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ON THE CLASSES OF FUNCTIONS OF BOUNDED PARTIAL AND TOTAL Λ-VARIATION

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Abstract. The inclusions of classes of functions with bounded partial Λ-variation into the classes of functions with bounded total harmonic variation are established. The result is applied to the problem of convergence of rectangular partial sums for multiple Fourier series.

1. Introduction

In 1970s, Goffman and Waterman studied the class of functions whose trigonometric Fourier series remained convergent after any change of variable. This led to the concept of Λ-bounded variation (or bounded Λ-variation) in one-dimensional case, which was first introduced in [10].

Later on, this class was generalized by Sahakian [9] for two-dimensional case, and by Sablin [7] for functions of three and more variables.

Unlike other classes of generalized bounded variation, the class ΛBV of functions of Λ-bounded variation has an important (in certain questions) subclass $C\Lambda V$ of functions continuous in Λ-variation. The continuity in Λ-variation was introduced by Waterman [11] for functions of one variable, and by the author [1] and Dragoshanskii [3] for functions of two and more variables.

The author obtained the following result [1, Theorems 3,4] on the convergence of multiple trigonometric Fourier series, that generalizes the one-dimensional result of Waterman [10] and the two-dimensional result of Sahakian [9].

Theorem 1.1. *Let $m \geqslant 3$. Let a continuous function f belong to the class $CHV(\mathbb{T}^m)$ of functions, continuous in harmonic variation (see Definition 2.5). Then its Fourier series converges uniformly in the sense of Pringsheim. There is a continuous function f in the class $HBV(\mathbb{T}^m)$ for any $m \geqslant 3$ (see Definition 2.3), such that the cubic partial sums of its Fourier series diverge at the point 0.*

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Goginava and Sahakian in a number of papers (see, e.g. [4–6]) introduced and studied the classes $P\Lambda BV(\mathbb{T}^m)$ of functions of bounded partial Λ-variation, i.e. the functions with bounded Λ-variation with respect to every single variable.

Our aim is to study the classes $P_d\Lambda BV(\mathbb{T}^m)$ of functions with bounded Λ-variation with respect to every set of q variables, $1 \leqslant q \leqslant d < m$. These classes are intermediate between $P\Lambda BV(\mathbb{T}^m)$ and $\Lambda BV(\mathbb{T}^m)$. We expand several results of Goginava and Sahakian to the classes $P_d\Lambda BV(\mathbb{T}^m)$.

The paper is organized in the following way. In Section 2 we give detailed definitions of the notions and precise statements of theorems that we use or generalize in our work. In Section 3 we prove the result on inclusion of the classes and the consequent result on convergence of Fourier series. In Section 4 we prove the result on divergence of Fourier series and the consequent result on non-inclusion of the classes.

2. Preliminaries

Let Δ be an interval on the real line. By $\Omega(\Delta)$ we denote the set of all finite systems $\{I_n\}_{n=1}^N$ of pairwise disjoint open intervals such that $\overline{I_n} \subset \Delta$.

Let $I^k = (a^k, b^k)$. Consider a function $f(\mathbf{x})$ on $\mathbb{R}^m$, $m \geqslant 1$. By definition, put

$$\Delta_{\mathbf{x},s,j}(f) = f(\mathbf{x} + s\mathbf{e}_j) - f(\mathbf{x})$$

and

$$f(I) = f(I^1 \times \cdots \times I^m) = \Delta_{\mathbf{a},b^1-a^1,1} \circ \cdots \circ \Delta_{\mathbf{a},b^m-a^m,m}(f).$$

It is well known that the operators $\Delta_{\mathbf{x},s,j}(f)$ commute with each other for different j's. Therefore, the mixed difference $f(I)$ is symmetric with respect to rearrangements of the variables.

Let the set $\{1,\dots,m\}$ be divided into two non-intersecting subsets ξ and τ with p and $m-p$ elements respectively (we write $|\xi| = p$). Given a $\mathbf{x} = (x^1,\dots,x^m)$, by x^ξ denote the vector from $\mathbb{R}^p$ with the coordinates x^j, $j \in \xi$. For a segment $I = \bigotimes_{j=1}^m I^j$ we write $I^\xi = \bigotimes_{j\in\xi} I^j$.

By $f(I^\xi, x^\tau)$ or $f(x^\tau, I^\xi)$ denote the mixed difference of f as a function of the variables x^j, $j \in \xi$, on I^ξ for the fixed values of x^k, $k \in \tau$, i.e., if $\xi = \{j_1,\dots,j_p\}$ and $I^{j_k} = (a^{j_k}, b^{j_k})$, then $f(I^\xi, a^\tau) = \Delta_{\mathbf{a},b^{j_1}-a^{j_1},j_1} \circ \cdots \circ \Delta_{\mathbf{a},b^{j_p}-a^{j_p},j_p}(f)$. We also denote $\mathbb{T} = [-\pi, \pi]$.

Definition 2.1. We say that a sequence $\Lambda = \{\lambda_n\}_{n=1}^\infty$ of positive numbers is a proper one, if it is non-decreasing (maybe for $n \geqslant n_0 > 1$), tends to infinity and $\sum\limits_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty$.

We denote $H = \{n\}_{n=1}^\infty$. It follows from the definition that the sequence H is a proper one.

Definition 2.2. Let $\Lambda^1,\dots,\Lambda^m$ be proper sequences. The value

$$V^{1,\dots,m}_{\Lambda^1,\dots,\Lambda^m}(f;\Delta) = \sup_{\substack{\{\{I^j_{k_j}\}\}_{j=1}^m \\ \{I^j_{k_j}\}\in\Omega(\Delta^j),\, j=1,\dots,m}} \sum_{k_1,\dots,k_m} \frac{|f(I^1_{k_1} \times \cdots \times I^m_{k_m})|}{\lambda^1_{k_1} \dots \lambda^m_{k_m}}$$

is called the $(\Lambda^1, \dots, \Lambda^m)$-variation of the function $f(x^1, \dots, x^m)$ with respect to the variables $x^1, \dots, x^m$ on the segment $\Delta = \Delta^1 \times \dots \times \Delta^m$.

Consider a nonempty set $\xi \subset \{1, \dots, m\}$ that consists of elements $j_1 < \dots < j_p$ and let $\tau = \{1, \dots, m\} \setminus \xi$. We write $I^{\xi}_{k^{\xi}} = \bigotimes_{l=1}^{p} I^{j_l}_{k_{j_l}}$, $x^{\xi} = (x^{j_1}, \dots, x^{j_p})$. By

$$V^{\xi}_{\Lambda^{\xi}}(f; \Delta^{\xi}, x^{\tau}) = V^{\xi}_{\Lambda^{\xi}}(f; x^{\tau}, \Delta^{\xi}) = V^{j_1,\dots,j_p}_{\Lambda^{\xi}}(f; x^{\tau}, \Delta^{\xi}) = V^{\xi}_{\Lambda^{j_1},\dots,\Lambda^{j_p}}(f; \Delta^{\xi}, x^{\tau})$$

denote the $(\Lambda^{j_1}, \dots, \Lambda^{j_p})$-variation of f as a function of the variables x^{ξ} with respect to these variables on the p-dimensional segment $\Delta^{\xi} = \Delta^{j_1} \times \dots \times \Delta^{j_p}$ for the fixed values x^{τ} of other variables (if $\tau \neq \varnothing$).

We define the $(\Lambda^{j_1}, \dots, \Lambda^{j_p})$-*variation of a function* $f(x^1, \dots, x^m)$ *with respect to the variables* x^{ξ} *on the segment* $\Delta = \Delta^1 \times \dots \times \Delta^m$ by the formula

$$V^{\xi}_{\Lambda^{\xi}}(f; \Delta) = V^{\xi}_{\Lambda^{j_1},\dots,\Lambda^{j_p}}(f; \Delta) = \sup_{x^{\tau} \in \Delta^{\tau}} V^{\xi}_{\Lambda^{j_1},\dots,\Lambda^{j_p}}(f; \Delta^{\xi}, x^{\tau}).$$

Definition 2.3. The total $(\Lambda^1, \dots, \Lambda^m)$-variation of a function $f(x^1, \dots, x^m)$ on the segment $\Delta = \Delta^1 \times \dots \times \Delta^m$ is defined as

$$V_{\Lambda^1,\dots,\Lambda^m}(f; \Delta) = \sum_{\substack{\xi \subseteq \{1,\dots,m\} \\ \xi \neq \varnothing}} V^{\xi}_{\Lambda^{\xi}}(f; \Delta)$$

We say that the function f has total bounded $(\Lambda^1, \dots, \Lambda^m)$-variation on Δ and write $f \in (\Lambda^1, \dots, \Lambda^m)BV(\Delta)$, if $V_{\Lambda^1,\dots,\Lambda^m}(f; \Delta) < \infty$. In the case $\Lambda^j = \Lambda$ for all j we write V^{ξ}_{Λ}, V_{Λ} and $\Lambda BV(\Delta)$ for short.

Definition 2.4. Let Λ be a proper sequence. We say that a function $f(x)$ from the class $\Lambda BV([a, b])$ is *continuous in* Λ-*variation* ($f \in C\Lambda V([a, b])$), if $\lim_{n\to\infty} V_{\Lambda_n}(f; [a, b]) = 0$ for the sequences $\Lambda_n = \{\lambda_{n+k}\}_{k=1}^{\infty}$.

Definition 2.5. Let $\Lambda^1, \dots, \Lambda^m$ be proper sequences. A function f from the class $(\Lambda^1, \dots, \Lambda^m)BV(\Delta)$ is said to be continuous in $(\Lambda^1, \dots, \Lambda^m)$-variation on Δ, if

$$\lim_{n\to\infty} V^{\xi}_{\Lambda^{j_1},\dots,\Lambda^{j_{k-1}},\Lambda^{j_k}_n,\Lambda^{j_{k+1}},\dots,\Lambda^{j_p}}(f; \Delta) = 0, \tag{1}$$

for any nonempty $\xi = \{j_1, \dots, j_p\} \subset \{1, \dots, m\}$ and for any $j_k \in \xi$. We write $f \in C(\Lambda^1, \dots, \Lambda^m)V(\Delta)$.

Definition 2.6. The function f has bounded partial $(\Lambda^1, \dots, \Lambda^m)$-variation on Δ, $f \in P(\Lambda^1, \dots, \Lambda^m)BV(\Delta)$, if $PV_{\Lambda^1,\dots,\Lambda^m}(f; \Delta) = \sum_{i=1}^{m} V^{i}_{\Lambda^i}(f; \Delta) < \infty$.

Definition 2.7. Let $d \in \{1, \dots, m\}$. Denote

$$V^{(d)}_{\Lambda^1,\dots,\Lambda^m}(f; \Delta) = \sum_{\substack{\xi \subseteq \{1,\dots,m\} \\ 1 \leqslant |\xi| \leqslant d}} V^{\xi}_{\Lambda^{\xi}}(f; \Delta).$$

We write $f \in P_d \Lambda BV(\mathbb{T}^m)$ if this sum is finite.

For $d = 1$, we have $P_1 \Lambda BV(\mathbb{T}^m) = P\Lambda BV(\mathbb{T}^m)$; on the other hand, for $d = m$, we have $P_m \Lambda BV(\mathbb{T}^m) = \Lambda BV(\mathbb{T}^m)$.

Though there are several results on application of generalized variation to different orthonormal systems, our work deals with (multilple) trigonometric Fourier series only, and we write "Fourier series" for "trigonometric Fourier series" everywhere below in the paper.

Goginava and Sahakian proved the following result.

Theorem 2.8 ([4, Theorem 1]). *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ be a proper sequence,*

$$\frac{\lambda_n}{n} \downarrow 0 \text{ for } n \to \infty, \text{ and } \sum_{n=1}^{\infty} \frac{\lambda_n \ln^{m-2}(n+1)}{n^2} < \infty.$$

Then $P\Lambda BV(\mathbb{T}^m) \subset CHV(\mathbb{T}^m)$.

Together with Theorem 1.1, this implies that the Fourier series of a continuous function from such a class converges uniformly ([4, Theorem 3a]).

On the other hand, in the same paper Goginava and Sahakian proved the following.

Theorem 2.9 ([4, Theorem 3b]). *Let Λ be a proper sequence, $\frac{\lambda_n}{n}$ be nonincreasing, and there exists a $\delta > 0$ such that*

$$\frac{\lambda_n}{n} = O\left(\frac{\lambda_{[n\delta]}}{[n\delta]}\right) \text{ for } n \to \infty, \text{ and } \sum_{n=1}^{\infty} \frac{\lambda_n \ln^{m-2}(n+1)}{n^2} = \infty.$$

Then there is a continuous function $f \in P\Lambda BV(\mathbb{T}^m)$, such that its Fourier series diverges at zero. Consequently, such a class $P\Lambda BV(\mathbb{T}^m)$ is not a subset of $CHV(\mathbb{T}^m)$.

For example, if $\Lambda = \{n \ln^a(n+1)\}$, then the inclusion $P\Lambda BV(\mathbb{T}^m) \subset CHV(\mathbb{T}^m)$ holds for $a < 1 - m$ and fails for $a \geqslant 1 - m$.

3. Inclusion of classes and convergence of Fourier series

Our first main result is the following theorem (for $d = 1$, this is Theorem 2.8).

Theorem 3.1. *Let $d \in \{2, \ldots, m-1\}$. Let a proper sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ satisfy the conditions*

$$\frac{\lambda_n}{n} \downarrow 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(\lambda_n)^d \ln^{m-d-1}(n+1)}{n^{d+1}} < \infty.$$

Then $P_d\Lambda BV(\mathbb{T}^m) \subset CHV(\mathbb{T}^m)$.

In particular, if $\Lambda = \{n \ln^a(n+1)\}$, then the inclusion holds for $a < 1 - \frac{m}{d}$.

Corollary 3.2. *Let $d \in \{2, \ldots, m-1\}$. Let a proper sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ satisfy the conditions*

$$\frac{\lambda_n}{n} \downarrow 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(\lambda_n)^d \ln^{m-d-1}(n+1)}{n^{d+1}} < \infty.$$

Then the Fourier series of a continuous function $f \in P_d\Lambda BV(\mathbb{T}^m)$ converges uniformly in Pringsheim sense.

The corollary follows immediately from Theorems 1.1 and 3.1.

We need the following result about the structure of the class $C\Lambda BV(\Delta)$.

Theorem 3.3 ([1, Theorem 1]). *The class $C(\Lambda^1,\dots,\Lambda^m)V(\Delta)$ of functions continuous in Λ-variation is the union of classes $(M^1,\dots,M^m)BV(\Delta)$ over all proper sequences $M^j=\{\mu_n^j\}$ such that $\dfrac{\mu_n^j}{\lambda_n^j}\to 0$ as $n\to\infty$.*

Proof (of Theorem 3.1). The idea of the proof is the same as in [4, Theorem 1]. We estimate the m-dimensional component of Γ-variation for a certain proper Γ, where $\frac{\gamma_n}{n}\downarrow 0$. It can be seen that the other components of Γ-variation are finite, therefore, f is continuous in harmonic variation due to Theorem 3.3.

We choose positive numbers $\{A_n\}$ such that $A_n\uparrow\infty$, $\frac{\lambda_n A_n}{n}\downarrow 0$ and

$$\sum_{n=1}^{\infty}\frac{(\lambda_n)^d(\ln^{m-d-1}(n+1))(A_n)^m}{n^{d+1}}<\infty. \tag{2}$$

This holds if $\{A_n\}$ increases slowly enough. Let $\gamma_n=n/A_n$, $\Gamma=\{\gamma_n\}$. Then $\frac{\lambda_n}{\gamma_n}\downarrow 0$. For $|\xi|=j\leqslant k$, this implies directly that

$$V_{\Gamma^\xi}^{\xi}(f;\Delta)\leqslant\left(\frac{\lambda_1}{\gamma_1}\right)^j V_{\Lambda^\xi}^{\xi}(f;\Delta)<\infty.$$

Therefore, by virtue of Theorem 3.3 the corresponding components of harmonic variation are finite and are continuous in variation.

Now we consider the case $\xi=\{1,\dots,m\}$. As $A_n\ln(n+1)\geqslant A_1\ln 2=C$, for $q=d+1,\dots,m-1$ we also have

$$\sum_{n=1}^{\infty}\frac{(\lambda_n)^d(\ln^{q-d-1}(n+1))(A_n)^q}{n^{d+1}}<\infty.$$

Hence the following argument is also valid for q-dimensional components of variation ($q=d+1,\dots,m-1$).

Consider arbitrary systems $\{I_{n_j}^j\}\in\Omega(\Delta^j)$, $j=1,\dots,m$ and the corresponding variational sum S for the Γ-variation. Then

$$S\leqslant\sum_{\sigma\in\Sigma_m}\sum_{n_{\sigma(1)}\leqslant\dots\leqslant n_{\sigma(m)}}\frac{|f(I_{n_1}^1\times\dots\times I_{n_m}^m)|}{\gamma_{n_1}\dots\gamma_{n_m}},$$

where Σ_m is the permutation group of an m-element set. As Σ_m is finite, it is sufficient to estimate the inner sum for a single σ. For convenience, we take $\sigma=\mathrm{id}$. We have

$$S_{id}=\sum_{n_1}\sum_{n_2\geqslant n_1}\cdots\sum_{n_{m-d}\geqslant n_{m-d-1}}\frac{1}{\gamma_{n_1}\dots\gamma_{n_{m-d}}}S(n_1,\dots,n_{m-d}), \tag{3}$$

where

$$S(n_1,\dots,n_{m-d})=\sum_{n_{m-d+1}\geqslant n_{m-d}}\cdots\sum_{n_m\geqslant n_{m-1}}\frac{|f(I_{n_1}^1\times\dots\times I_{n_m}^m)|}{\gamma_{n_{m-d+1}}\dots\gamma_{n_m}}.$$

By definition, $\frac{\lambda_n}{\gamma_n} = \frac{\lambda_n A_n}{n} \downarrow 0$, hence

$$\begin{aligned} S(n_1,\dots,n_{m-d}) &= \sum_{n_{m-d+1}\geqslant n_{m-d}} \cdots \sum_{n_m \geqslant n_{m-1}} \frac{|f(I^1_{n_1}\times\cdots\times I^m_{n_m})|}{\lambda_{n_{m-d+1}}\dots\lambda_{n_m}} \cdot \frac{\lambda_{n_{m-d+1}}\dots\lambda_{n_m}}{\gamma_{n_{m-d+1}}\dots\gamma_{n_m}} \\ &\leqslant \sum_{n_{m-d+1}\geqslant n_{m-d}} \cdots \sum_{n_m \geqslant n_{m-1}} \frac{|f(I^1_{n_1}\times\cdots\times I^m_{n_m})|}{\lambda_{n_{m-d+1}}\dots\lambda_{n_m}} \cdot \left(\frac{\lambda_{n_{m-d}}}{\gamma_{n_{m-d}}}\right)^d \\ &\leqslant V_\Lambda^{m-d+1,\dots,m}(f;\mathbb{T}^m)\cdot \left(\frac{\lambda_{n_{m-d}}}{\gamma_{n_{m-d}}}\right)^d. \end{aligned}$$

Now we substitute this estimate to (3) and replace n_{m-d} by n for short. Then

$$S_{id} \leqslant V_\Lambda^{m-d+1,\dots,m}(f;\mathbb{T}^m)\cdot \sum_{n_1}\sum_{n_2\geqslant n_1}\cdots\sum_{n\geqslant n_{m-d-1}} \frac{1}{\gamma_{n_1}\dots\gamma_{n_{m-d-1}}\gamma_n}\cdot\left(\frac{\lambda_n}{\gamma_n}\right)^d.$$

By the definition of Γ this implies

$$S_{id} \leqslant V_\Lambda^{m-d+1,\dots,m}(f;\mathbb{T}^m)\cdot \sum_{n_1}\sum_{n_2\geqslant n_1}\cdots\sum_{n\geqslant n_{m-d-1}} \frac{A_{n_1}\dots A_{n_{m-d-1}}A_n}{n_1\dots n_{m-d-1}n}\cdot\left(\frac{\lambda_n A_n}{n}\right)^d.$$

As the sequence $\{A_j\}$ is nondecreasing, the product of its elements in the numerator does not exceed $(A_n)^m$. Changing the order of summation, we obtain that

$$S_{id} \leqslant V_\Lambda^{m-d+1,\dots,m}(f;\mathbb{T}^m)\cdot \sum_n \frac{(A_n)^m(\lambda_n)^d}{n^{d+1}} \sum_{n_{m-d-1}=1}^{n}\cdots\sum_{n_1=1}^{n_2}\frac{1}{n_1\dots n_{m-d-1}}.$$

Now notice that

$$\begin{aligned} &\sum_{n_{m-d-1}=1}^{n}\sum_{n_{m-d-2}=1}^{n_{m-d-1}}\cdots\sum_{n_1=1}^{n_2}\frac{1}{n_1\dots n_{m-d-1}} \\ &\qquad\leqslant \sum_{n_{m-d-1}=1}^{n}\sum_{n_{m-d-1}=1}^{n}\cdots\sum_{n_1=1}^{n}\frac{1}{n_1\dots n_{m-d-1}} \leqslant C(m,d)(\ln(n+1))^{m-d-1}. \end{aligned}$$

Substituting this into the previous estimate and then summing by σ, we obtain the final inequality

$$S \leqslant C(m,d)V_\Lambda^{(d)}(f;\mathbb{T}^m)\cdot\sum_n \frac{(A_n)^m(\lambda_n)^d(\ln(n+1))^{m-d-1}}{n^{d+1}}$$

for the variation sum S. By virtue of (2) this implies that all S's are uniformly bounded, i.e., that the Γ-variation is finite. Due to Theorem 3.3, this gives the continuity in harmonic variation. □

4. Divergence of Fourier series

Now we turn to the generalization of Theorem 2.9.

THEOREM 4.1. *Let $d \in \{2,\dots,m-1\}$. Let $\frac{\lambda_n}{n} = O\left(\frac{\lambda_{[n^\delta]}}{[n^\delta]}\right)$ for some $\delta > 0$ when $n \to \infty$, and*

$$\sum_{n=1}^{\infty} \frac{(\lambda_n)^d \ln^{m-1-d}(n+1)}{n^{d+1}} = \infty. \tag{4}$$

Let also $\frac{\lambda_m}{m} \leqslant C\frac{\lambda_n}{n}$ for all $m > n$ and for some C. Then there exists a continuous function in $P_d\Lambda BV(\mathbb{T}^m)$ such that its Fourier series diverges unboundedly at zero.

COROLLARY 4.2. *Let $d \in \{1,\dots,m-1\}$. Let $\frac{\lambda_n}{n} = O\left(\frac{\lambda_{[n^\delta]}}{[n^\delta]}\right)$ for some $\delta > 0$ when $n \to \infty$, and*

$$\sum_{n=1}^{\infty} \frac{(\lambda_n)^d \ln^{m-1-d}(n+1)}{n^{d+1}} = \infty.$$

Let also $\frac{\lambda_m}{m} \leqslant C\frac{\lambda_n}{n}$ for all $m > n$ for some C. Then the class $P_d\Lambda BV(\mathbb{T}^m)$ is not a subset of $HBV(\mathbb{T}^m)$.

For example, if $\Lambda = \{n \ln^a(n+1)\}$, then the inclusion $P_d\Lambda BV(\mathbb{T}^m) \subset HBV(\mathbb{T}^m)$ fails for $a \geqslant 1 - \frac{m}{d}$.

Previuosly, Sablin [8] proved that for any function from $HBV(\mathbb{T}^m)$ the rectangular partial sums of its Fourier series are uniformly bounded. A stronger variant of this result is proved in [2].

The corollary follows immediately from this fact and Theorem 4.1 for $d > 1$, or from this fact and Theorem 2.9 for $d = 1$.

Proof (of Theorem 4.1). The idea of the proof is the same as in [4, Theorem 3b].

Let an integer N be sufficiently large. We define the cubes

$$A_{i_1,\dots i_m} = \left[\frac{\pi i_1}{N+\frac{1}{2}}, \frac{\pi(i_1+1)}{N+\frac{1}{2}}\right) \times \cdots \times \left[\frac{\pi i_m}{N+\frac{1}{2}}, \frac{\pi(i_m+1)}{N+\frac{1}{2}}\right).$$

Let $N_\delta = \lfloor (N/2)^{1/\delta} \rfloor$ and $p_j = \lfloor j^\delta \rfloor$, where $\lfloor x \rfloor$ is the integer part of x. Let $W_N = \{(i_{,1},\dots i_m) \in \mathbb{N}^m : 1 \leqslant i_m \leqslant N_\delta, i_j \in (i_m, i_m + p_{i_m}) \text{ for } j < m\}$ and let $\Lambda(j) = \sum_{i=1}^{j} \frac{1}{\lambda_i}$.

Consider the function

$$f_N(\mathbf{x}) = \sum_{(i_1,\dots i_m) \in W_N} \frac{1}{(\Lambda(p_{i_m}))^d} \chi_{A_{i_1,\dots i_m}}(\mathbf{x}) \prod_{s=1}^{m} \sin(N+\tfrac{1}{2})x^s.$$

Now we estimate the Λ-variation of this function with respect to any q variables, $1 \leqslant q \leqslant d$. Without loss of generality it is enough to estimate the Λ-variation with respect to $(x^1,\dots,x^q)$ and with respect to $(x^{m-q+1},\dots,x^m)$.

In the first case, let us fix a point $x^* = (x^{q+1},\dots,x^m)$.

If $x^m \notin \bigcup_{i_m=1}^{N_\delta} \left[\frac{\pi i_m}{N+\frac{1}{2}}, \frac{\pi(i_m+1)}{N+\frac{1}{2}}\right)$, then $f_N(x^1,\dots,x^q,x^*) \equiv 0$, and there is nothing to estimate.

Now suppose that $x_m \in \left[\frac{\pi i_m}{N+\frac12}, \frac{\pi(i_m+1)}{N+\frac12}\right)$. Then

$$f(\mathbf{x}) = \sin(N+\tfrac12)x^m \frac{1}{(\Lambda(p_{i_m}))^d} \prod_{j=1}^{m-1} g_{N,i_m}(x^j),$$

where
$$g_{N,i_m}(t) = \sin(N+\tfrac12)t \cdot \chi_{\left[\frac{\pi(i_m+1)}{N+\frac12}, \frac{\pi(i_m+p_{i_m})}{N+\frac12}\right)}(t).$$

Obviously, $|g_{N,i_m}| \leqslant 1$. Hence for arbitrary systems of intervals $\{I^j_{i_j}\}$ we have

$$\left| \sum_{i_1,\dots,i_q} \frac{|f(I^1_{i_1} \times \dots \times I^q_{i_q}, x^*)|}{\lambda_{i_1} \dots \lambda_{i_q}} \right| \leqslant \frac{1}{(\Lambda(p_{i_m}))^d} \left(V_\Lambda(g_{N,i_m}; \mathbb{T})\right)^q.$$

It is easy to see that
$$V_\Lambda(g_{N,i_m}; \mathbb{T}) \leqslant 2 \sum_{i=1}^{p_{i_m}} \frac{1}{\lambda_j} \leqslant 2\Lambda(p_{i_m}).$$

This implies that
$$V_\Lambda^{1,\dots,q}(f_N; \mathbb{T}^m) \leqslant 2^q (\Lambda(p_{i_m}))^q \cdot \frac{1}{(\Lambda(p_{i_m}))^d} \leqslant 2^d.$$

Now we turn to the second case. Consider a point $x^* = (x^1, \dots, x^{m-q})$. If there exists a j in $\{1,\dots,m-q\}$ such that x^j does not belong to any $\left[\frac{\pi i}{N+\frac12}, \frac{\pi(i+1)}{N+\frac12}\right)$, where $1 < i < N_\delta + p_{N_\delta}$, then by definition of W_N the function $f(x^*, x^{m-q+1}, \dots, x^m)$ vanishes as a function of $(x^{m-q+1}, \dots, x^m)$. Otherwise, there exist i_l, $l = 1,\dots,m-q$ such that

$$x_l \in \left[\frac{\pi i_l}{N+\frac12}, \frac{\pi(i_l+1)}{N+\frac12}\right).$$

Let $\tilde{\tilde{i}}_m = \min\{i_l \,|\, l = 1,\dots,m-q\}$, $\tilde{i}_m = \max_{l=1,\dots,m-q} \min\{i_m \,|\, i_m + p_{i_m} > i_l\}$. If $\tilde{i}_m > \tilde{\tilde{i}}_m$, then again $f(x^*, x^{m-q+1}, \dots, x^m) \equiv 0$ as a function of $(x^{m-q+1}, \dots, x^m)$. Otherwise,

$$\begin{aligned} g(x^{m-q+1}, \dots, x^m) &= f(x^*, x^{m-q+1}, \dots, x^m) \\ &= \prod_{s=1}^{m-q} \sin(N+\tfrac12)x^s \sum_{i_m=\tilde{i}_m}^{\tilde{\tilde{i}}_m} \frac{\chi_{\left[\frac{\pi i_m}{N+\frac12}, \frac{\pi(i_m+1)}{N+\frac12}\right)}(x^m) \sin(N+\frac12)x^m}{(\Lambda(p_{i_m}))^d} \prod_{s=m-q+1}^{m-1} g_{N,i_m}(x^j). \end{aligned}$$

Notice that its q-dimensional Λ-variation does not exceed

$$V = 2^q \sum_{i_m=\tilde{i}_m}^{\tilde{\tilde{i}}_m} \frac{1}{(\Lambda(p_{i_m}))^d} \frac{1}{\lambda_{i_m-\tilde{i}_m}} \sum_{i_{m-q+1}=i_m+1}^{i_m+p_{i_m}-1} \cdots \sum_{i_{m-1}=i_m+1}^{i_m+p_{i_m}-1} \frac{1}{\lambda_{i_{m-q+1}-i_m}} \cdots \frac{1}{\lambda_{i_{m-1}-i_m}}.$$

Here the inner sum is estimated by $(\Lambda(p_{i_m}))^{q-1}$, hence

$$V \leqslant 2^q \sum_{i_m=\tilde{i}_m}^{\tilde{\tilde{i}}_m} \frac{1}{(\Lambda(p_{i_m}))} \frac{1}{\lambda_{i_m-\tilde{i}_m}} \leqslant 2^q \frac{1}{(\Lambda(p_{\tilde{i}_m}))} \sum_{j=1}^{\tilde{\tilde{i}}_m - \tilde{i}_m} \frac{1}{\lambda_j}.$$

Let the minimum in the definition of $\tilde{\tilde{i}}_m$ be for $l = l_0$; then $\tilde{\tilde{i}}_m = i_{l_0} < \tilde{i}_m + p_{\tilde{i}_m}$.

Therefore, $\tilde{\tilde{i}}_m - \tilde{i}_m \leqslant p_{\tilde{i}_m}$ and

$$V \leqslant 2^q \frac{1}{(\Lambda(p_{\tilde{i}_m}))} \sum_{j=1}^{p_{\tilde{i}_m}} \frac{1}{\lambda_j} = 2^q.$$

This finally gives $V_\Lambda^{m-q+1,\dots,m}(f_N;\mathbb{T}^m) \leqslant 2^q \leqslant 2^d$.

Now we estimate the cubic partial sum $S_{N,\dots,N}(f_N,0,\dots,0)$. By definition

$$\begin{aligned}
\pi^m S_{N,\dots,N}(f_N,0,\dots,0) &= \int_{\mathbb{T}^m} f_N(\mathbf{x}) \prod_{s=1}^m D_N(x^s)\,d\mathbf{x} \\
&= \sum_{(i_1,\dots,i_m)\in W_N} \frac{1}{\Lambda^d(p_{i_m})} \int_{A_{i_1,\dots i_m}} \prod_{s=1}^m \sin^2(N+\tfrac12)x^s\,d\mathbf{x} \\
&\geqslant c \sum_{(i_1,\dots,i_m)\in W} \frac{1}{i_m\Lambda^d(p_{i_m})} \frac{1}{i_1\dots i_{m-1}} \\
&\geqslant c \sum_{i_m=1}^{N_\delta} \frac{1}{i_m\Lambda^d(p_{i_m})} \sum_{i_1=i_m}^{i_m+p_{i_m}} \cdots \sum_{i_{m-1}=i_m}^{i_m+p_{i_m}} \frac{1}{i_1\dots i_{m-1}} \\
&\geqslant c \sum_{i_m=1}^{N_\delta} \frac{1}{i_m\Lambda^d(p_{i_m})} \ln^{m-1}\left(\frac{i_m+p_{i_m}}{i_m}\right) \geqslant c(\delta-1)^{m-1} \sum_{i_m=1}^{N_\delta} \frac{\ln^{m-1} i_m}{i_m\Lambda^d(p_{i_m})}.
\end{aligned} \tag{5}$$

Due to the conditions posed on Λ, the estimates

$$\Lambda(p_j) = \sum_{i=1}^{p_j} \frac{1}{\lambda_i} = \sum_{i=1}^{p_j} \frac{1}{i}\cdot\frac{i}{\lambda_i} \leqslant C\frac{p_j}{\lambda_{p_j}} \ln(p_j+1) \leqslant C\frac{j\ln(j+1)}{\lambda_j}$$

hold, hence, $\dfrac{\ln(j+1)}{\Lambda(p_j)} \geqslant c\dfrac{\lambda_j}{j}$.

Raising this inequality to the dth power and substituting it into (5), we obtain

$$\pi^m S_{N,\dots,N}(f_N,0,\dots,0) \geqslant c(\delta-1)^{m-1} \sum_{n=1}^{N_\delta} \frac{\lambda_n^d \ln^{m-d-1}(n+1)}{n^{d+1}} \to \infty$$

for $N\to\infty$ due to (4).

Using a standard argument we obtain that $P_d\Lambda BV(\mathbb{T}^m)$ is a Banach space with respect to the norm $\|f\| = |f(0,\dots,0)| + V_\Lambda^{(d)}(f;\Delta)$, and $P_d\Lambda BV(\mathbb{T}^m)\cap C(\mathbb{T}^m)$ is a closed subset in it. As $\|f_N\| \leqslant C$ and $S_{N,\dots,N}(f_N,0,\dots,0) \to \infty$ for $N\to\infty$, by the Banach–Steinhaus theorem we obtain that there exists a continuous function $f \in P_d\Lambda BV(\mathbb{T}^m)$ such that $S_{N,\dots,N}(f,0,\dots,0)$ are unbounded. Thus the proof is complete. □

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