A STUDY ON ELLIPTIC PDE INVOLVING THE p-HARMONIC AND THE p-BIHARMONIC OPERATORS WITH STEEP POTENTIAL WELL

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Abstract. In this paper, we give an existence result pertaining to a nontrivial solution to the problem $\Delta_p^2 u - \Delta_p u + \lambda V(x)|u|^{p-2}u = f(x,u)$, $x \in \mathbb{R}^N$, $u \in W^{2,p}(\mathbb{R}^N)$, where $p > 1$, $\lambda > 0$, $V \in C(\mathbb{R}^N, \mathbb{R}^+)$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $N > 2p$. We also explore the problem in the limiting case of $\lambda \to \infty$.

1. Introduction

The problem we will address in this article is

$$\begin{cases} \Delta_p^2 u - \Delta_p u + \lambda V(x)|u|^{p-2}u = f(x,u) \,, \ x \in \mathbb{R}^N, \\ u \in W^{2,p}(\mathbb{R}^N), \end{cases} \tag{1}$$

where $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$, $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ and $\lambda > 0$ is a parameter, with $p > 1$, $N > 2p$. The potential function $V(x)$ is a real valued continuous function on $\mathbb{R}^N$ satisfying the following conditions:

(V1) $V(x) \geq 0$ on $\mathbb{R}^N$.

(V2) There exists $b > 0$ such that the set $V_b = \{x \in \mathbb{R}^n : V(x) < b\}$ is nonempty and has finite Lebesgue measure in $\mathbb{R}^N$.

(V3) $\Omega = \text{int} \, V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

This type of assumptions were introduced by Bartsch et al [1] (see also [3]), referred to as the steep well potential for the potential function $V(x)$, in the study of a nonlinear Schrödinger equation. Further, to study the existence of nontrivial solution and the limiting case, $\lambda \to \infty$, of the problem 1, we make the following assumptions on the nonlinear function f:

(F1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists constants $1 < \gamma_1 < \gamma_2 < \ldots < \gamma_m < p$ and functions $\xi_i \in L^{\frac{p}{p-\gamma_i}}(\mathbb{R}^N, \mathbb{R}^+)$ such that $|f(x, u)| \leq \sum_{i=1}^{m} \gamma_i \xi_i(x)|u|^{\gamma_i - 1}$, $\forall (x, u) \in \mathbb{R}^N \times \mathbb{R}$.

(F2) There exists constants $\eta, \delta > 0, \gamma_0 \in (1, p)$ such that $|F(x, u)| \geq \eta|u|^{\gamma_0}$ for all $x \in \Omega$ and for all such u such that $|u| \leq \delta$, where $F(x, u) = \int_0^u f(x, s) ds$.

In the recent years problems related to the kind in (1), for the case of $p = 2$, the equations with biharmonic operator have been investigated. Readers may refer to [5, 7, 9–15] and the references there in. The present work in this article draws its motivation from W. Zhang et al [15], Ye and Tang [11] and Liu et al [5]. In all these articles, they have considered the problem (1) for $p = 2$. We address the problem for $p \neq 2$, $N > 2p$. The notion of p-biharmonic operator is introduced in the recent work of Bhakta [2]. For $p \neq 2$, things seemed to become more complicated not only due to the lack of linearity of both p-Laplacian and p-biharmonic operator but also because of the fact that the associated energy functional is defined on a Banach space which is not a Hilbert space. We further have to deal with lack of compact embedding, since the domain considered here is $\mathbb{R}^N$. The main two results proved in this article are the following.

THEOREM 1.1. *Assume the conditions (V1)-(V3), (F1), (F2) hold. Then there exists $\Lambda_0 > 0$ such that for each $\lambda > \Lambda_0$, problem (1) has at least one non trivial solution u_λ.*

THEOREM 1.2. *Let $u_n = u_{\lambda_n}$ be a solution of the problem (1) corresponding to $\lambda = \lambda_n$. If $\lambda_n \to \infty$, then $||u_n||_{\lambda_n} \leq c$, for some $c > 0$ and for $p \leq q < p^*$, $u_n \to \tilde{u}$ in $L^q(\mathbb{R}^N)$ up to a subsequence. Further, this $\tilde{u}$ is a solution of the problem*

$$\Delta_p^2 u - \Delta_p u = f(x, u), \ in \ \Omega$$
$$u = 0, \ on \ \mathbb{R}^N \setminus \Omega. \tag{2}$$

and $u_n \to \tilde{u}$ in $W^{2,p}(\mathbb{R}^N)$.

The paper has been organized as follows. In Section 2, we discuss the notations which will be used in the theorems. In Section 3, we give the proof of Theorem 1.1 and in Section 4, we prove the Theorem 2.

2. Preliminaries and Notations

We will denote a Sobolev space of order 2 as $W^{2,p}(\mathbb{R}^N)$, which is given by
$$W^{2,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : |\nabla u|, \Delta u \in L^p(\mathbb{R}^N)\}$$
endowed with the norm
$$||u||_{W^{2,p}(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + |u|^p) dx.$$

Let $$X = \left\{ u \in W^{2,p}(\mathbb{R}^N) : \int_{\mathbb{R}^n} (|\Delta u|^p + |\nabla u|^p + V(x)|u|^p) dx < \infty \right\}$$

be endowed with the norm

$$||u||^p = \int_{\mathbb{R}^n} (|\Delta u|^p + |\nabla u|^p + V(x)|u|^p) dx.$$

For $\lambda > 0$, we set

$$E_\lambda = \{u \in W^{2,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + \lambda V(x)|u|^p) dx < \infty\}$$

with $$||u||_\lambda^p = \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + \lambda V(x)|u|^p) dx.$$

It is easy to verify that $(E_\lambda, || \cdot ||_\lambda)$ is a closed in $W^{2,p}(\mathbb{R}^N)$ and $||u|| \leq ||u||_\lambda$ for any $\lambda \geq 1$. We will denote μ to be the Lebesgue measure on $\mathbb{R}^N$.

LEMMA 2.1. *If (V1)-(V2) hold, then there exists positive constants λ_0, c_0 such that $||u||_{W^{2,p}(\mathbb{R}^N)} \leq c_0||u||_\lambda$, for all $u \in E_\lambda, \lambda \geq \lambda_0$.*

Proof. By using conditions (V1)-(V2) and the Sobolev inequality, we have

$$||u||_{W^{2,p}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + |u|^p) dx$$

$$= \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p) dx + \int_{V_b} |u|^p dx + \int_{\mathbb{R}^N \setminus V_b} |u|^p dx$$

$$\leq \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p) dx$$

$$+ (\mu(V_b))^{\frac{P^*-p}{p^*}} \left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{\frac{p}{p^*}} + \int_{\mathbb{R}^N \setminus V_b} |u|^p dx$$

$$\leq \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p) dx$$

$$+ S_\alpha^{-1} (\mu(V_b))^{\frac{P^*-p}{p^*}} \int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{\lambda b} \int_{\mathbb{R}^N \setminus V_b} \lambda V(x)|u|^p dx$$

$$\leq \int_{\mathbb{R}^N} (|\Delta u|^p + (1 + S_\alpha^{-1} (\mu(V_b))^{\frac{P^*-p}{p^*}})|\nabla u|^p) dx + \frac{1}{\lambda b} \int_{\mathbb{R}^N} V(x)|u|^p dx$$

$$\leq \max \left\{ 1, 1 + S_\alpha^{-1} (\mu(V_b))^{\frac{P^*-p}{p^*}}, \frac{1}{\lambda b} \right\} \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + \lambda V(x)|u|^p) dx.$$

where S_α denote the Sobolev constant, $p^* = \frac{Np}{N-2p}$. Take $\lambda_0 = \frac{1}{b\left(1 + S_\alpha^{-1}(\mu(V_b))^{\frac{P^*-p}{p^*}}\right)}$.

Then for all $\lambda \geq \lambda_0$, we have

$$\max \left\{ 1, 1 + S_\alpha^{-1} (\mu(V_b))^{\frac{P^*-p}{p^*}}, \frac{1}{\lambda b} \right\} = \max \left\{ 1, 1 + S_\alpha^{-1} (\mu(V_b))^{\frac{P^*-p}{p^*}} \right\} = c_0 \text{ (say)}.$$

Hence for all $\lambda \geq \lambda_0$ and $u \in E_\lambda$, we have $||u||_{W^{2,p}(\mathbb{R}^N)} \leq c_0||u||_\lambda$. $\square$

This shows that the embedding $E_\lambda \hookrightarrow W^{2,p}(\mathbb{R}^N)$ is continuous. By the Sobolev embedding results for $2p < N$, the embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for $q \in [p, p^*]$. Hence there exists $c_q > 0$ such that $||u||_q \leq c_q||u||_{W^{2,p}(\mathbb{R}^N)} \leq c_0 c_q ||u||_\lambda$, for all $\lambda \geq \lambda_0, q \in [p, p^*]$. Moreover, $E_\lambda \hookrightarrow L_{loc}^q(\mathbb{R}^N)$ is compact for $q \in [p, p^*)$.

3. Existence of non trivial solutions

Let
$$J_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\Delta u|^p + |\nabla u|^p + \lambda V(x)|u|^p)dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

Then it can be seen that $J_\lambda \in C^1(E_\lambda, \mathbb{R})$ and its Fréchet derivative is given by

$$\langle J_\lambda'(u), v \rangle = \int_{\mathbb{R}^N} (|\Delta u|^{p-2}\Delta u \Delta v + |\nabla u|^{p-2}\nabla u \cdot \nabla v + \lambda V(x)|u|^{p-2}uv)dx$$

$$- \int_{\mathbb{R}^N} f(x,u)vdx,$$

for all $v \in E_\lambda$. Thus $u \in E_\lambda$ is a critical point of J_λ if and only if it is a weak solution of the problem (1). In order to prove the existence of non trivial solutions of the problem (1) we use the following theorem.

THEOREM 3.1. *[6] Let B be a real Banach space and $J \in C^1(B, \mathbb{R})$ satisfy the Palais-Smale (PS) condition. If J is bounded below, then $c = \inf_B J$ is a critical value of J.*

We now prove the following lemmas.

LEMMA 3.2. *Suppose that (V1)-(V3), (F1),(F2) are satisfied. Then there exists $\Lambda_0 > 0$ such that for every $\lambda \geq \Lambda_0$, J_λ is bounded below in E_λ.*

Proof. Using the Hölder's inequality we have,

$$J_\lambda(u) = \frac{1}{p}||u||_\lambda^p - \int_{\mathbb{R}^N} F(x,u)dx \geq \frac{1}{p}||u||_\lambda^p - \sum_{i=1}^m \int_{\mathbb{R}^N} \xi_i(x)|u|^{\gamma_i}dx$$

$$\geq \frac{1}{p}||u||_\lambda^p - \sum_{i=1}^m \left(\int_{\mathbb{R}^N} |\xi_i(x)|^{\frac{p}{p-\gamma_i}} dx \right)^{\frac{p-\gamma_i}{p}} \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{\gamma_i}{p}}$$

$$\geq \frac{1}{p}||u||_\lambda^p - \sum_{i=1}^m c_p^{\gamma_i} c_0^{\gamma_i} ||\xi||_{\frac{p}{p-\gamma_i}} ||u||_\lambda^{\gamma_i}.$$

Since $1 < \gamma_1 < \ldots < \gamma_m < p$, the above inequality implies that $J_\lambda(u) \to +\infty$, when $||u||_\lambda \to +\infty$. Consequently, there exists $\Lambda_0 = \max\{1, \lambda_0\} > 0$ such that for every $\lambda \geq \Lambda_0$, J_λ is bounded from below. $\square$

LEMMA 3.3. *Assume that the conditions (V1)-(V3), (F1), (F2) are satisfied. Then J_λ satisfies the Palais-Smale (PS) condition for every $\lambda \geq \Lambda_0$.*

Proof. Suppose that $(u_n) \subset E_\lambda$ be a sequence such that $J_\lambda(u_n)$ is bounded and $J_\lambda'(u_n) \to 0$ as $n \to \infty$. Then by Lemma 3.2, (u_n) is bounded below in E_λ. Thus there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$, $||u_n||_q \leq c_q c_0 ||u_n||_\lambda \leq c$, for all $u \in E_\lambda, \lambda \geq \lambda_0, p \leq q \leq p^*$. Hence by Eberlein-Smulian theorem, passing on to a subsequence (the subsequence is still denoted by u_n), we may assume that $u_n \rightharpoonup u_0$ in E_λ. Since the inclusion $E_\lambda \hookrightarrow L_{loc}^q(\mathbb{R}^N)$ is compact for $q \in [p, p^*)$, we have $u_n \to u_0$

in $L_{loc}^p(\mathbb{R}^N)$. Since $\xi_i(x) \in L^{\frac{p}{p-\gamma_i}}(\mathbb{R}^N, \mathbb{R}^+)$, we can choose $R_\epsilon > 0$ such that

$$\left(\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |\xi_i(x)|^{\frac{p}{p-\gamma_i}} dx \right)^{\frac{p-\gamma_i}{p}} < \epsilon, \quad 1 \le i \le m. \tag{3}$$

Since $u_n \to u_0$ in $L_{loc}^p(\mathbb{R}^N)$, there exists $N_0 \in \mathbb{N}$ such that

$$\left(\int_{B_{R_\epsilon}} |u_n - u_0|^p dx \right)^{\frac{\gamma_i}{p}} < \epsilon, \tag{4}$$

for $n \ge N_0$ and for all $1 \le i \le m$. By (4) and Hölder inequality, we have,

$$\int_{B_{R_\epsilon}} |f(x, u_n - u_0)||u_n - u_0| dx \le \sum_{i=1}^m \gamma_i \int_{B_{R_\epsilon}} |\xi_i(x)||u_n - u_0|^{\gamma_i} dx$$

$$\le \sum_{i=1}^m \gamma_i \left(\int_{B_{R_\epsilon}} |\xi_i(x)|^{\frac{p}{p-\gamma_i}} dx \right)^{\frac{p-\gamma_i}{p}} \left(\int_{B_{R_\epsilon}} |u_n - u_0|^p dx \right)^{\frac{\gamma_i}{p}}$$

$$\le \left(\sum_{i=1}^m \gamma_i \|\xi_i\|_{\frac{p}{p-\gamma_i}} \right) \epsilon, \forall n \ge N_0.$$

Hence it follows that

$$\int_{B_{R_\epsilon}} |f(x, u_n - u_0)||u_n - u_0| dx \to 0, \quad \text{as} \quad n \to \infty. \tag{5}$$

On the other hand, by (3) and boundedness of (u_n) in $L^p(\mathbb{R}^N)$ we have,

$$\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |f(x, u_n - u_0)||u_n - u_0| dx \le \sum_{i=1}^m \gamma_i \|\xi_i\|_{\frac{p}{p-\gamma_i}, \mathbb{R}^N \setminus B_{R_\epsilon}} \|u_n - u_0\|_{p, \mathbb{R}^N \setminus B_{R_\epsilon}}^{\gamma_i}$$

$$\le \epsilon \sum_{i=1}^m \gamma_i \|u_n - u_0\|_p^{\gamma_i} \le \epsilon \sum_{i=1}^m \gamma_i \left(\|u_n\|_p + \|u_0\|_p \right)_i^\gamma \le \epsilon \sum_{i=1}^m \gamma_i \left(c + \|u_0\|_p \right)^{\gamma_i}.$$

Therefore,

$$\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |f(x, u_n - u_0)||u_n - u_0| dx \to 0, \quad \text{as} \quad n \to \infty. \tag{6}$$

Combining (5) and (6), we have

$$\int_{\mathbb{R}^N} |f(x, u_n - u_0)||u_n - u_0| dx \to 0, \quad \text{as} \quad n \to \infty. \tag{7}$$

Since $u_n \rightharpoonup u_0$ in E_λ, hence $\langle J_\lambda'(u_n - u_0), u_n - u_0 \rangle \to 0$ as $n \to \infty$. But

$$0 \le \|u_n - u_0\|_\lambda^p = \langle J_\lambda'(u_n - u_0), u_n - u_0 \rangle + \int_{\mathbb{R}^N} (f(x, u_n - u_0))(u_n - u_0) dx$$

$$\le \langle J_\lambda'(u_n - u_0), u_n - u_0 \rangle + \int_{\mathbb{R}^N} |f(x, u_n - u_0)||u_n - u_0| dx$$

By (7) and $\langle J_\lambda'(u_n - u_0), u_n - u_0 \rangle \to 0$, it follows that $\|u_n - u_0\|_\lambda^p \to 0$ as $n \to \infty$. This shows that $u_n \to u_0$ in E_λ. $\qquad \square$

Proof (**Proof of the Theorem 1.1**). By Lemmas 3.2, 3.3 and Theorem 3.1, it follows that $c_\lambda = \inf_{E_\lambda} J_\lambda(u)$ is a critical value of J_λ, that is there exists a critical point $u_\lambda \in E_\lambda$ such that $J_\lambda(u_\lambda) = c_\lambda$. Therefore, u_λ is a solution for the problem (1) for $\lambda > \Lambda_0$. Now we will show that $u_\lambda \neq 0$. Let $u^\# \in \left(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \right) \setminus \{0\}$ such that $||u^\#||_\infty \leq 1$, where Ω is given in the condition (V3). Then by the condition (F2), we have

$$J_\lambda(tu^\#) = \frac{1}{p}||tu^\#||_\lambda^p - \int_{\mathbb{R}^N} F(x, tu^*)dx$$
$$= \frac{t^p}{p}||u^\#||_\lambda^P - \int_\Omega F(x, tu^\#)dx \leq \frac{t^p}{p}||u^\#||_\lambda^p - \eta t^{\gamma_0}\int_\Omega |u^\#|^{\gamma_0}dx,$$

$\forall t \in (0, \delta)$, where δ is defined in (F2). Since $1 < \gamma_0 < p$, it follows that $J_\lambda(tu^\#) < 0$ for $t > 0$ small enough. Hence $J_\lambda(u_\lambda) = c_\lambda < 0$. Therefore, u_λ is a nontrivial solution of the problem (1). $\square$

4. Limiting case $\lambda \to \infty$

We now consider the limiting case, $\lambda \to \infty$, of the problem (1) on the set $V^{-1}(0)$. Define $\tilde{W}(\Omega) = \begin{cases} u, u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \\ u = 0 \ a.e., \ \text{in} \ R^N \setminus \Omega. \end{cases}$, where Ω is given in the condition (V3). Then $\tilde{W}(\Omega) \subset E_\lambda$ for all $\lambda > 0$. Define $\tilde{c} = \inf_{u \in \tilde{W}(\Omega)} J_\lambda|_{\tilde{W}(\Omega)}$, where $J_\lambda|_{\tilde{W}(\Omega)}$ is a restriction of J_λ on $\tilde{W}(\Omega)$, that is

$$J_\lambda|_{\tilde{W}(\Omega)} = \frac{1}{p}\int_\Omega \left(|\Delta u|^p + |\nabla u|^p \right) dx - \int_\Omega F(x, u)dx,$$

for $u \in \tilde{W}(\Omega)$). Similar to the proof of the Theorem 1.1, it can be seen that $\tilde{c} < 0$ is achieved and $c_\lambda \leq \tilde{c} < 0$ for all $\lambda > \Lambda_0$.

Proof (**Proof of the Theorem 1.2**). For any sequence $\lambda_n \to \infty$, let $u_n = u_{\lambda_n}$ be the critical points of J_{λ_n}. Thus we have,

$$c_{\lambda_n} = J_{\lambda_n}(u_n) \leq \tilde{c} < 0. \tag{8}$$

Also in the Lemma 3.2, we have seen that

$$J_{\lambda_n}(u_n) \geq \frac{1}{p}||u_n||_{\lambda_n}^p - \sum_{i=1}^m c_p^{\gamma_i} c_0^{\gamma_i} ||\xi_i||_{\frac{p}{p-\gamma_i}} ||u_n||_{\lambda_n}^{\gamma_i}.$$

Therefore, (8) and the above inequality implies that

$$||u_n||_{\lambda_n} \leq c, \tag{9}$$

where the constant $c > 0$ is independent of λ_n. Therefore, passing on to a subsequence we may assume that $u_n \rightharpoonup \tilde{u}$ in E_λ. This implies that $u_n \to \tilde{u}$ in $L_{loc}^q(\mathbb{R}^N)$ for $p \leq q < p^*$. Then by Fatou's lemma we have,

$$\int_{\mathbb{R}^N} V(x)|\tilde{u}|^p dx \leq \lim_{n \to \infty} \inf \int_{\mathbb{R}^N} V(x)|u_n|^p dx \leq \lim_{n \to \infty} \inf \frac{||u_n||_{\lambda_n}^p}{\lambda_n} = 0,$$

which implies that $\tilde{u} = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$. Since $\Omega = \text{int}V^{-1}(0)$ and Ω has smooth boundary, hence $\tilde{u} \in \tilde{W}(\Omega)$. Now for any $\varphi \in C_0^\infty(\Omega)$, since $\langle J'_{\lambda_n}(u_n), \varphi \rangle = 0$, it follows that

$$\int_\Omega \left(|\Delta \tilde{u}|^{p-2} \Delta \tilde{u} \Delta \varphi + |\nabla \tilde{u}|^{p-2} \nabla u \cdot \nabla \varphi \right) dx - \int_\Omega f(x, \tilde{u}) \varphi dx = 0,$$

which implies that $\tilde{u}$ is a weak solution of the problem (2), where we have used the density of $C_0^\infty(\Omega)$ in $\tilde{W}(\Omega)$.

Next we show that $u_n \to \tilde{u}$ in $L^q(\mathbb{R}^N)$. If not, then by Lions Vanishing lemma [4,8], there exists $\delta > 0$, $\rho > 0$ and sequence $\{x_n\} \in \mathbb{R}^N$ with $|x_n| \to \infty$ such that

$$\int_{B_\rho(x_n)} |u_n - \tilde{u}|^p dx \geq \delta.$$

Since $|x_n| \to \infty$, hence $\mu\left(B_\rho(x_n) \cap V_b\right) \to 0$ as $n \to \infty$. Therefore, by Hölder inequality, we have

$$\int_{B_\rho(x_n) \cap V_b} |u_n - \tilde{u}|^p dx \leq \mu\left(B_\rho(x_n) \cap V_b\right)^{\frac{p^*-p}{p^*}} \left(\int_{\mathbb{R}^N} |u_n - \tilde{u}|^{p^*} dx \right)^{\frac{p}{p^*}} \to 0,$$

as $n \to \infty$. Consequently

$$\begin{aligned}
\|u_n\|_{\lambda_n}^p &\geq \lambda_n b \int_{B_\rho(x_n) \cap \{x \in \mathbb{R}^N : V(x) \geq b\}} |u_n|^p dx \\
&= \lambda_n b \int_{B_\rho(x_n) \cap \{x \in \mathbb{R}^N : V(x) \geq b\}} |u_n - \tilde{u}|^p dx, \quad [\text{as } \tilde{u} = 0 \text{ a.e. in } \mathbb{R}^N \setminus V^{-1}(0)] \\
&= \lambda_n b \left(\int_{B_\rho(x_n)} |u_n - \tilde{u}|^p dx - \int_{B_\rho(x_n) \cap V_b} |u_n - \tilde{u}|^p dx + o(1) \right) \overset{n \to \infty}{\longrightarrow} \infty,
\end{aligned}$$

which contradicts to (9). Therefore, $u_n \to \tilde{u}$ in $L^q(\mathbb{R}^N)$ for $p \leq q < p^*$.

Next, we show that $u_n \to \tilde{u}$ in $W^{2,p}(\mathbb{R}^N)$. Since $u_n \rightharpoonup \tilde{u}$ in E_λ and E_λ compactly embedded in $L_{loc}^q(\mathbb{R}^N)$, hence by the similar method as in Lemma 3.3, it follows that

$$\int_{\mathbb{R}^N} |f(x, u_n - \tilde{u})||u_n - \tilde{u}| dx \to 0, \text{ as } n \to \infty. \tag{10}$$

Since $u_n \rightharpoonup \tilde{u}$ in E_λ, hence $\langle J'_\lambda(u_n - \tilde{u}), u_n - \tilde{u} \rangle \to 0$ as $n \to \infty$. But

$$\begin{aligned}
\|u_n - \tilde{u}\|_\lambda^p &= \langle J'_\lambda(u_n - \tilde{u}), u_n - \tilde{u} \rangle + \int_{\mathbb{R}^N} (f(x, u_n - \tilde{u}))(u_n - \tilde{u}) dx \\
&\leq \langle J'_\lambda(u_n - \tilde{u}), u_n - \tilde{u} \rangle + \int_{\mathbb{R}^N} |f(x, u_n - \tilde{u})||u_n - \tilde{u}| dx
\end{aligned}$$

Therefore, by (10) and the above inequality, it follows that $\|u_n - \tilde{u}\|_\lambda \to 0$ as $n \to \infty$. Again, by the Lemma 2.1, $\|u_n - \tilde{u}\|_{W^{2,p}(\mathbb{R}^N)} \leq c_0 \|u_n - \tilde{u}\|_\lambda$, hence we have $u_n \to \tilde{u}$ in $W^{2,p}(\mathbb{R}^N)$.

From (8), we have

$$\frac{1}{p} \int_\Omega \left(|\Delta \tilde{u}|^p + |\nabla \tilde{u}|^p \right) dx - \int_\Omega F(x, \tilde{u}) dx \leq \tilde{c} < 0,$$

which implies that $\tilde{u} \neq 0$. This completes the theorem. $\qquad \square$

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