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A NOTE ON SOME OPERATORS ACTING ON CENTRAL MORREY SPACES

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Abstract. We prove boundedness of maximal commutators and convolution operators with generalized Poisson kernels on central Morrey spaces.

1. Introduction

Central Morrey-Campanato spaces have been extensively studied during the last years. We may highlight the contributions made by Chen and Lau [3], García-Cuerva [4], Guliyev [7], Guliyev and Aliyev [8], Lu and Yang [12], and many others.

In this note, we will be mainly interested to prove, on central Morrey spaces, some results related to continuity of maximal commutators and certain convolution operators with kernels that generalize the classical Poisson kernel for the upper half-space $\mathbb{R}^{n+1}_+$. In order to prove the continuity of maximal commutators we will lean on results proved by Komori-Furuya et al. in [9]. Concerning the boundedness of the generalized Poisson transform, called Weinstein transform, basically, we will employ properties of the kernel involved, which has been recently studied by J. Wittsten in [15]. We finally obtain as a corollary the continuity of the Weinstein transform on weighted versions of local Morrey spaces.

We will use standard notation along this work, and as usual, we shall denote by the letter C a constant that could be changing line by line.

2. Preliminaries

The Morrey spaces were introduced by C. Morrey in [14]. Here, we will consider central versions of these spaces. They are defined as follows ([1,3,4,12]): for $1 < p < \infty$

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and $\lambda \in \mathbb{R}$, a function $f \in L^p_{loc}(\mathbb{R}^n)$ belongs to the central Morrey space $B^{p,\lambda}$ if $\|f\|_{B^{p,\lambda}} < \infty$, where

$$\|f\|_{B^{p,\lambda}} := \sup_{r>0} \left(\frac{1}{|B_r(0)|^{1+p\lambda}} \int_{B_r(0)} |f(x)|^p \, dx \right)^{1/p}.$$

It is well known that $\left(B^{p,\lambda}, \|\cdot\|_{B^{p,\lambda}}\right)$ are Banach spaces. We will restrict to the case $-1/p \leq \lambda$, since $B^{p,\lambda}$ reduces to zero for $\lambda < -1/p$. Moreover, if $\lambda < \mu$ then $B^{p,\lambda}$ is properly contained in $B^{p,\mu}$. Also, for $\lambda = -1/p$, $B^{p,-1/p} = L^p$.

An alternative way to describe the spaces $B^{p,\lambda}$ is the following ([1,3,4]): $f \in B^{p,\lambda}$ if and only if

$$\sup_{k\in\mathbb{Z}} 2^{-nk/p(1+p\lambda)} \|f\chi_{C_k}\|_p < \infty, \tag{1}$$

where $C_k = \left\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\right\}$. The quantity in (1) defines an equivalent norm in $B^{p,\lambda}$.

Clearly, for $-1/p < \lambda < 0$, the classical Morrey spaces $L^{p,\lambda}$ on $\mathbb{R}^n$ defined by means of the condition

$$\|f\|_{L^{p,\lambda}} := \sup_{r>0,\ a\in\mathbb{R}^n} \left(\frac{1}{|B_r(a)|^{1+p\lambda}} \int_{B_r(a)} |f(x)|^p \, dx \right)^{1/p}$$

are included in $B^{p,\lambda}$, however, this inclusion is proper as the following example shows.

Let us consider $n = 1$, although appropriate modifications will work for arbitrary n. Define $\varphi(x) = \sum_{k=-\infty}^{\infty} 2^{k\left(\frac{1}{p(1+p\lambda)} - \frac{1}{p}\right)} \chi_{C_k}(x)$. Since for every $k \in \mathbb{Z}$ we have $2^{-k/p(1+p\lambda)} 2^{k\left(\frac{1}{p(1+p\lambda)} - \frac{1}{p}\right)} 2^{k/p} = 1$, it is immediate that (1) is finite, which shows that $\varphi \in B^{p,\lambda}$. However, given $k \in \mathbb{N}$, $k \geq 2$, let I be an interval whose length is 2^{k-1} and it is completely contained in C_k. In this way

$$\begin{aligned} \frac{1}{|I|^{1+p\lambda}} \int_I |\varphi(x)|^p \, dx &= \frac{1}{|I|^{1+p\lambda}} 2^{k\left(\frac{1}{p(1+p\lambda)} - \frac{1}{p}\right)p} |I| \\ &= 2^{(k-1)(-p\lambda)} 2^{k\left(\frac{1}{1+p\lambda} - 1\right)} = 2^{p\lambda} 2^{k\left(-p\lambda + \frac{1}{1+p\lambda} - 1\right)}, \end{aligned} \tag{2}$$

and noticing that $-p\lambda + \frac{1}{1+p\lambda} - 1 > 0$ since $-1/p < \lambda < 0$, we see that the integrals in the left-hand side of (2) grow without bound as $k \to \infty$. This shows that $\varphi \notin L^{p,\lambda}$.

It is also possible to identify the preduals of the spaces $B^{p,\lambda}$, in a similar fashion as the case of Morrey spaces. For $1 \leq p < r \leq \infty$, a function $b : \mathbb{R}^n \to \mathbb{R}$ is called a (p,r)-central block, if there exists $t > 0$ such that $\operatorname{supp} b \subset B_t(0)$ and $\|b\|_r \leq t^{n\left(\frac{1}{r} - \frac{1}{p}\right)}$.

If we define

$$h^0_{p,r} := \left\{ f = \sum_{j=1}^{\infty} \beta_j b_j : b_j \text{ is a } (p,r)\text{-central block and } \|f\|_{h^0_{p,r}} < \infty \right\}, \tag{3}$$

where

$$\|f\|_{h^0_{p,r}} = \inf \sum_{j=1}^{\infty} |\beta_j|, \tag{4}$$

we obtain a Banach space normed by (4). Moreover, the convergence of the series in

(3) is in L^p and absolutely a.e.

With the same proof given in [11] (see also [10]) we can obtain the following result.

Proposition 2.1. *For* $1 \leq p < r' < \infty$, $-\frac{1}{r} < \lambda < 0$, $\frac{1}{r} + \frac{1}{r'} = 1$ *and* $p = \frac{1}{1+\lambda}$ *we have* $\left(h^0_{p,r'}\right)^* \simeq B^{r,\lambda}$.

3. Maximal commutators

In [9], Komori-Furuya et al. studied the continuity of several classical operators on Morrey-Campanato type spaces. In particular, they considered the fractional maximal operator M_α, for $0 \leq \alpha < n$, which is defined as

$$M_\alpha f(x) := \sup_{x\in Q} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)|\, dy,$$

where Q is a cube with sides parallel to the coordinate axes (instead of cubes, we could also consider balls). Notice that for $\alpha = 0$, we recover the classical Hardy-Littlewood maximal function. They proved the following results:

Theorem 3.1. *([9], Theorem 7) For* $1 < p < \infty$, $0 \leq \sigma \leq \frac{n}{p}$, $\frac{\sigma}{n} - \frac{1}{p} \leq \nu \leq 0$, *the Hardy-Littlewood maximal operator is continuous from* $B^{p,\nu}$ *to* $B^{p,v}$.

It is interesting to observe that Theorem 3.1 generalizes the case $\nu = 0$ proved in [4] and [3].

Theorem 3.2. *([9, Theorem 7]) For* $0 < \alpha < n$, $\sigma \geq 0$, $1 < p \leq \frac{n}{\sigma+\alpha}$, $\frac{\sigma}{n} - \frac{1}{p} \leq \nu \leq -\frac{\alpha}{n}$, $1 < q \leq \left(\frac{n\nu-\sigma}{n\nu-\sigma+\alpha}\right)p$, *the fractional maximal operator* M_α *is continuous from* $B^{p,\nu}$ *to* $B^{q,\nu+\alpha/n}$.

Now, we shall use Theorem 3.2 in order to examine the action of the maximal commutator with a Lipschitz function in the spaces $B^{p,\lambda}$. Maximal commutators with a given function have played an important role in the study of continuity properties of some classical operators. We highlight the work done by García-Cuerva et al. in [5].

Given $b \in L^1_{loc}$, the maximal commutator of the Hardy-Littlewood maximal operator M with b is defined as

$$C_b(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |b(x) - b(y)|\, |f(y)|\, dy.$$

If we allow an appropriate smoothness condition on the function b, let us say, that b is a Lipschitz function, that is, for every x, $y \in \mathbb{R}^n$

$$|b(x) - b(y)| \leq C\, |x-y|^\beta \tag{5}$$

where C is a positive constant, and $\beta \in (0,1)$, we can proceed as follows:

$$C_b(f)(x) = \sup_{x\in Q} \frac{1}{|Q|^{1-\frac{\beta}{n}+\frac{\beta}{n}}} \int_Q |b(x) - b(y)|\, |f(y)|\, dy$$

$$\leq C \left\|b\right\|_{\Lambda_\beta} \sup_{x\in Q} \frac{\left|Q\right|^{\frac{\beta}{n}}}{\left|Q\right|^{1-\frac{\beta}{n}} \left|Q\right|^{\frac{\beta}{n}}} \int_Q \left|f\left(y\right)\right| dy$$
$$\leq C \left\|b\right\|_{\Lambda_\beta} M_\beta f\left(x\right), \tag{6}$$

where $\left\|b\right\|_{\Lambda_\beta}$ denotes the Lipschitz norm of b, i.e., the infimum of the constants C satisfying (5).

Now, if we choose $\sigma \geq 0$, $1 < p \leq \frac{n}{\sigma+\beta}$, $\frac{\sigma}{n} - \frac{1}{p} \leq \nu \leq -\frac{\beta}{n}$, $1 < q \leq \left(\frac{n\nu-\sigma}{n\nu-\sigma+\beta}\right)p$, by estimate (6) and Theorem 3.2 we obtain the continuity of $C_b : B^{p,\nu} \to B^{q,\nu+\beta/n}$ and $\left\|C_b\right\|_{B^{p,\nu}\to B^{q,\nu+\beta/n}} \leq C \left\|b\right\|_{\Lambda_\beta}$. Thus, we have proved

Proposition 3.3. *For $0 < \beta < 1$, $b \in \Lambda_\beta$, $\sigma \geq 0$, $1 < p \leq \frac{n}{\sigma+\beta}$, $\frac{\sigma}{n} - \frac{1}{p} \leq \nu \leq -\frac{\beta}{n}$, and $1 < q \leq \left(\frac{n\nu-\sigma}{n\nu-\sigma+\beta}\right)p$, the maximal commutator C_b is continuous from $B^{p,\nu}$ to $B^{q,\nu+\beta/n}$ with norm $\left\|C_b\right\|_{B^{p,\nu}\to B^{q,\nu+\beta/n}} \leq C \left\|b\right\|_{\Lambda_\beta}$.*

4. Weinstein transform

Recently, J. Wittsten [15] has studied boundary values of convolutions of weighted distributions with the kernels $\mathcal{K}_\alpha$, $\alpha > -1$, defined by

$$\mathcal{K}_\alpha\left(x,t\right) := \frac{\Gamma\left(\left(\alpha+n+1\right)/2\right)}{\Gamma\left(\left(\alpha+1\right)/2\right)\pi^{n/2}} \frac{t^{\alpha+1}}{\left(\left|x\right|^2+t^2\right)^{\left(\alpha+n+1\right)/2}}, \text{ for } \left(x,t\right) \in \mathbb{R}^{n+1}_+.$$

These kernels are related to the elliptic partial differential equation

$$D_\alpha u := t^{-\alpha}\left(\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial t^2} - \frac{\alpha}{t}\frac{\partial u}{\partial t}\right) = 0, \tag{7}$$

with $\alpha > -1$. Solutions to (7) are called generalized axially symmetric potentials. Notice that when $\alpha = 0$ we recover the Laplace equation.

In the paper [15], it was proven that $\left\|\mathcal{K}_{\alpha,t}\right\|_1 = 1$, where $\mathcal{K}_{\alpha,t}\left(x\right) := \mathcal{K}_\alpha\left(x,t\right)$, that $\mathcal{K}_\alpha$ is a solution to the equation (7) in $\mathbb{R}^{n+1}_+$, and $\mathcal{K}_{\alpha,t} \to \delta_0$ in $\mathcal{S}'$ as $t \to 0$.

In this section, we will examine the behavior of the family of kernels $\mathcal{K}_{\alpha,t}$ when they act by convolution in the central Morrey spaces B^p, $1 < p < \infty$.

For $\theta > n$, we have the inclusion $B^p \subset L^p\left(\left(1+\left|x\right|^2\right)^{-\theta/2} dx\right)$, as it has been proved in [1, Corollary 2.5]. If we denote by Λ_α the convolution operator with the kernel $\mathcal{K}_{\alpha,t}$, this operator preserves the space $L^1\left(\left(1+\left|x\right|^2\right)^{-\left(\alpha+n+1\right)/2} dx\right)$ (see [15] and also [2, Remark 3.2]). However, we can say more, as the following result shows.

Proposition 4.1. *The operator Λ_α is bounded from $L^p\left(\left(1+\left|x\right|^2\right)^{-\left(\alpha+n+1\right)/2} dx\right)$ to itself for $1 \leq p < \infty$.*

Proof. Let us denote $w_\alpha(x) = (1+|x|^2)^{(\alpha+n+1)/2}$. Using Jensen's inequality, Tonelli theorem and radiality of $\mathcal{K}_{\alpha,t}$ we obtain

$$\begin{aligned}\|\Lambda_\alpha(f)\|^p_{L^p\left(w_\alpha^{-1}\,dx\right)} &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\mathcal{K}_{\alpha,t}(x-y)\,f(y)\,dy\right|^p w_\alpha^{-1}(x)\,dx \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(y)|^p\left(\mathcal{K}_{\alpha,t}*w_\alpha^{-1}\right)(y)\,dy.\end{aligned}$$

The next step is to estimate $\mathcal{K}_{\alpha,t} * w_\alpha^{-1}(y)$. According to [15, pp. 912–913], $\mathcal{K}_{\alpha,t} * w_\alpha^{-1}(y) \le C_{\alpha,n} w_\alpha^{-1}(y)$, where $C_{\alpha,n}$ is a constant only depending on n and α.

Therefore, $\quad \|\Lambda_\alpha(f)\|^p_{L^p\left(w_\alpha^{-1}\,dx\right)} \le C_{\alpha,n}\int_{\mathbb{R}^n}|f(y)|^p\,w_\alpha^{-1}(y)\,dy = C_{\alpha,n}\|f\|^p_{L^p\left(w_\alpha^{-1}\,dx\right)}.$

This concludes the proof. □

Remark 4.2. It is also true that for $f \in L^p\left(w_\alpha^{-1}\,dx\right)$, $1 \le p < \infty$, we have $\mathcal{K}_{\alpha,t} * f \to f$ in $L^p\left(w_\alpha^{-1}\,dx\right)$ as $t \to 0$. The proof of this assertion is basically the same as that given in [15, Theorem 4.3], (see also [2, Theorem 3.6]).

Now, we will prove the desired continuity.

Theorem 4.3. *The Weinstein transform Λ_α is bounded from B^p into itself, $1 < p < \infty$.*

Proof. As in the case of the Poisson kernel, we can obtain the following estimate (see [6, pp. 154 and 177]) for each $t > 0$

$$\begin{aligned}|\mathcal{K}_{\alpha,t} * f(x)| &\le C_{n,\alpha}\Bigg\{\int_{|y-x|\le t}\frac{t^{\alpha+1}\,|f(y)|}{\left(t^2+|y-x|^2\right)^{(\alpha+n+1)/2}}\,dy \\ &\qquad + \sum_{k=0}^{\infty}\int_{2^k t<|y-x|\le 2^{k+1}t}\frac{t^{\alpha+1}\,|f(y)|}{\left(t^2+|y-x|^2\right)^{(\alpha+n+1)/2}}\,dy\Bigg\} \\ &\le C_{n,\alpha}\left\{\frac{1}{t^n}\int_{|y-x|\le t}|f(y)|\,dy + \sum_{k=0}^{\infty}\frac{1}{(2^k t)^n}\int_{|y-x|\le 2^{k+1}t}|f(y)|\,dy\right\} \\ &\le C_{n,\alpha} Mf(x). \end{aligned} \tag{8}$$

Using the fact that $M : B^p \to B^p$ is a continuous operator (Theorem 3.1), we obtain the boundedness of the operator Λ_α from B^p to B^p. □

We can improve the previous result using weighted versions of the central Morrey spaces B^p. These are defined as follows.

Let w be a weight on $\mathbb{R}^n$, that is, $w \in L^1_{loc}$ and $0 < w < \infty$ a.e. on $\mathbb{R}^n$. We will denote by $B^p(w)$ the space $B^p(w) := \left\{f \in L^p_{loc,w} : \|f\|_{B^p(w)} < \infty\right\}$, where $L^p_{loc,w}$ is the space of functions locally in $L^p(w)$ and $\|f\|_{B^p(w)} = \sup_{k\in\mathbb{Z}} w(C_k)^{-1/p}\|f\chi_{C_k}\|_{L^p(w)}$.

Matsuoka has proved in [13]:

Proposition 4.4. *([13]) Let $1 < p < \infty$ and $w \in A_p$. Then, the Hardy-Littlewood maximal operator M is bounded from $B^p(w)$ into $B^p(w)$.*

Here, the set A_p denotes the classical family of weights w satisfying the condition

$$\left(\frac{1}{|B|}\int_B w\left(x\right)\,dx\right)\left(\frac{1}{|B|}\int_B w\left(x\right)^{-1/(p-1)}\,dx\right)^{p-1}\leq C$$

for every ball B and some positive constant C.

In view of Proposition 4.4 and estimate (8) we can obtain

Corollary 4.5. *If $w\in A_p$ then the Weinstein transform Λ_α is bounded from $B^p\left(w\right)$ into $B^p\left(w\right)$.*

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