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FILTERED LAGRANGIAN FLOER HOMOLOGY OF PRODUCT MANIFOLDS

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Abstract. In this note we construct a commutative diagram in filtered Lagrangian Floer homology that involves a product of certain Lagrangian submanifolds. As a corollary, we prove the Künneth formula for Lagrangian Floer homology. We also prove that the Künneth formula for Lagrangian Floer homology lifts through a Lagrangian type of Piunikhin-Salamon-Schwarz map to the Künneth formula for Morse homology.

1. Introduction

Floer [5] introduced Lagrangian Floer homology as an infinite dimensional version of Morse homology. He used this theory to prove the Arnol'd conjecture. This homology was further developed by Oh in [18] and Fukaya, Oh, Ohta and Ono [6,7] in a very general situation. It is well known that, in the case when it is defined, Lagrangian Floer homology neither depends on the choice of a Hamiltonian nor of a compatible almost complex structure. The existence of the action functional allows us to define filtered Lagrangian Floer homology as a homology of an appropriate (filtered) subcomplex. However, this homology does depend on the choice of a Hamiltonian. Filtered homology plays an important role in the study of Hamiltonian spectral invariants (see [4, 13, 15, 18, 19, 23]).

The Künneth formula is a well known and useful algebraic tool. Here, we prove this formula in the Lagrangian Floer homology context (we work with $\mathbb{Z}_2$ coefficients) as a corollary of our main theorem. Our proof is rather elementary. The existence of an appropriate commutative diagram between the chain maps (see Theorem 1.3) and the direct limit as our parameters tend to infinity gives us the Künneth formula in Lagrangian Floer homology. This formula was also proved by Li [16] in the monotone case and by Amorim [2] in the more general Fukaya-Oh-Ohta-Ono setup. Both of them, Li and Amorim, used the framework of spectral sequences.

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Before stating the main result we define the class of Lagrangian submanifolds we are considering in this paper.

Definition 1.1. Let (M, ω_M) be a closed symplectic manifold and let L_M be a closed Lagrangian submanifold of M. We say that L_M is relatively symplectically aspherical if

$$[\omega_M]|_{\pi_2(M,L_M)} = 0, \quad \mu_{L_M}|_{\pi_2(M,L_M)} = 0, \tag{1}$$

where μ_{L_M} is the Maslov class.

One very nice example of a relatively symplectically aspherical submanifold is the subtorus $L = \mathbb{T}^k \times \{0\}$ of the torus $\mathbb{T}^{2k}$. In this case (1) holds since $\pi_2(\mathbb{T}^{2k}, L) = 0$. More examples of such submanifolds can be derived using the plumbing construction (see [24]).

Remark 1.2. All our statements also hold in the monotone case. We say that a Lagrangian submanifold is monotone if it satisfies $[\omega_M]|_{\pi_2(M,L_M)} = \tau\mu_{L_M}|_{\pi_2(M,L_M)}$ for some positive constant τ. In that case the PSS map (see Section 4 for the definition) is only a homomorphism in general. For the sake of brevity, we use the stronger assumption (1).

Throughout this paper we assume that $L_M \subset M$ and $L_N \subset N$ are relatively symplectically aspherical submanifolds of closed symplectic manifolds (M, ω_M) and (N, ω_N). Their product $L := L_M \times L_N$ is a relatively symplectically aspherical submanifold of $(P = M \times N, \omega_M \oplus \omega_N)$ (see Subsection 2.4 for the precise definition).

Let $H : M \times [0, 1] \to \mathbb{R}$ be a smooth Hamiltonian such that the intersection $L_M \cap \phi_H^1(L_M)$ is transversal. The first condition in (1) gives us a well defined action functional (see Subsection 2.1). The latter provides a filtration on the Floer homology for Lagrangian intersections, $CF^{(-\infty,a)}(L_M, \phi_H^1(L_M))$ (see Subsection 2.2). We also pick a smooth time-dependent Hamiltonian K on the manifold N such that the intersection $L_N \cap \phi_K^1(L_N)$ is transverse.

The following theorem is the main result of this paper.

Theorem 1.3. *For all $a, b \in \mathbb{R}$ there exists a commutative diagram*

$$\begin{tikzcd}
\bigoplus_{r+s=p} CF_r^{(-\infty,a)}(L_M, \phi_H^1(L_M)) \otimes CF_s^{(-\infty,b)}(L_N, \phi_K^1(L_N)) \arrow[r, "\Psi^{a,b}"] \arrow[d, "\iota_M^{a,\widetilde{a}} \otimes \iota_N^{b,\widetilde{b}}"'] & CF_p^{(-\infty,a+b)}(L, \phi^1_{H\oplus K}(L)) \arrow[d, "\iota_P^{a+b,\widetilde{a}+\widetilde{b}}"] \arrow[dl, "\Phi^{a+b,\widetilde{a},\widetilde{b}}"'] \\
\bigoplus_{r+s=p} CF_r^{(-\infty,\widetilde{a})}(L_M, \phi_H^1(L_M)) \otimes CF_s^{(-\infty,\widetilde{b})}(L_N, \phi_K^1(L_N)) \arrow[r, "\Psi^{\widetilde{a},\widetilde{b}}"] & CF_p^{(-\infty,\widetilde{a}+\widetilde{b})}(L, \phi^1_{H\oplus K}(L)),
\end{tikzcd} \tag{2}$$

where $\widetilde{a} \geq \max\{a, a+b-\min \operatorname{Crit} \mathcal{A}_K\}$ and $\widetilde{b} \geq \max\{b, a+b-\min \operatorname{Crit} \mathcal{A}_H\}$. The vertical arrows are inclusions of chain complexes and the maps $\Psi^{\cdot,\cdot}$ and $\Phi^{\cdot,\cdot,\cdot}$ are chain maps.

We provide explicit definitions of the maps $\Psi^{\cdot,\cdot}$ and $\Phi^{\cdot,\cdot,\cdot}$ in Section 3 and prove Theorem 1.3 therein. The existence of the commutative diagram (2) is potentially important in the theory of spectral invariants.

Since Ψ and Φ are chain maps there is a commutative diagram in homology

$$\begin{tikzcd}
H_p\left[CF^a(L_M)\otimes CF^b(L_N)\right] \arrow[r, "\Psi_*^{a,b}"] \arrow[d, "(\imath_M^{a,\widetilde{a}}\otimes\imath_N^{b,\widetilde{b}})_*"'] & HF_p^{a+b}(L) \arrow[d, "\imath_{P*}^{a+b,\widetilde{a}+\widetilde{b}}"] \arrow[dl, "\Phi_*^{a+b,\widetilde{a},\widetilde{b}}"'] \\
H_p\left[CF^{\widetilde{a}}(L_M)\otimes CF^{\widetilde{b}}(L_N)\right] \arrow[r, "\Psi_*^{\widetilde{a},\widetilde{b}}"] & HF_p^{\widetilde{a}+\widetilde{b}}(L).
\end{tikzcd}$$

For brevity, we use the notation $CF^a(L_M)$ instead of $CF^{(-\infty,a)}(L_M,\phi^1_H(L_M))$ and $HF_p^{a+b}(L)$ instead of $HF_p^{(-\infty,a+b)}(L,\phi^1_{H\oplus K}(L))$. Taking the direct limit as $a\to+\infty$ and $b\to+\infty$ we obtain the diagram

$$\begin{tikzcd}
\varinjlim_{a\to+\infty}\varinjlim_{b\to+\infty} H_p\left[CF^a(L_M)\otimes CF^b(L_N)\right] \arrow[r, "\Psi_*"] \arrow[d, "\cong"'] & HF_p(L,\phi^1_{H\oplus K}(L)) \arrow[d, "\cong"] \arrow[dl, "\Phi_*"'] \\
\varinjlim_{\widetilde{a}\to+\infty}\varinjlim_{\widetilde{b}\to+\infty} H_p\left[CF^{\widetilde{a}}(L_M)\otimes CF^{\widetilde{b}}(L_N)\right] \arrow[r, "\Psi_*"] & HF_p(L,\phi^1_{H\oplus K}(L)).
\end{tikzcd}$$

Since we work with $\mathbb{Z}_2$ coefficients and our homology groups are finite dimensional vector spaces the direct limit commutes with homology. For parameters a and b big enough the vertical arrows are therefore induced by the identity and so in homology they are isomorphisms. Thus, the diagonal arrow is an isomorphism, as well as $\Psi_*^{a,b}$, for a and b big enough. As a corollary of this commutative diagram we obtain the Künneth formula for Lagrangian Floer homology.

Corollary 1.4. *Let (M,ω_M) and (N,ω_N) be closed symplectic manifolds and $L_M\subset M$, $L_N\subset N$ closed relatively symplectically aspherical Lagrangian submanifolds. For generic Hamiltonians $H\in C^\infty(M\times[0,1])$ and $K\in C^\infty(N\times[0,1])$ it holds*

$$HF_p(L_M\times L_N,\phi^1_{H\oplus K}(L_M\times L_N))\cong\bigoplus_{r+s=p}HF_r(L_M,\phi^1_H(L_M))\otimes HF_s(L_N,\phi^1_K(L_N)).$$

This isomorphism provides us some examples of non-displaceable Lagrangian submanifolds.

Definition 1.5. We say that a closed Lagrangian submanifold L_M is displaceable if there exists a Hamiltonian diffeomorphism $\phi_H : M\to M$ such that $L_M\cap\phi_H(L_M)=\emptyset$.

We know that the Floer homology of a displaceable Lagrangian submanifold equals zero. Combining this with Corollary 1.4 we obtain non-displaceable Lagrangian submanifolds.

Corollary 1.6. *Given symplectic manifolds with Lagrangian submanifolds as in Corollary 1.4, if L_M is not displaceable in M, then $L_M\times L_N$ is not displaceable in $M\times N$.*

Specially, the Lagrangian subtorus $\mathbb{T}^k\times\{0\}$ is not displaceable in $\mathbb{T}^{2k}$. It follows because $\mathbb{T}^1\times\{0\}$ is not displaceable in $\mathbb{T}^2$ ($\mathbb{T}^1\times\{0\}$ has non-vanishing Floer homology).

A commutative diagram similar to the one from Theorem 1.3 was obtained by Oancea [17]. Oancea proved the Künneth formula for Floer homology for periodic

Hamiltonian orbits for manifolds with restricted contact type boundary. The Künneth formula was also discussed in [9].

In general, Lagrangian Floer homology is not isomorphic to the singular homology of the Lagrangian submanifold (or to Morse homology, which is isomorphic to the singular one). In [1] Albers constructed, for certain degrees, homomorphisms between these two homologies. He generalized the well known construction carrried out by Piunikhin, Salamon and Schwarz [20] (see also [3,10–12] for other generalisations). We recall the definition of Albers' homomorphism (which we denote by PSS throughout the rest of the paper) in Section 4. In our setup, the PSS homomorphisms are actually isomorphisms. Using the direct sum of Morse functions Schwarz [22] proved the Künneth formula for Morse homology (see Section 4 for details).

The next theorem states that the PSS isomorphism lifts the Künneth formula in Morse homology to the Künneth formula in Lagrangian Floer homology.

THEOREM 1.7. *There exists a commutative diagram*

$$\begin{tikzcd} \bigoplus_{r+s=p} HF_r(L_M, \phi^1_H(L_M)) \otimes HF_s(L_N, \phi^1_K(L_N)) \arrow[r, "\Psi_*"] & HF_p(L, \phi^1_{H\oplus K}(L)) \\ \bigoplus_{r+s=p} HM_r(f_M) \otimes HM_s(f_N) \arrow[r, "\overline{\Psi}_*"] \arrow[u, "PSS_*^{L_M} \otimes PSS_*^{L_N}"] & HM_p(f_M \oplus f_N), \arrow[u, "PSS_*^{L}"'] \end{tikzcd} \tag{3}$$

where $PSS^{\cdot}$ denotes the PSS isomorphism of the appropriate Lagrangian submanifold and $\overline{\Psi}_*$ is an isomorphism of Künneth type in Morse homology.

We prove this theorem in Section 4. Leclercq [14] proved the same statement but in a different context. He proved that the PSS isomorphism agrees with the Künneth formula in Floer homology of periodic orbits.

2. Definitions

Let $H : M \times [0,1] \to \mathbb{R}$ be a smooth Hamiltonian, and X_H the corresponding Hamiltonian vector field, defined by $\omega_M(X_H, \cdot) = -dH(\cdot)$.

Denote by ϕ^t_H the family of symplectomorphisms, $\phi_H : M \times [0,1] \to M$, which represents the flow of the vector field X_H, $\frac{d}{dt}\phi^t_H(x) = X_H(\phi^t_H(x))$, $\phi^0_H = \mathrm{Id}$. Throughout this paper we assume that L_M is transverse to $\phi^1_H(L_M)$.

We also pick an almost complex structure J^M which is ω_M-compatible, meaning that $\omega_M(\cdot, J^M\cdot)$ defines Riemannian metric. In addition, the pair (H, J^M) satisfies a regularity condition. We say that a pair (H, J^M) is a regular pair if the linearization of the perturbed Cauchy-Riemann operator at every holomorphic strip with Lagrangian boundary condition is surjective (see [5]).

2.1 Action functional

Define the action functional $\mathcal{A}_H : \Omega_0(M, L_M) \to \mathbb{R}$ on

$$\Omega_0(M, L_M) := \{\gamma \in C^\infty([0,1], M) \mid \gamma(0), \gamma(1) \in L_M,\ [\gamma] = 0 \in \pi_1(M, L_M)\}$$

as
$$\mathcal{A}_H(\gamma) := -\iint_{\mathbb{D}^2_+} h^*\omega + \int_0^1 H(\gamma(t),t)\,dt. \tag{4}$$

Here, an extension h is any map from the upper half-disc
$$h : \mathbb{D}^2_+ = \{z \in \mathbb{C} \mid |z| \le 1, \mathtt{Im} z \ge 0\} \to M,$$
such that $h(e^{i\pi t}) = \gamma(t)$ and $h(t) \in L_M$ for $t \in [-1,1]$. Since $[\omega_M]|_{\pi_2(M,L_M)} = 0$, the first integral in (4) does not depend on the extension h.

2.2 (Filtered) Lagrangian Floer homology

The set of critical points of $\mathcal{A}_H$, denoted by $\operatorname{Crit}\mathcal{A}_H = \{\gamma \in \Omega_0(M,L_M) \mid \dot\gamma = X_H(\gamma)\}$, is in one-to-one correspondence with the set $L_M \cap \phi^1_H(L_M)$ (which is finite). The condition $\mu_{L_M}|_{\pi_2(M,L_M)} = 0$ ensures that the relative Maslov index $\mu_{L_M}(x)$ is well defined for all $x \in \operatorname{Crit}\mathcal{A}_H$ (see [21] for details). For $x, y \in \operatorname{Crit}\mathcal{A}_H$ we form the moduli space
$$\mathcal{M}(x,y;H,J^M) := \left\{ u : \mathbb{R}\times[0,1] \to M \;\middle|\; \begin{array}{l} \partial_s u + J^M(\partial_t u - X_H(u)) = 0, \\ u(\mathbb{R}\times\{0\}), u(\mathbb{R}\times\{1\}) \subseteq L_M, \\ u(-\infty,t) = x(t),\, u(+\infty,t) = y(t) \end{array} \right\}.$$
For a generic choice of the pair (H, J^M) these moduli spaces are smooth manifolds of dimension $\mu_{L_M}(x) - \mu_{L_M}(y)$ carrying a free $\mathbb{R}-$action if $x \neq y$. We set $\mathcal{M}(x,y;H,J^M)_{[d]}$ to be the union of the $d-$dimensional components. The moduli space $\widehat{\mathcal{M}}(x,y;H,J^M)_{[d-1]} := \mathcal{M}(x,y;H,J^M)_{[d]}/\mathbb{R}$ (modulo $\mathbb{R}$-action) is compact if $d = 1$. If $d = 2$ it is compact up to a simple breaking, i.e. it admits a compactification (denoted by the same symbol) such that the boundary decomposes as follows $\partial\widehat{\mathcal{M}}(x,z;H,J^M)_{[1]} = \bigcup_{y\in\operatorname{Crit}\mathcal{A}_H} \widehat{\mathcal{M}}(x,y;H,J^M)_{[0]} \times \widehat{\mathcal{M}}(y,z;H,J^M)_{[0]}$, see [5] for details.

The Floer complex $CF_k(L_M, \phi^1_H(L_M); H, J^M)$ is generated over $\mathbb{Z}_2$ by the set of Hamiltonian chords (i.e. by the set of critical values of the action functional)
$$CF_k(L_M, \phi^1_H(L_M); H, J^M) := \mathbb{Z}_2\langle\, x \in \operatorname{Crit}\mathcal{A}_H \mid \mu_{L_M}(x) = k\,\rangle.$$
The boundary operator ∂ is defined on generators $x \in CF_k(L_M, \phi^1_H(L_M); H, J^M)$ by $\partial(x) := \sum_{y\in\operatorname{Crit}\mathcal{A}_H} \sharp_2\widehat{\mathcal{M}}(x,y;H,J^M)_{[0]}\cdot y$, where $\sharp_2\widehat{\mathcal{M}}(x,y;H,J^M)_{[0]}$ denotes the (mod 2) number of elements in $\widehat{\mathcal{M}}(x,y;H,J^M)_{[0]}$. The Lagrangian Floer homology groups are $HF_*(L_M,\phi^1_H(L_M)) := H_*(CF(L_M,\phi^1_H(L_M);H,J^M),\partial)$. Floer homology does not depend on a generic choice of compatible almost complex structure and it is invariant under Hamiltonian perturbations, i.e. $HF_*(L_M,\phi^1_H(L_M)) \cong HF_*(L_M,\phi^1_{H'}(L_M))$ for any two Hamiltonians H, H' as far as $\phi^1_H(L_M)$ and $\phi^1_{H'}(L_M)$ are transverse to L_M.

Since the action functional decreases along the perturbed holomorphic strips $u \in \mathcal{M}(x,y;H,J^M)$, the differential ∂ preserves the filtration given by $\mathcal{A}_H$. Thus we can define filtered Lagrangian Floer homology as
$$CF_k^{(-\infty,\lambda)}(L_M,\phi^1_H(L_M)) := \mathbb{Z}_2\langle\, x \in \operatorname{Crit}\mathcal{A}_H \mid \mu_{L_M}(x) = k,\, \mathcal{A}_H(x) < \lambda\,\rangle,$$
$$\partial^\lambda := \partial|_{CF_*^{(-\infty,\lambda)}(L_M,\phi^1_H(L_M))},$$
$$HF_*^{(-\infty,\lambda)}(L_M,\phi^1_H(L_M)) := H_*(CF^{(-\infty,\lambda)}(L_M,\phi^1_H(L_M)),\partial^\lambda).$$

2.3 Morse homology

We only sketch the construction of the Morse homology and we refer to [22] for details. Let X be a closed, smooth manifold and $f : X \to \mathbb{R}$ a Morse function. For a critical point p of f, $m_f(p)$ denotes its Morse index. We define the Morse chain complex as the $\mathbb{Z}_2$-vector space generated by the set of critical points of f

$$CM_k(f) = \mathbb{Z}_2\langle\, p \,|\, p \in \mathrm{Crit}(f),\, m_f(p) = k\,\rangle.$$

It is graded by the Morse index of a critical point. The boundary operator is given by $\partial_M(p) = \sum_{q\in CM_*(f)} n(p,q)\, q$, where $n(p,q)$ is the (mod 2) number of gradient trajectories $\gamma : \mathbb{R} \to X$, $\dot\gamma = -\nabla f(\gamma)$, such that $\gamma(-\infty) = p$ and $\gamma(+\infty) = q$. Morse homology is defined as the homology of the complex CM_* with respect to the boundary operator ∂_M $HM_*(f) = H_*(CM_*(f), \partial_M)$.

2.4 Product of symplectic manifolds

Let (M, ω_M) and (N, ω_N) be closed symplectic manifolds and $L_M \subset M$, $L_N \subset N$ its closed Lagrangian submanifolds. Suppose that the condition (1) holds for both pairs (M, L_M) and (N, L_N). Denote by $P := M \times N$, and $\pi_1 : P \to M$, $\pi_2 : P \to N$ the corresponding projections. Then $\omega(X,Y) = \omega_M(\pi_{1*}X, \pi_{1*}Y) + \omega_N(\pi_{2*}X, \pi_{2*}Y)$, $X, Y \in TP \cong TM \times TN$, defines a symplectic form on P, usually denoted by $\omega = \omega_M \oplus \omega_N$. It is obvious that $L := L_M \times L_N$ is a closed Lagrangian submanifold of (P, ω).

Let K be a time-dependent Hamiltonian function on N, X_K the corresponding vector field and ϕ_K^t its flow. We can defined a time-dependent Hamiltonian function on P by $(H \oplus K)(x, y, t) = H(x, t) + K(y, t)$. Its Hamiltonian vector field $X_{H\oplus K}$ satisfies $X_{H\oplus K}(a, b) = (X_H(a), X_K(b)) \in T_{(a,b)}P$, for all $a \in M$ and $b \in N$. Obviously, its flow is $\phi^t_{H\oplus K}(a, b) = (\phi^t_H(a), \phi^t_K(b))$.

If J^N is an almost complex structure on N which is ω_N-compatible and (K, J^N) is a regular pair, then $J(X, Y) := (J^M(X), J^N(Y))$, is an $\omega-$compatible almost complex structure on P and $(H \oplus K, J)$ is a regular pair.

In Subsection 2.1 we saw that we can define the action functionals $\mathcal{A}_H$ on $\Omega_0(M, L_M)$ and $\mathcal{A}_K$ on $\Omega_0(N, L_N)$. Since $[\omega]|_{\pi_2(P,L)} = 0$ we can define $\mathcal{A}_{H\oplus K}$ on $\Omega_0(P, L)$, too. Let us find the connection between these three functionals. Take $x \in \Omega_0(M, L_M)$, $y \in \Omega_0(N, L_N)$ and define $z(t) = (x(t), y(t)) \in \Omega_0(P, L)$. We know that

$$\begin{aligned}\mathcal{A}_{H\oplus K}(z) &= -\iint_{\mathbb{D}^2_+} h^*\omega + \int_0^1 (H \oplus K)(z(t), t)\, dt,\\ &= -\iint_{\mathbb{D}^2_+} h^*\omega + \int_0^1 \left(H(x(t), t) + K(y(t), t)\right) dt,\end{aligned}$$

where h is any map from the upper half-disc $\mathbb{D}^2_+$ to P that restricts to z on the upper half-circle. Denote by $h_x = \pi_1 \circ h$ and $h_y = \pi_2 \circ h$. We see that h_x is a map from the upper half-disc $\mathbb{D}^2_+$ to M that restricts to x on the upper half-circle. The same holds for h_y and y. For the 2-form $h^*\omega$ we compute

$$h^*\omega(X,Y) = \omega(h_*X, h_*Y) = \omega_M(\pi_{1*}h_*X, \pi_{1*}h_*Y) + \omega_N(\pi_{2*}h_*X, \pi_{2*}h_*Y)$$
$$= \omega_M(h_{x*}X, h_{x*}Y) + \omega_N(h_{y*}X, h_{y*}Y) = h_x^*\omega_M(X,Y) + h_y^*\omega_N(X,Y).$$

Now,

$$\mathcal{A}_{H\oplus K}(z) = -\iint_{\mathbb{D}^2_+} h_x^*\omega_M - \iint_{\mathbb{D}^2_+} h_y^*\omega_N + \int_0^1 H(x(t),t)\,dt + \int_0^1 K(y(t),t)\,dt$$
$$= \mathcal{A}_H(x) + \mathcal{A}_K(y). \tag{5}$$

The critical points of all three action functionals are related by $\dot{z}(t) = (\dot{x}(t), \dot{y}(t)) = (X_H(x), X_K(y)) = X_{H\oplus K}(z)$, so

$$\operatorname{Crit} \mathcal{A}_{H\oplus K} = \operatorname{Crit} \mathcal{A}_H \times \operatorname{Crit} \mathcal{A}_K. \tag{6}$$

If we denote by μ_L the relative Maslov index of Hamiltonian chords on P it holds $\mu_L(z) = \mu_{L_M}(x) + \mu_{L_N}(y)$, (see [21] for details). It also holds $\mu_L|_{\pi_2(P,L)} = 0$, since $\mu_{L_M}|_{\pi_2(M,L_M)} = 0$ and $\mu_{L_N}|_{\pi_2(N,L_N)} = 0$. Therefore, L is a relatively simplectically aspherical submanifold of P.

2.5 Product of chain complexes and the algebraic Künneth formula

Suppose we are given chain complexes of Abelian groups (C, ∂^C) and (D, ∂^D). Their tensor product is defined with

$$(C\otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j,$$

with differential operator given by $\partial(c_i\otimes d_j) = \partial^C c_i \otimes d_j + (-1)^i c_i \otimes \partial^D d_j$, on generators $c_i \otimes d_j \in C_i \otimes D_j$. Since we work with chain complexes over $\mathbb{Z}_2$ the factor $(-1)^i$ in the last formula can be omitted.

The Künneth Formula says that for every n there is a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} (H_i(C)\otimes H_j(D)) \to H_n(C\otimes D, \partial) \to \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(C), H_j(D)) \to 0$$

and this sequence splits (see [8] for details). We are discussing here Floer homologies over $\mathbb{Z}_2$ so for any two Floer homologies we have $\operatorname{Tor}(HF_*, HF_*) = 0$.

3. Commutative diagrams

In this section we prove Theorem 1.3. Given the relations (5) and (6), the definitions of the maps $\Psi^{\cdot,\cdot}$ and $\Phi^{\cdot,\cdot}$ are rather straightforward.

By what we have seen in Section 2.4, we can define the map $\Psi^{a,b}$ on generators $x\otimes y \in CF_r^{(-\infty,a)}(L_M, \phi_H^1(L_M)) \otimes CF_s^{(-\infty,b)}(L_N, \phi_K^1(L_N))$, by $\Psi^{a,b}(x\otimes y) = (x,y)$. The following lemma states that $\Psi^{a,b}$ is a chain map.

LEMMA 3.1. *The following diagram commutes.*

$$\begin{tikzcd} \bigoplus_{r+s=p}\left[CF_r^a(L_M)\otimes CF_s^b(L_N)\right] \arrow[r, "\Psi^{a,b}"] \arrow[d, "\partial=\partial^{M,a}\otimes\partial^{N,b}"'] & CF_p^{a+b}(L) \arrow[d, "\partial^{P,a+b}"] \\ \bigoplus_{r+s=p-1}\left[CF_r^a(L_M))\otimes CF_s^b(L_N)\right] \arrow[r, "\Psi^{a,b}"] & CF_{p-1}^{a+b}(L), \end{tikzcd} \tag{7}$$

Proof. It suffices to prove this property on generators $x\otimes y$:

$$\begin{aligned}(\Psi^{a,b}\circ\partial)(x\otimes y) &= \Psi^{a,b}(\partial^{M,a}x\otimes y + x\otimes\partial^{N,b}y)\\ &= \Psi^{a,b}\left\{\left[\sum_{x'}\sharp_2\widehat{\mathcal{M}}(x,x')_{[0]}\cdot x'\right]\otimes y + x\otimes\left[\sum_{y'}\sharp_2\widehat{\mathcal{M}}(y,y')_{[0]}\cdot y'\right]\right\}\\ &= \Psi^{a,b}[\sum_{x'}\sharp_2\widehat{\mathcal{M}}(x,x';H,J^M)_{[0]}x'\otimes y + \sum_{y'}\sharp_2\widehat{\mathcal{M}}(y,y';K,J^N)_{[0]}x\otimes y']\\ &= \sum_{x'\in CF^a_{r-1}}\sharp_2\widehat{\mathcal{M}}(x,x';H,J^M)_{[0]}(x',y) + \sum_{y'\in CF^b_{s-1}}\sharp_2\widehat{\mathcal{M}}(y,y';K,J^N)_{[0]}(x,y'). \end{aligned}\tag{8}$$

On the other side

$$(\partial^{P,a+b}\circ\Psi^{a,b})(x\otimes y) = \partial^{P,a+b}(x,y) = \sum_{z'\in CF^{a+b}_{p-1}(L)}\sharp_2\widehat{\mathcal{M}}(z,z';H\oplus K,J)_{[0]}\cdot z', \tag{9}$$

where $z(t)=(x(t),y(t))$.

If $u\in\mathcal{M}(z,z';H\oplus K,J)$ is a holomorphic strip that connects $z=(x,y)$ and $z'=(x',y')$ then it splits into two holomorphic strips $u^H=\pi_1\circ u\in\mathcal{M}(x,x';H,J^M)$ and $u^K=\pi_2\circ u\in\mathcal{M}(y,y';K,J^N)$. And vice versa, if $u^H\in\mathcal{M}(x,x';H,J^M)$ and $u^K\in\mathcal{M}(y,y';K,J^N)$ then $u=(u^H,u^K)\in\mathcal{M}(z,z';H\oplus K,J)$. We conclude that $\mathcal{M}(z,z';H\oplus K,J)=\mathcal{M}(x,x';H,J^M)\times\mathcal{M}(y,y';K,J^N)$. We want to describe the free $\mathbb{R}-$action on the 1-dimensional components of the previous moduli space in order to count the number of elements of the set $\widehat{\mathcal{M}}(z,z';H\oplus K,J)_{[0]}$. From the dimension formula of product spaces we conclude that

$$\begin{aligned}\mathcal{M}(z,z';H\oplus K,J)_{[1]} =&\mathcal{M}(x,x';H,J^M)_{[0]}\times\mathcal{M}(y,y';K,J^N)_{[1]}\\ &\bigcup\mathcal{M}(x,x';H,J^M)_{[1]}\times\mathcal{M}(y,y';K,J^N)_{[0]}.\end{aligned}$$

After dividing by the $\mathbb{R}-$action we are only left with those strips u for which $x=x'$ or $y=y'$. Since $\mu_L(z)-\mu_L(z')=1$, in the case when $x=x'$ we get $\mu_{L_N}(y')=\mu_L(z')-\mu_{L_M}(x')=p-1-r=s-1$.

Similarly, in the case $y=y'$ we get $\mu_{L_M}(x')=r-1$. So,

$$\begin{aligned}&\sum_{z'\in CF^{a+b}_{p-1}(L)}\sharp_2\widehat{\mathcal{M}}(z,z';H\oplus K,J)_{[0]}\cdot z' =\\ &\sum_{x'\in CF^a_{r-1}}\sharp_2\widehat{\mathcal{M}}(x,x';H,J^M)_{[0]}(x',y) + \sum_{y'\in CF^b_{s-1}}\sharp_2\widehat{\mathcal{M}}(y,y';K,J^N)_{[0]}(x,y').\end{aligned}\tag{10}$$

From equations (8), (9) and (10) we get the commutativity of diagram (7). □

The chain map $\Psi^{a,b}$ induces the map at the homology level

$$\Psi_*^{a,b} : H_p\left[CF^a(L_M) \otimes CF^b(L_N)\right] \to HF_p^{a+b}(L).$$

It is obvious that after a direct limit as $a, b \to +\infty$ the map $\Psi_* = \varinjlim_{a\to+\infty} \varinjlim_{b\to+\infty} \Psi_*^{a,b}$ becomes $\Psi_*([x] \otimes [y]) = [(x, y)]$.

Let us define the map $\Phi^{a+b,\widetilde{a},\widetilde{b}}$. Take $z \in CF_p^{(-\infty,a+b)}(L, \phi^1_{H\oplus K}(L))$. Define $x = \pi_1 \circ z$, $y = \pi_2 \circ z$. We know that $\mu_L(z) = \mu_{L_M}(x) + \mu_{L_N}(y) = p$, so $x \in CF_r(L_M, \phi^1_H(L_M))$ and $y \in CF_s(L_N, \phi^1_K(L_N))$ for some $r + s = p$. Since $\mathcal{A}_H(x) + \mathcal{A}_K(y) < a+b$ and $\mathcal{A}_K(y) \geq \min \operatorname{Crit} \mathcal{A}_K$ it follows $\mathcal{A}_H(x) < a+b-\min \operatorname{Crit} \mathcal{A}_K \leq \widetilde{a}$. Hence, $x \in CF_r^{(-\infty,\widetilde{a})}(L_M, \phi^1_H(L_M))$. In the same way one can show that y is in the proper filtered complex. We define $\Phi^{a+b,\widetilde{a},\widetilde{b}}(z) = x \otimes y$. The proof of the fact that $\Phi^{a+b,\widetilde{a},\widetilde{b}}$ is a chain map is similar to the proof of Lemma 3.1.

4. PSS isomorphism commutes with the Künneth formula

In this section we prove Theorem 1.7. First, we recall Albers' construction of an isomorphism between the Lagrangian Floer homology and the singular homology of a Lagrangian submanifold (the so called PSS isomorphism).

Let g_M be a metric and f_M a Morse function on L_M such that the pair (f_M, g_M) is Morse-Smale. To $p \in \operatorname{Crit} f_M$ and $x \in \operatorname{Crit} \mathcal{A}_H$ we associate a moduli space

$$\mathcal{M}_{p,x}^{f_M,H} := \left\{ (\gamma, u) \,\middle|\, \begin{array}{l} -\nabla f_M(\gamma(t)) = \frac{d\gamma}{dt}, \gamma(-\infty) = p, \\ \partial_s u + J^M(\partial_t u - \beta(s) X_H \circ u) = 0, \\ u(\mathbb{R} \times \{0\}), u(\mathbb{R} \times \{1\}) \subseteq L_M, \\ u(+\infty, t) = x(t),\, u(-\infty, t) = \gamma(0), \end{array} \right\},$$

where ∇f_M denotes gradient of f_M with respect to g_M and $\beta(s)$ is a smooth, increasing function whose value is 0 for $s \leq 1/2$ and 1 for $s \geq 1$. The dimension of the manifold $\mathcal{M}_{p,x}^{f_M,H}$ is $m_{f_M}(p) - \mu_{L_M}(x)$. The morphism

$$PSS^{L_M} : CM_k(f_M) \to CF_k(L_M, \phi^1_H(L_M)),$$

defined on generators by

$$PSS^{L_M}(p) := \sum_{x \in CF_k} \sharp_2 \mathcal{M}_{p,x}^{f_M,H} \cdot x,$$

induces an isomorphism $PSS_*^{L_M} : HM_k(L_M) \to HF_k(L_M, \phi^1_H(L_M))$.

Now, we give a sketch of Schwarz's construction of a Künneth type isomorphism in Morse homology. Let f_N and g_N satisfy the same condition as f_M and g_M. We define a Morse function $f_M \oplus f_N$ on $L_M \times L_N$ by $(f_M \oplus f_N)(x, y) = f_M(x) + f_N(y)$. Then clearly $\operatorname{Crit}_p(f_M \oplus f_N) = \bigcup_{r+s=p} \operatorname{Crit}_r(f_M) \times \operatorname{Crit}_s(f_N)$. In [22] Schwarz proved that the map $\overline{\Psi} : (CM_*(f_M) \otimes CM_*(f_N))_p \to CM_p(f_M \oplus f_N)$ $\overline{\Psi}(x_r \otimes y_s) = (x_r, y_s)$, $r + s = p$, defines a chain group isomorphism.

We are left to prove Theorem 4. One composition in the diagram (3) is equal to

$$\left(\Psi_* \circ (PSS_*^{L_M} \otimes PSS_*^{L_N})\right)([p] \otimes [q]) = \sum_{x \in CF_r(L_M),\, y \in CF_s(L_N)} \left(\sharp_2 \mathcal{M}_{p,x}^{f_M,H} \cdot \sharp_2 \mathcal{M}_{q,y}^{f_N,K}\right) \cdot [(x,y)].$$

Here $[p]$ is an element of the homology group $HM_r(f_M)$ and $[q]$ is an element of the homology group $HM_s(f_N)$. The other composition in the diagram (3) gives

$$\left(PSS_*^{L} \circ \overline{\Psi}_*\right)([p] \otimes [q]) = \sum_{z \in CF_{r+s}(L)} \sharp_2 \mathcal{M}_{(p,q),z}^{f_M \oplus f_N, H \oplus K} \cdot [z].$$

Since $\mathcal{M}_{(p,q),z}^{f_M \oplus f_N, H \oplus K} = \mathcal{M}_{p,x}^{f_M,H} \times \mathcal{M}_{q,y}^{f_N,K}$ for $z = (x,y) \in CF_{r+s}(L)$, we get the commutativity of the diagram (3).

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