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STRONG LINEAR PRESERVERS OF UT-TOEPLITZ WEAK MAJORIZATION ON $\mathbb{R}^n$

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Abstract. Let $x, y \in \mathbb{R}^n$, we say x is ut-Toeplitz weak majorized by y (written as $x \prec_{uT} y$) if there exists an upper triangular substochastic Toeplitz matrix A such that $x = Ay$. In this paper, we characterize all linear functions that strongly preserve $\prec_{uT}$ on $\mathbb{R}^n$.

1. Introduction

Majorization is one of the interesting concepts in matrix analysis and there are special researches on it and its linear preservers in recent years. Considering $M_n(\mathbb{R})$ as the space of all real $n \times n$ matrices, $D \in M_n(\mathbb{R})$ is called doubly (sub)stochastic if its entries are all nonnegative and the sum of its entries in each row and column is (less than or) equal to 1. Let $\mathbb{R}^n$ be the vector space of all real $n \times 1$ vectors. For $x, y \in \mathbb{R}^n$, it is said that x is (weak) majorized by y and denoted by $(x \prec_w y) x \prec y$ if there is a doubly (sub)stochastic matrix D such that $x = Dy$. It is well known that $x \prec y$ if and only if $\sum_{j=1}^{k} x_{[j]} \leq \sum_{j=1}^{k} y_{[j]}$, for $k = 1, 2, \ldots, n-1$, and $\sum_{j=1}^{n} x_{[j]} = \sum_{j=1}^{n} y_{[j]}$, and $x \prec_w y$ if and only if $\sum_{j=1}^{k} x_{[j]} \leq \sum_{j=1}^{k} y_{[j]}$, for $k = 1, 2, \ldots, n$, where $x_{[j]}$ is the j^{th} largest element of vector x. For more study see [8].

Definition 1.1. A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is called a linear preserver of a relation $\sim$ on $\mathbb{R}^n$ if for all $x, y \in \mathbb{R}^n$ $x \sim y \Rightarrow Tx \sim Ty$. and it is called a strongly linear preserver of the relation if $x \sim y \Leftrightarrow Tx \sim Ty$.

There are some researches on characterization of linear or nonlinear preservers of special kinds of (weak) majorization. For example, in [1, 3] authors have characterized strong linear preservers and linear preservers of g-tridiagonal majorization

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respectively. In [10] authors have characterized strong linear preservers and linear preservers of circulant majorization. In [9] authors have characterized nonlinear preserver of some special weak majorization, and also in [2, 4, 5, 7] authors have characterized linear preservers of some other special majorizations.

In this paper we introduce ut-Toeplitz weak majorization and characterize all linear maps that strongly preserve upper triangular Toeplitz weak majorization. Actually this kind of majorization is a particular case of that introduced by Ilkhanizadeh Manesh in [6].

2. Preliminaries and notations

The k^{th} diagonal of a matrix $A = [a_{i,j}]$ is the collection of entries $a_{i,j}$ where $j-i=k$. The 0^{th} diagonal of a matrix is known as the main diagonal. A matrix A is called Toeplitz if all entries of each diagonal are equal. We denote a Toeplitz matrix by $A = [a_{-(n-1)} \setminus \cdots \setminus a_0 \setminus a_1 \setminus \cdots a_{n-1}]$ where a_i is the amount of the i^{th} diagonal, and if the Toeplitz matrix is upper triangular we use the notation $A = [a_0 \setminus a_1 \setminus \cdots a_{n-1}]$.

Definition 2.1. Let $x, y \in \mathbb{R}^n$. We say that x is ut-Toeplitz weak majorized by y (written as $x \prec_{uT} y$) if there exists an upper triangular substochastic Toeplitz matrix $D \in M_n(\mathbb{R})$ such that $x = Dy$.

For $x \in \mathbb{R}^n$ we use the notation $x \geq 0$ if all entries of x are nonnegative. Obviously if x is weak majorized by y and $y \geq 0$, then $x \geq 0$. Also if $x \prec_{uT} 0$, then $x = 0$.

We use $\phi(x)$ for the vector space generated by $\{y \in \mathbb{R}^n : y \prec_{uT} x\}$. Also the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is identified with its matrix representation under the canonical basis, $e_1, \ldots e_n$, in $\mathbb{R}^n$.

In this paper we also use the following special upper triangular substochastic Toeplitz matrices.

$$U_0 = I,\ U_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ & 0 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & 0 & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \ldots, U_{n-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ & 0 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ & 0 & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

Actually every upper triangular substochastic Toeplitz matrix is the form of $\sum_{i=0}^{n-1} c_i U_i$, where $0 \leq c_i \leq 1$ and $\sum_{i=0}^{n-1} c_i \leq 1$.

3. Linear preservers of ut-Toeplitz majorization

We start this section by stating some preliminaries and properties of ut-Toeplitz weak majorization on $\mathbb{R}^n$. We will use these properties to prove our main theorem. The following lemma describes vectors that are ut-Toeplitz weak majorized by some special vectors in $\mathbb{R}^n$.

LEMMA 3.1. *Let* $x, y \in \mathbb{R}^n$ *and* $x \prec_{uT} y$. *If* $y \geq 0$ *and* k *is the largest index such that* $y_k \neq 0$, *then:*
(*i*) $x_i = 0, \quad \forall i > k$; (*ii*) $\sum_{i=l}^{k} x_i \leq \sum_{i=l}^{k} y_i \quad \forall 1 \leq l \leq k$.

Proof. Since $x \prec_{uT} y$ there is a substochastic upper triangular Toeplitz matrix $T = [t_0 \setminus t_1 \setminus \cdots t_{n-1}]$ such that $x = Ty$.

Obviously $x_i = 0$ for each $i \geq k$ and $x_j = \sum_{i=1}^{k-j+1} t_{i-1} y_{i+j-1}$. Considering $\sum_{i=1}^{n} t_{i-1} \leq 1$, we have

$$\sum_{i=l}^{k} x_i = t_0 y_l + \cdots + t_{k-l} y_k + \cdots + t_0 y_{k-1} + t_1 y_k + t_0 y_k$$

$$= t_0 y_l + (t_0 + t_1) y_{l+1} + \cdots + (\sum_{i=1}^{k-l+1} t_{i-1}) y_k \leq \sum_{i=l}^{k} y_i. \qquad \square$$

LEMMA 3.2. *Let* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. *Then* k *is the largest index that* $x_k \neq 0$ *if and only if* $\phi(x) = \langle e_1, \ldots, e_k \rangle$.

Proof. Let k be the largest index such that $x_k \neq 0$. We know $U_i x \prec_{uT} x$. Since $x_j = 0$ for each $j > k$, $U_0 x = x_1 e_1 + \cdots + x_k e_k, \ldots, U_{k-2} x = x_{k-1} e_1 + x_k e_2, U_{k-1} x = x_k e_1$ and $U_j x = 0, \forall j \geq k$. Hence $\phi(x)$ contains $e_1, \ldots, e_k$, which means $\langle e_1, \ldots, e_k \rangle \subseteq \phi(x)$. On the other hand by part (i) of Lemma 3.1 if $y \prec_{uT} x$, then $y_i = 0$ for each $i > k$, which means that each $y \in \phi(x)$ is a linear combination of $e_1, \ldots, e_k$. Hence $\phi(x) = \langle e_1, \ldots, e_k \rangle$. Proof of the converse is obvious. $\square$

LEMMA 3.3. *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be a linear map strongly preserves* $\prec_{uT}$. *Then* T *is an invertible upper triangular matrix.*

Proof. First we prove that T is invertible. Let $Tx = 0$. Since T is a linear operator $T(0) = 0 = T(x)$. Considering that T strongly preserves $\prec_{uT}$, implies $x \prec_{uT} 0$. Hence $x = 0$.

To prove that T is an upper triangular matrix we apply the induction principle. By Lemma 3.2 we know that $\phi(e_1) = \langle e_1 \rangle$. Since T is invertible, $dim T\phi(e_1) = dim T\langle e_1 \rangle = 1$. Since T strongly preserves $\prec_{uT}$, we have

$$T\phi(e_1) = \langle \{Tx : x \prec_{uT} e_1\} \rangle = \langle \{Tx : Tx \prec_{uT} Te_1\} \rangle = \phi T((e_1)).$$

Hence considering $dim T\phi(e_1) = 1$ and Lemma 3.2, we have $Te_1 = (a_{1,1}, 0, \cdots, 0)^t$.

Suppose that $Te_i = (a_{1,i}, \ldots, a_{i,i}, 0, \ldots, 0)^t$, for each $i < k$. Now we prove for k. By lemma 3.2 we have $\phi(e_k) = \langle e_1, \ldots, e_k \rangle$. Since T is invertible, $dim T\phi(e_k) = dim T\langle e_1, \cdots, e_k \rangle = k$. Obviously $e_i \prec_{uT} e_k$, for each $i < k$, hence $e_1, \ldots, e_{k-1} \in \phi(e_k)$, that means $T\langle e_1, \ldots, e_{k-1} \rangle \subseteq T\phi(e_k)$.

Considering the hypothesis of induction, we have $Te_i = (a_{1,i}, \ldots, a_{i,i}, 0, \ldots, 0)^t$, for each $i < k$, which means $T\langle e_1, \ldots, e_{k-1} \rangle = \langle e_1, \ldots, e_{k-1} \rangle$. Now if the index of the largest nonzero entry of Te_k is less than k, then $Te_k \in T\langle e_1, \ldots, e_{k-1} \rangle$ and consequently $dimT\phi(e_k) < k$ that is not true. On the other hand let the index of the largest nonzero entry of Te_k be greater than k. Since T strongly preserves $\prec_{uT}$, $T\phi(e_k) = \langle\{Tx : x \prec_{uT} e_k\}\rangle = \langle\{Tx : Tx \prec_{uT} Te_k\}\rangle = \phi(T(e_k))$ which implies $dimT\phi(e_k) > k$ that is impossible.

Hence $Te_k = (a_{1,k}, \ldots, a_{k,k}, 0, \ldots, 0)^t$ for each $1 \leq k \leq n$, that means T is an upper triangular matrix. □

THEOREM 3.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. If T is an upper triangular Toeplitz matrix then T preserves $\prec_{uT}$. Moreover T strongly preserves $\prec_{uT}$ if and only if T is an invertible upper triangular Toeplitz matrix.*

Proof. Let T be an upper triangular Toeplitz matrix and $\mathcal{T}_n$ be the set of all nonsingular, upper triangular Toeplitz matrices of size n. It is well known that $\mathcal{T}_n$ is an Abelian group. Let $T \in \mathcal{T}_n$ and $x, y \in \mathbb{R}^n$ be such that $x \prec_{uT} y$. Then $x = Dy$ for some substochastic matrix $D \in \mathcal{T}_n$. We obtain $Tx = TDy = DTy$ so that $Tx \prec_{uT} Ty$, that is T is a linear preserver of $\prec_{uT}$.

Now let T be an invertible upper triangular Toeplitz matrix. To prove T strongly preserves $\prec_{uT}$, it suffices to show that if $Tx \prec_{uT} Ty$, then $x \prec_{uT} y$. $Tx \prec_{uT} Ty$ implies $Tx = DTy$ for some substochastic matrix $D \in \mathcal{T}_n$, hence $Tx = TDy$. Since T is invertible we have $x = Dy$, hence $x \prec_{uT} y$, and the proof is complete.

To prove the converse of the theorem, let T strongly preserves $\prec_{uT}$. Then by Theorem 3.3, T is an invertible upper triangular matrix. To show T is Toeplitz, first we show that all entries on the main diagonal are equal. Since T is an invertible upper triangular matrix $a_{i,i} \neq 0$, for each $1 \leq i \leq n$. We assume that $a_{n,n} > 0$ (proof for the case $a_{n,n} < 0$ is similar). Consider an arbitrary natural number $1 \leq k \leq n$. Obviously $e_k \prec_{uT} e_n$, hence $Te_k \prec_{uT} Te_n$ that means there is an upper triangular substochastic Toeplitz matrix $W = [w_0 \setminus \cdots \setminus w_{n-1}]$ such that $Te_k = WTe_n$.

$$\begin{aligned} Te_k &= (a_{1,k}, a_{2,k}, \ldots, a_{k,k}, 0, \ldots, 0)^t \\ &= (\sum_{j=1}^{n} w_{j-1}a_{j,n}, \ldots, \sum_{j=1}^{n-k+1} w_{j-1}a_{j+k-1,n}, \ldots, w_0 a_{n,n})^t \end{aligned} \tag{1}$$

We have $w_0 a_{n,n} = 0$. Since $a_{n,n} \neq 0$, we obtain $w_0 = 0$. Considering the $(n-1)th$ entry of WTe_n, i.e. $w_0 a_{n-1,n} + w_1 a_{n,n} = w_1 a_{n,n} = 0$, implies $w_1 = 0$. Continuing this process we have $w_{i-1} = 0$ for each $1 \leq i \leq n-k$. Consequently the k^{th} entry of TWe_n is equal to $w_{n-k}a_{n,n}$. Hence by the equation (1) we have $a_{kk} = w_{n-k}a_{n,n}$ which implies that $a_{k,k} \leq a_{n,n}$.

Since T is onto, there is $y \in \mathbb{R}^n$ such that $Ty = U_k Te_n$. Also since T strongly preserves $\prec_{uT}$ and $Ty \prec_{uT} Te_n$, we have $y \prec_{uT} e_n$. Hence there is an upper triangular substochastic Toeplitz matrix $W = [w_0 \setminus \cdots \setminus w_{n-1}]$ such that $y = We_n$ which implies that $U_k Te_n = Ty = TWe_n$. We have the following equation:

$$(a_{k,n}, \ldots, a_{n,n}, 0, \ldots, 0)^t = (\sum_{j=1}^{n} a_{1,j}w_{n-j}, \ldots, \sum_{j=k}^{n} a_{k,j}w_{n-j}, \ldots, a_{n,n}w_0)$$

Like the above argument we have $w_{i-1} = 0$ for each $1 \leq i \leq n-k$ and hence

$a_{n,n} = w_{n-k}a_{kk}$ which implies that $a_{n,n} \le a_{k,k}$. Hence we proved $a_{k,k} = a_{n,n}$ for each $1 \le k \le n$.

Suppose that the entries of ith diagonal for each $1 \le i \le k$ are all equal to a constant number a_i. We show that the entries of $(k+1)th$ diagonal are equal. To reach this aim we show that $a_{n-k,n} = a_{j-k,j}$ for each $k+1 \le j \le n-1$. We know $Te_j \prec_{uT} Te_n$, hence $(a_{1,j}, \dots, a_{j-k,j}, a_k, \dots, a_1, 0, \dots, 0)^t \prec_{uT} (a_{1,n}, \dots, a_{n-k,n}, a_k, \dots, a_1)^t$ for $j \ge k+1$. Hence we have $w_0a_1 = 0$. Since T is invertible, $a_1 \ne 0$ and this implies $w_0 = 0$. In a similar way we have $w_0a_2 + w_1a_1 = 0$ which implies $w_1 = 0$ and continuing this process we have $w_{i-1} = 0$ for $1 \le i \le n-j$. Now we have $w_0a_{j,n} + w_1a_{j+1,n} + \cdots + w_{n-j-1}a_2 + w_{n-j}a_1 = a_1$ hence $w_{n-j} = 1$. Also $w_0a_{j-1,n} + w_1a_{j,n} + \cdots + w_{n-j}a_2 + w_{n-j+1}a_1 = a_2$ which implies $w_{n-j+1} = 0$. Again continuing this process we have $w_{n-j} = \cdots = w_{n-j+k-1} = 0$. Hence $W = [0 \setminus \cdots \setminus 0 \setminus 1 \setminus 0 \setminus \cdots \setminus 0 \setminus w_{n-j+k} \setminus w_{n-1}]$, where 1 is in $(n-j)$th position. Now we have $w_0a_{j-k,n} + \cdots + w_{n-j}a_{n-k,n} + w_{n-j+1}a_k + \cdots + w_{n-j+k}a_1 = a_{j-k,j}$. Hence $a_{n-k,n} + w_{n-j+k}a_1 = a_{j-k,j}$, which implies

$$a_{n-k,n} \le a_{j-k,j} \tag{2}$$

Since T is onto, there is $y \in \mathbb{R}^n$ such that $Ty = U_tTe_n$, where $1 \le t \le n-k$. Since T strongly preserves $\prec_{uT}$ and $Ty \prec_{uT} Te_n$, we have $y \prec_{uT} e_n$. Hence there is an upper triangular substochastic Toeplitz matrix $W = [w_0 \setminus \cdots \setminus w_{n-1}]$ such that $y = We_n$. Consequently $U_kTe_n = Ty = TWe_n$. We have:

$$TWe_n = \begin{pmatrix} \sum_{j=1}^{k} a_jw_{n-j+1} + \sum_{j=k+1}^{n} a_{1,j}w_{n-j+1} \\ \sum_{j=2}^{k+1} a_{j-1}w_{n-j+1} + \sum_{j=k+2}^{n} a_{2,j}w_{n-j+1} \\ \vdots \\ a_1w_{k+1} + a_2w_k + \cdots + a_kw_2 + a_{n-k,n}w_1 \\ a_1w_k + a_2w_{k-1} + \cdots + a_kw_1 \\ \vdots \\ a_1w_2 + a_2w_1 \\ a_1w_1 \end{pmatrix} = \begin{pmatrix} a_{tn} \\ \vdots \\ a_{n-k,n} \\ a_k \\ \vdots \\ a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = U_tTe_n$$

Since $a_1w_1 = 0$ implies $w_1 = 0$ and $a_1w_2 + a_2w_1 = 0$ implies $w_2 = 0$, continuing this process, we have $w_1 = \cdots = w_{t-1} = 0$. Now $a_1w_t + a_2w_{t-1} + \cdots + a_kw_{t-k+1} + a_{n-t+1,n-t+k+1}w_{t-k} + \cdots + a_{n-t+1,n}w_1 = a_1$. Hence $w_t = 1$ and like the above argument we conclude $w_{t+1} = \cdots = w_{t+k-1} = 0$. We have $a_1w_{t+k} + a_2w_{t+k-1} + \cdots + a_kw_{t+1} + a_{n-t-k+1,n-t+1}w_t + \cdots + a_{n-t-k+1,n}w_1 = a_{n-k,n}$. Hence $a_{n-t-k+1,n-t+1} \le a_{n-k,n}$. If we consider $j = n-t+1$, then

$$a_{j-k,j} \le a_{n-k,n}. \tag{3}$$

By inequalities (2) and (3) we have $a_{j-k,j} = a_{n-k,n}$, $\forall k+1 \le j \le n$, and the proof is completed. □

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