. In a PM space a double sequence which is strongly statistically convergent is clearly strongly I-statistically convergent but converse is not true. For this we consider the following example.

Example 3.4. Let $F\in\Delta^+$ be fixed and distinct from ϵ_0 and ϵ_∞. We consider the equilateral PM space $(S,\mathfrak{F},M)$ where $\mathfrak{F}$ is defined by

$$\mathfrak{F}_{pq} = \begin{cases} F, & \text{if } p\neq q \\ \epsilon_0, & \text{if } p=q \end{cases}$$

and M is the maximal triangle function. Let I be any nontrivial admissible ideal of $\mathbb{N}\times\mathbb{N}$ and $A\in I$ be such that $A=\{(t_m,t_n) : (m,n)\in\mathbb{N}\times\mathbb{N}\}$. Let B be a subset of $\mathbb{N}\times\mathbb{N}$ such that $d_2(B)=0$. Now for fixed $p,q\in S$ we define

$$x_{t_mt_n} = \begin{cases} p, & \text{if } (m,n)\in B \\ q, & \text{if } (m,n)\notin B \end{cases}$$

and $x_{mn}=p$ if $(m,n)\notin A$. Then $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is strongly I-statistically convergent to p but is not strongly statistically convergent.

Theorem 3.5. *Every strongly I-statistically convergent double sequence is strongly I-statistically pre-Cauchy in a PM space $(S,\mathfrak{F},\tau)$.*

Proof. Let $\{x_{jk}\}_{j,k\in\mathbb{N}}$ be a double sequence in a PM space $(S,\mathfrak{F},\tau)$ which is strongly I-statistically convergent to $a\in S$. Let $t>0$ and $\delta>0$. Choose $\delta_1>0$ such that $1-(1-\delta_1)^2<\delta$.

Now, for the chosen $t>0$, there exists $\eta>0$ such that for all $a,b,c\in S$ we have

$$d_L(F_{ac},\epsilon_0)<t \quad \text{whenever} \quad d_L(F_{ab},\epsilon_0)<\eta \quad \text{and} \quad d_L(F_{bc},\epsilon_0)<\eta. \tag{1}$$

Let $C=\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\{(j,k) : x_{jk}\notin\mathcal{N}_a(\eta), j\leq m, k\leq n\}\right| \geq \delta_1\right\}$. Then $C\in I$ and for $(m,n)\in C^c$ we have,

$$\frac{1}{mn}\left|\{(j,k) : x_{jk}\notin\mathcal{N}_a(\eta), j\leq m, k\leq n\}\right| < \delta_1$$
$$\Rightarrow \frac{1}{mn}\left|\{(j,k) : x_{jk}\in\mathcal{N}_a(\eta), j\leq m, k\leq n\}\right| > 1-\delta_1.$$

Let $B_{mn} = \{(j,k) : x_{jk}\in\mathcal{N}_a(\eta), j\leq m, k\leq n\}$. Then for $(j,k),(p,q)\in B_{mn}$, $d_L(F_{x_{jk}a},\epsilon_0)<\eta$ and $d_L(F_{x_{pq}a},\epsilon_0)<\eta$. Then from (1) we get $d_L(F_{x_{jk}x_{pq}},\epsilon_0)<t$. This gives $[|B_{mn}|^2/m^2n^2] \leq \frac{1}{m^2n^2}\left|\{(j,k)_{pq} : (x_{jk},x_{pq})\in\mathfrak{U}(t); j,p\leq m; k,q\leq n\}\right|$. Thus for all $(m,n)\in C^c$ we have,

$$(1-\delta_1)^2 < [|B_{mn}|^2/m^2n^2] \leq \frac{1}{m^2n^2}\left|\{(j,k)_{pq} : (x_{jk},x_{pq})\in\mathfrak{U}(t); j,p\leq m; k,q\leq n\}\right|.$$

This gives

$$\frac{1}{m^2n^2}\left|\{(j,k)_{pq} : (x_{jk}, x_{pq}) \notin \mathfrak{U}(t); j, p \leq m; k, q \leq n\}\right| \leq 1-(1-\delta_1)^2 < \delta$$

$$\Rightarrow \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2}\left|\{(j,k)_{pq} : (x_{jk}, x_{pq}) \notin \mathfrak{U}(t); j, p \leq m; k, q \leq n\}\right| \geq \delta\right\} \subset C.$$

Since $C \in I$, then

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2}\left|\{(j,k)_{pq} : (x_{jk}, x_{pq}) \notin \mathfrak{U}(t); j, p \leq m; k, q \leq n\}\right| \geq \delta\right\} \in I.$$

Hence x is strongly I-statistically pre-Cauchy. □

REMARK 3.6. The converse of the above theorem is not true. To show this we present Example 3.9.

DEFINITION 3.7. Let $\{t_k\}_{k\in\mathbb{N}}$ and $\{t_j\}_{j\in\mathbb{N}}$ be two strictly increasing sequences of natural numbers. If $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ is a double sequence in a PM space $(S, \mathfrak{F}, \tau)$, then we define $\{x_{t_jt_k}\}_{j,k\in\mathbb{N}}$ as a subsequence of x.

LEMMA 3.8. *Let I be a strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$. Then every strongly I-statistically convergent double sequence in a PM space has a strongly convergent subsequence possessing the same limit.*

Proof. Let $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ be a strongly I-statistically convergent double sequence in a PM space $(S, \mathfrak{F}, \tau)$ converging to $a \in S$. Choose $\delta = t = 1$. Then we have

$$C = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\left\{(j,k) : d_L(F_{x_{jk}a}, \epsilon_0) \geq 1; j \leq m, k \leq n\right\}\right| \geq 1\right\} \in I.$$

Since I is an non-trivial ideal of $\mathbb{N}\times\mathbb{N}$, then $C \neq \mathbb{N}\times\mathbb{N}$. So there exists $(m_1, n_1) \in C^c$ such that

$$\frac{1}{m_1n_1}\left|\left\{(j,k) : d_L(F_{x_{jk}a}, \epsilon_0) \geq 1; j \leq m_1, k \leq n_1\right\}\right| < 1$$

$$\Rightarrow \frac{1}{m_1n_1}\left|\left\{(j,k) : d_L(F_{x_{jk}a}, \epsilon_0) < 1; j \leq m_1, k \leq n_1\right\}\right| > 0.$$

So there exists $j_1 \leq m_1$ and $k_1 \leq n_1$ such that $d_L(F_{x_{j_1k_1}a}, \epsilon_0) < 1$. Again, by choosing $\delta = t = \frac{1}{2}$, we have

$$D = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\left|\left\{(j,k) : d_L(F_{x_{jk}a}, \epsilon_0) \geq \frac{1}{2}; j \leq m, k \leq n\right\}\right| \geq \frac{1}{2}\right\} \in I.$$

Since I is strongly admissible ideal, $D \cup (\mathbb{N}\times\{1, 2, \ldots, 4n_1\}) \cup (\{1, 2, \ldots, 4m_1\}\times\mathbb{N}) \in I$. Since I is non-trivial, we choose $(m_2, n_2) \in \mathbb{N}\times\mathbb{N}$ such that $(m_2, n_2) \notin D$ and $m_2 > 4m_1, n_2 > 4n_1$. Then $\frac{1}{m_2n_2}\left|\left\{(j,k) : d_L(F_{x_{jk}a}, \epsilon_0) \geq \frac{1}{2}; j \leq m_2, k \leq n_2\right\}\right| < \frac{1}{2}$. This gives,

$$\frac{1}{m_2n_2}\left|\left\{(j,k) : d_L(F_{x_{jk}a}, \epsilon_0) < \frac{1}{2}; j \leq m_2, k \leq n_2\right\}\right| > \frac{1}{2}. \tag{2}$$

Now if $d_L(F_{x_{jk}a}, \epsilon_0) \geq \frac{1}{2}$ for all $m_1 < j \leq m_2$ and for all $n_1 < k \leq n_2$, then $\frac{1}{m_2n_2}\left|\left\{(j,k) : d_L(F_{x_{jk}a}, \epsilon_0) < \frac{1}{2}; j \leq m_2, k \leq n_2\right\}\right| \leq \frac{m_1n_1}{m_2n_2} < \frac{1}{16}$, which contradicts

(2). So there exist $m_1 < j_2 \leq m_2$ and $n_1 < k_2 \leq n_2$ such that $d_L(F_{x_{j_2k_2}a}, \epsilon_0) < \frac{1}{2}$. Then clearly $j_1 < j_2$ and $k_1 < k_2$. Proceeding in this way we get a set $K = \{(j_1,k_1),(j_2,k_2),\ldots\}$ with $j_1 < j_2 < \ldots$, $k_1 < k_2 < \ldots$ and $d_L(F_{x_{j_ik_i}a}, \epsilon_0) < \frac{1}{i}$. This shows that the subsequence $\{x\}_K$ of x is strongly convergent to a. $\square$

We now consider the following example to show that the converse of Theorem 3.5 is not true.

EXAMPLE 3.9. Let (S,d) be the Euclidean line and $H(x) = 1 - e^{-x}$, where $H \in \Delta^+$. Consider the simple space (S,d,H) which is generated by (S,d) and H. Then this space becomes a PM space $(S,\mathfrak{F})$ under the continuous triangle function τ_M, which is in fact a Menger space, where $\mathfrak{F}$ is defined on $S \times S$ by $\mathfrak{F}(p,q)(t) = F_{pq}(t) = H(\frac{t}{d(p,q)}) = 1 - e^{\frac{-t}{|p-q|}}$ for all $p,q \in S$ and $t \in \mathbb{R}^+$. Here we make the convention that $H(t/0) = H(\infty) = 1$ for $t > 0$, and $H(0) = H(\frac{0}{0}) = 0$. Now define the double sequence x in S in the following way. For $m,n,j,k \in \mathbb{N}$ such that $(m-1)! < j \leq m!$ and $(n-1)! < k \leq n!$ we define $x_{jk} = \sum_{u=1}^{m} \frac{1}{u} + \sum_{v=1}^{n} \frac{1}{v}$ and let $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$.

Clearly, x has no strongly convergent subsequence by construction of the sequence. But x is strongly statistically pre-Cauchy because for given $t_1 > 0$ we have the following, we first define $G_m(t) = 1 - e^{\frac{-t}{2/m}}$ then clearly $G_m(t)$ is a d.d.f and for $t > 0$, $G_m(t)$ weakly converges to ϵ_0 as $m \to \infty$. So for that $t_1 > 0$ there exists a positive integer M such that for all $m \geq M$ we have $d_L(G_m(t), \epsilon_0) < t_1$. Choose $m > M$ and then choose $n > m$. Then if $m! < m_1 \leq (m+1)!$; $n! < n_1 \leq (n+1)!$ and $(m-1)! < j,p \leq m_1$; $(n-1)! < k,q \leq n_1$ then we have, $|x_{jk} - x_{pq}| < \frac{2}{m}$. It follows that for $t_1 > 0$ and $m! < m_1 \leq (m+1)!$, $n! < n_1 \leq (n+1)!$, we have

$$\begin{aligned}&\frac{1}{m_1^2n_1^2}\left|\{(j,k) : d_L(F_{x_{jk}x_{pq},\epsilon_0}) < t_1; j,p \leq m_1, k,q \leq n_1\}\right| \\ &\geq \frac{1}{m_1^2n_1^2}[m_1 - (m-1)!]^2[n_1 - (n-1)!]^2 \geq [1 - \frac{1}{m}]^2[1 - \frac{1}{n}]^2.\end{aligned}$$

Since $\lim_{\substack{m\to\infty\\ n\to\infty}} [1 - \frac{1}{m}]^2[1 - \frac{1}{n}]^2 = 1$, it follows that x is strongly statistically pre-Cauchy, hence strongly I-statistically pre-Cauchy for any admissible ideal I.

The next result gives a necessary and sufficient condition for a double sequence to be I-statistically pre-Cauchy.

THEOREM 3.10. *Let $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ be a double sequence in a PM space $(S,\mathfrak{F},\tau)$. Then x is strongly I-statistically pre-Cauchy if and only if*

$$I - \lim_{\substack{m\to\infty\\ n\to\infty}} \frac{1}{m^2n^2} \sum_{j,p\leq m} \sum_{k,q\leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) = 0. \quad (3)$$

Proof. First we assume that (3) holds. Note that for $t > 0$ and $(m,n) \in \mathbb{N}\times\mathbb{N}$ we have

$$\frac{1}{m^2n^2} \sum_{j,p\leq m} \sum_{k,q\leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0)$$

$$\geq t\bigg(\frac{1}{m^2n^2}\left|\left\{(j,k)_{pq} : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq t; j,p \leq m; k,q \leq n\right\}\right|\bigg).$$

Therefore, for any $\delta > 0$,

$$A = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2}\left|\left\{(j,k)_{pq} : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq t; j,p \leq m; k,q \leq n\right\}\right| \geq \delta\right\}$$
$$\subset \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2}\sum_{j,p\leq m}\sum_{k,q\leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq \delta t\right\} = B \text{ (say)}.$$

From (3) follows that $B \in I$. This implies $A \in I$. This shows that x is strongly I-statistically pre-Cauchy.

Conversely, let x be strongly I-statistically pre-Cauchy in S and let $\delta > 0$ be given. Choose $t > 0$ and $\delta_1 > 0$ such that $\frac{t}{2} + \delta_1 < \delta$. Since $d_L(F_{ab}, \epsilon_0) \leq 1$ for all $a, b \in S$, for each $(m,n) \in \mathbb{N}\times\mathbb{N}$ we have

$$\frac{1}{m^2n^2}\sum_{j,p\leq m}\sum_{k,q\leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0)$$
$$= \frac{1}{m^2n^2}\sum_{\substack{d_L(F_{x_{jk}x_{pq}},\epsilon_0)<\frac{t}{2}\\ j,p\leq m,k,q\leq n}} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) + \frac{1}{m^2n^2}\sum_{\substack{d_L(F_{x_{jk}x_{pq}},\epsilon_0)\geq\frac{t}{2}\\ j,p\leq m,k,q\leq n}} d_L(F_{x_{jk}x_{pq}}, \epsilon_0)$$
$$< \frac{t}{2} + \frac{1}{m^2n^2}\left|\left\{(j,k)_{pq} : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq \frac{t}{2}; j,p \leq m; k,q \leq n\right\}\right|.$$

Now since x is strongly I-statistically pre-Cauchy, then for that $\delta_1 > 0$, $A = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2}\left|\left\{(j,k)_{pq} : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq \frac{t}{2}; j,p \leq m; k,q \leq n\right\}\right| \geq \delta_1\right\} \in I$. For $(m,n) \in A^c$, $\frac{1}{m^2n^2}\left|\left\{(j,k)_{pq} : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq t; j,p \leq m; k,q \leq n\right\}\right| < \delta_1$ and so $\frac{1}{m^2n^2}\sum\limits_{j,p\leq m}\sum\limits_{k,q\leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) < \frac{t}{2} + \delta_1 < \delta$. This implies

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2n^2}\sum_{j,p\leq m}\sum_{k,q\leq n} d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq \delta\right\} \subset A.$$

Since $A \in I$, (3) holds. □

Definition 3.11. ([3]) Let I be an admissible ideal of $\mathbb{N}\times\mathbb{N}$ and $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ be a real double sequence. Let $A_x = \{\alpha \in \mathbb{R} : \{(j,k) : x_{jk} < \alpha\} \notin I\}$. Then I-limit inferior of x is given by

$$I - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \phi, \\ \infty, & \text{if } A_x = \phi. \end{cases}$$

It is known [3, Theorem 3] that I-lim inf $x = \alpha$ (finite) if and only if for arbitrary $\epsilon > 0$, $\{(j,k) : x_{jk} < \epsilon + \alpha\} \notin I$ and $\{(j,k) : x_{jk} < \alpha - \epsilon\} \in I$.

We now provide a sufficient condition under which a strongly I-statistically pre-Cauchy double sequence can be strongly I-statistically convergent.

Theorem 3.12. *Let $(S, \mathfrak{F}, \tau)$ be a PM space and let $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ be a strongly I-statistically pre-Cauchy double sequence in S. If $x = \{x_{jk}\}_{j,k\in\mathbb{N}}$ has a subsequence*

$\{x_{t_j t_k}\}_{j,k\in\mathbb{N}}$ *which is strongly convergent to L and*

$$0 < I - \lim_{\substack{m\to\infty\\ n\to\infty}} inf \frac{1}{mn} \left|\{(t_j, t_k) : t_j \leq m, t_k \leq n; j, k \in \mathbb{N}\}\right| < \infty,$$

then x is strongly I-statistically convergent to L.

Proof. Let $I - \lim\limits_{\substack{m\to\infty\\ n\to\infty}} inf \frac{1}{mn} \left|\{(t_j, t_k) : t_j \leq m, t_k \leq n; j, k \in \mathbb{N}\}\right| = r$. Then $0 < r < \infty$. Let $t > 0$ and $\delta > 0$ be given. We choose $\delta_1 > 0$ such that $\frac{2\delta_1}{r} < \delta$. Now for that $t > 0$ there exists $\eta > 0$ such that for all $a, b, c \in S$ we have

$$d_L(F_{ac}, \epsilon_0) < t \quad \text{whenever} \quad d_L(F_{ab}, \epsilon_0) < \eta \quad \text{and} \quad d_L(F_{bc}, \epsilon_0) < \eta. \tag{4}$$

Since $\{x_{t_j t_k}\}_{j,k\in\mathbb{N}}$ is strongly convergent to L, there exists $n_0 \in \mathbb{N}$ such that for all $t_j, t_k > n_0$, we have $d_L(F_{x_{t_j t_k} L}, \epsilon_0) < \eta$. Let $A = \{(t_j, t_k) : t_j > n_0, t_k > n_0; j, k \in \mathbb{N}\}$ and $B(t) = \{(j,k) : d_L(F_{x_{jk} L}, \epsilon_0) \geq t\}$. Then from (4) we have,

$$\begin{aligned}
&\frac{1}{m^2 n^2} \left|\{(j,k)_{pq} : d_L(F_{x_{jk} x_{pq}}, \epsilon_0) \geq \eta; j, p \leq m; k, q \leq n\}\right| \\
&\geq \frac{1}{m^2 n^2} \sum_{j,p\leq m} \sum_{k,q\leq n} \chi_{A\times B(t)}((j,k),(p,q)) \\
&= \frac{1}{mn} \left|\{(t_j, t_k) \in A : t_j \leq m, t_k \leq n\}\right| \\
&\quad \times \frac{1}{mn} \left|\{(p,q) \in B(t) : d_L(F_{x_{pq} L}, \epsilon_0) \geq t; p \leq m, q \leq n\}\right|.
\end{aligned}$$

Since x is strongly I-statistically pre-Cauchy, for $\delta_1 > 0$ and $\eta > 0$ we have $C = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{m^2 n^2} \left|\{(j,k)_{pq} : d_L(F_{x_{jk} x_{pq}}, \epsilon_0) \geq \eta; j, p \leq m; k, q \leq n\}\right| \geq \delta_1\right\} \in I$. Therefore, for every $(m,n) \in C^c$ we have

$$\frac{1}{m^2 n^2} \left|\{(j,k)_{pq} : d_L(F_{x_{jk} x_{pq}}, \epsilon_0) \geq \eta; j, p \leq m; k, q \leq n\}\right| < \delta_1. \tag{5}$$

Now since $I - \lim\limits_{\substack{m\to\infty\\ n\to\infty}} inf \frac{1}{mn} \left|\{(t_j, t_k) : t_j \leq m, t_k \leq n; j, k \in \mathbb{N}\}\right| = r$, the set $D = \left\{(m,n) : \frac{1}{mn} \left|\{(t_j, t_k) : t_j \leq m, t_k \leq n; j, k \in \mathbb{N}\}\right| < \frac{r}{2}\right\} \in I$. So for every $(m,n) \in D^c$ we have

$$\frac{1}{mn} \left|\{(t_j, t_k) : t_j \leq m, t_k \leq n; j, k \in \mathbb{N}\}\right| \geq \frac{r}{2}. \tag{6}$$

Now from (5), (6) we get for every $(m,n) \in C^c \cap D^c = (C \cup D)^c$:

$$\frac{1}{mn} \left|\{(p,q) : d_L(F_{x_{pq} L}, \epsilon_0) \geq t; p \leq m, q \leq n\}\right| < \frac{2\delta_1}{r} < \delta.$$

This implies $\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn} \left|\{(j,k) : d_L(F_{x_{jk} L}, \epsilon_0) \geq t; j \leq m, k \leq n\}\right| \geq \delta\right\} \subset (C \cup D)$. Since $C, D \in I$, the set $C \cup D \in I$ and so

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn} \left|\{(j,k) : d_L(F_{x_{jk} L}, \epsilon_0) \geq t; j \leq m, k \leq n\}\right| \geq \delta\right\} \in I.$$

This shows that x is strongly I-statistically convergent to L. □

Definition 3.13. Let $(S, \mathfrak{F}, \tau)$ be a PM space and I be a strongly admissible ideal of

$\mathbb{N}\times\mathbb{N}$. A double sequence $x=\{x_{jk}\}_{j,k\in\mathbb{N}}$ in S is said to be strongly I^*-statistically pre-Cauchy if there exists a set $M\in\mathbb{F}(I)$ such that $\{x\}_M$ is strongly statistically pre-Cauchy, i.e for $t>0:\lim_{\substack{m\to\infty\\ n\to\infty\\ (m,n)\in M}}\frac{1}{m^2n^2}\left|\{(j,k)_{pq}:d_L(F_{x_{jk}x_{pq}},\epsilon_0)\geq t; j,p\leq m; k,q\leq n\}\right|=0.$

Theorem 3.14. *Let $(S,\mathfrak{F},\tau)$ be a PM space and $x=\{x_{jk}\}_{j,k\in\mathbb{N}}$ be a strongly I^*-statistically pre-Cauchy double sequence in S. Then x is strongly I-statistically pre-Cauchy.*

Proof. Let $t>0$ and $\delta>0$ be given. Since $x=\{x_{jk}\}_{j,k\in\mathbb{N}}$ is strongly I^*-statistically pre-Cauchy, there exists a set $M\in\mathbb{F}(I)$ such that

$$\lim_{\substack{m\to\infty\\ n\to\infty\\ (m,n)\in M}}\frac{1}{m^2n^2}\left|\{(j,k)_{pq}:d_L(F_{x_{jk}x_{pq}},\epsilon_0)\geq t; j,p\leq m; k,q\leq n\}\right|=0.$$

Then there exists $n_0\in\mathbb{N}$ such that for $(m,n)\in M$ with $m,n\geq n_0$, we have $\frac{1}{m^2n^2}\left|\{(j,k)_{pq}:d_L(F_{x_{jk}x_{pq}},\epsilon_0)\geq t; j,p\leq m; k,q\leq n\}\right|<\delta$. Let $K=\{1,\ldots,n_0-1\}$. Then clearly

$$\begin{aligned}A=&\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{m^2n^2}\left|\{(j,k)_{pq}:d_L(F_{x_{jk}x_{pq}},\epsilon_0)\geq t; j,p\leq m; k,q\leq n\}\right|\geq\delta\right\}\\ &\subset(\mathbb{N}\times\mathbb{N}\setminus M)\cup(K\times\mathbb{N})\cup(\mathbb{N}\times K).\qquad(7)\end{aligned}$$

Since I is a strongly admissible ideal, the set on the right-hand side of (7) belongs to I and $A\in I$. Hence $x=\{x_{jk}\}_{j,k\in\mathbb{N}}$ is strongly I-statistically pre-Cauchy. □

Definition 3.15. ([9]) We say taht an admissible ideal $I\subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property $(AP2)$ if for every countable family of mutually disjoint sets $\{A_1,A_2,\ldots\}$ belonging to I, there exists a countable family of sets $\{B_1,B_2,\ldots\}$ such that $A_j\Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j\in\mathbb{N}$ and $B=\bigcup_{j=1}^{\infty}B_j\in I$. Here Δ denotes the symmetric difference between two sets.

Lemma 3.16. *([9]) Let $\{P_i\}$ be a countable collection of subsets of $\mathbb{N}\times\mathbb{N}$ such that $P_i\in\mathbb{F}(I)$ for each i, where $\mathbb{F}(I)$ is a filter associated with a strongly admissible ideal I with the property $(AP2)$. Then there exists a set $P\in\mathbb{N}\times\mathbb{N}$ such that $P\in\mathbb{F}(I)$ and the set $P\setminus P_i$ is finite for all i.*

Theorem 3.17. *Let $(S,\mathfrak{F},\tau)$ be a PM space. If I is a strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$ with the property $(AP2)$ then the notions of strong I-statistical pre-Cauchy and I^*-statistical pre-Cauchy coincide.*

Proof. Because of Theorem 3.14, it is sufficient to prove that a strongly I-statistically pre-Cauchy double sequence $x=\{x_{jk}\}_{j,k\in\mathbb{N}}$ in S is strongly I^*-statistically pre-Cauchy. Let $x=\{x_{jk}\}_{j,k\in\mathbb{N}}$ be a strongly I-statistically pre-Cauchy double sequence in S. Let $t>0$ be given. For each $i\in\mathbb{N}$, let

$$P_i=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{m^2n^2}\left|\{(j,k)_{pq}:d_L(F_{x_{jk}x_{pq}},\epsilon_0)\geq t; j,p\leq m; k,q\leq n\}\right|<\frac{1}{i}\}.$$

Then $P_i \in \mathbb{F}(I)$ for each $i \in \mathbb{N}$. Since I has the property $(AP2)$, then by Lemma 3.16 there exists a set $P \in \mathbb{N}\times\mathbb{N}$ such that $P \in \mathbb{F}(I)$ and $P \setminus P_i$ is finite for all $i \in \mathbb{N}$. Now we show that

$$\lim_{\substack{m\to\infty\\ n\to\infty\\ (m,n)\in P}} \frac{1}{m^2n^2}\left|\left\{(j,k)_{pq} : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq t; j,p \leq m; k,q \leq n\right\}\right| = 0. \tag{8}$$

Let $\epsilon > 0$ be given. Then there exists $j \in \mathbb{N}$ such that $j > \frac{1}{\epsilon}$. Since $P \setminus P_j$ is a finite set, there exists $r = r(j) \in \mathbb{N}$ such that $(m,n) \in P \cap P_j$ for all $m,n \geq r$. Therefore for all $(m,n) \in P$ with $m,n \geq r$ we have,

$$\frac{1}{m^2n^2}\left|\left\{(j,k)_{pq} : d_L(F_{x_{jk}x_{pq}}, \epsilon_0) \geq t; j,p \leq m; k,q \leq n\right\}\right| < \frac{1}{j} < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, (8) holds. Therefore x is strongly I^*-statistically pre-Cauchy. □

Acknowledgement. The second author is thankful to University Grants Commission, New Delhi, India for his research fund.

References

[1] C. Belen, M. Yildirim, *On generalized statistical convergence of double sequences via ideals*, Ann. Univ. Ferrara 58(1)(2012), 11–20.

[2] J. Conor, J. A. Fridy, J. Kline, *Statistically pre-Cauchy Sequences*, Analysis 14 (1994), 311–317.

[3] P. Das, P.Malik, *On extremal I-limit points of double sequences*, Tatra Mt. Math. Publ. 40 (2008), 91–102.

[4] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, *$\mathcal{I}$ and $\mathcal{I}^*$ convergence of double sequences*, Math. Slovaca **58**(5) (2008), 605–620.

[5] P. Das, E. Savas, S.Kr Ghosal, *On generalizations of certain summability methods using ideals*, Appl. Math. Lett. 24 (2011), 1509–1514.

[6] P. Das, E. Savas, *A generalized statistical convergence via ideals*, Appl. Math. Lett. 24 (2011), 826–830.

[7] P. Das, E. Savas, *On I-statistically pre-Cauchy sequences*, Taiwanese J. Math. 18(1) (2014), 115–126.

[8] K. Dutta, P. Malik, M. Maity, *Statistical convergence of double sequences in probabilistic metric spaces*, Selcuk J. Math. 14(1) (2013), 57–70.

[9] E. Dundar, B. Altay, *On some properties of I_2-convergence and I_2 Cauchy of double sequences*, Fen. Math. Notes 7(1) (2011), 1–12.

[10] H. Fast, *Sur la convergence statistique*, Colloq. Math. 2 (1951), 241–244.

[11] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[12] K. Menger, *Statistical metric spaces*, Proc. Nat. Acad of Sci., USA **28** (1942), 535–537.

[13] M. Mursaleen, O. O. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288**(1) (2003), 223–231.

[14] A. Pringsheim, *Zur theorie der Zweifach unendlichen Zahlenfolgen*, Math. Ann. 53(3) (1900), 289–321.

[15] T. Sálat, P. Kostyrko, W. Wilczyński, *I-convergence*, Real Anal. Exchange 26(2) (2000-2001), 669–685.

[16] E. Savas, *On certain double A-summability methods*, Filomat 30(2) (2016), 395–401.

[17] E. Savas, R. Savas Eran, *On double sequence spaces defined by an Orlicz function on a semi-normed space*, Filomat 30(3) (2016), 631–638.

[18] I. J Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66(1959) 361-375.

[19] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math. 10 (1960), 313–334.

[20] B. Schweizer, A. Sklar, E. Thorp, *The metrization of statistical metric spaces*, Pacific J. Math. 10 (1960), 673–675.

[21] B. Schweizer, A. Sklar, *Statistical metric spaces arising from sets of random variables in Euclidean n-space*, Theory Probability Appl. 7 (1962), 447–456.

[22] B. Schweizer, A. Sklar, *Tringle inequalities in a class of statistical metric spaces* , J. London Math. Soc. **38** (1963), 401–406.

[23] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland Publishing Co., New York (1983).

[24] C. Şençimen, S. Pehlivan, *Strongly ideal convergence in Probabilistic metric spaces*, Proc. Indian Acad. Sci. (Math. Sci). **119**(3) (2009), 401–410.

[25] H. Steinhus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–74.

[26] R. M. Tardiff, *Topologies for probabilistic metric spaces*, Pacific J. Math. **65** (1976), 233–251.

[27] E. Thorp, *Generalized topologies for statistical metric spaces*, Fundamenta Math. 51 (1962), 9–21.

(received 23.09.2016; in revised form 31.01.2017; available online 09.03.2018)

Department of Mathematics, The University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India.

E-mail: pmjupm@yahoo.co.in

Department of Mathematics, The University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India.

E-mail: buagbu@yahoo.co.in

Boral High School, Kolkata-700154, India.

E-mail: mepsilon@gmail.com