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A NEW EXTENSION OF BESSEL-MAITLAND FUNCTION AND ITS PROPERTIES

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Abstract. This paper deals with a new extended Bessel-Maitland function. The m^{th} differentiation, Beta transform, Laplace transform, Whittaker transform and various other transforms for our new extended Bessel-Maitland function are presented here. Further, the Riemann-Liouville fractional integration and differentiation for the function introduced here are also indicated.

1. Introduction

The special function of the form defined by the series representation

$$J_{\nu}^{\mu}(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(\mu n+\nu+1)}=\phi(\mu,\nu+1;-z) \tag{1}$$

is known as Bessel-Maitland function, or the Wright generalized function (see, e.g., [8, Eq.(8.3)]). It has many application in a wide range of research fields such as physics, chemistry, biology, engineering and applied sciences. The theory of Bessel functions is intimately connected with the theory of certain types of differential equations. For a detailed account of applications of Bessel functions, the reader may be referred to [15].

The generalized hypergeometric function is defined as follows (see, e.g., [10]):

$${}_pF_q\left[\begin{array}{l}(\alpha_p)\,;\\(\beta_q)\,;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{p}\,(\alpha_j)_n}{\Pi_{j=1}^{q}\,(\beta_j)_n}\frac{z^n}{n!},$$

provided $p\leq q$; $p=q+1$ and $|z|<1$ and where $(\alpha)_n$ is the familiar Pochhammer symbol.

The Fox-Wright function ${}_p\Psi_q(z)$ of a generalization of the generalized hypergeometric function ${}_pF_q$ is given by (see, e.g., [2, 6, 17, 18]):

$${}_p\Psi_q\left[\begin{array}{l}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\,;\\(\beta_1,B_1),\ldots,(\beta_q,B_q)\,;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+A_1n)\ldots\Gamma(\alpha_p+A_pn)}{\Gamma(\beta_1+B_1n)\ldots\Gamma(\beta_q+B_qn)}\frac{z^n}{n!}, \tag{2}$$

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where $A_j > 0\ \ (j = 1, 2 \ldots, p)\,; B_j > 0\ \ (j = 1, 2, \ldots, q)$ and $1 + \sum\limits_{j=1}^{q} B_j - \sum\limits_{j=1}^{p} A_j \geq 0$.

In recent years, various extensions of some well-known special functions have been investigated (see, e.g., [1, 3, 4, 12, 13]).

In particular, Singh et al. [14] introduced the following generalization of Bessel-Maitland function:

$$J_{\nu,q}^{\mu,\gamma} = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n + \nu + 1)n!}, \tag{3}$$

where $\mu, \nu, \gamma \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\gamma) > 0$ and $q \in (0, 1) \cup \mathbb{N}$ and $(\gamma)_0 = 1$, $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol (see, e.g., [10]).

In the sequel of the above mentioned works, in this paper, we introduce a new extension of Bessel-Maitland function and investigate various properties and identities involving this newly introduced function.

For our purpose, we recall the following definitions.

Euler Transform: The Euler transform of the function $f(z)$ is defined by

$$B\left\{f(z)\,;\alpha,\beta\right\} = \int_0^1 z^{\alpha-1}\left(1-z\right)^{\beta-1} f(z)\ dz, \quad (\min\{\Re(\alpha), \Re(\beta)\} > 0). \tag{4}$$

Laplace Transform: The Laplace transform of the function $f(t)$ is defined as

$$F(s) = L\left\{f(t)\,;s\right\} = \int_0^{\infty} e^{-st} f(t)\ dt, \quad \Re(s) > 0.$$

Riemann-Liouville fractional derivative and integral: The Riemann-Liouville fractional integral and derivative operators are, for $\Re(\alpha) > 0$, respectively, defined as follows (see, e.g., [11, Section 2]):

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt; \tag{5}$$

$$(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}}\, dt; \tag{6}$$

$$(D_{0+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} (I_{0+}^{1-\{\alpha\}} f)(x) = \frac{1}{\Gamma(1-\{\alpha\})} \left(\frac{d}{dx}\right)^{[\alpha]+1} \int_0^x \frac{f(t)}{(x-t)^{\{\alpha\}}}\, dt; \tag{7}$$

$$(D_{-}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} (I_{-}^{1-\{\alpha\}} f)(x) = \frac{1}{\Gamma(1-\{\alpha\})} \left(-\frac{d}{dx}\right)^{[\alpha]+1} \int_x^{\infty} \frac{f(t)}{(t-x)^{\{\alpha\}}}\, dt.$$

Here $[\alpha]$ and $\{\alpha\}$ denote the greatest integer less than or equal to the real number α and the fractional part of the real number α with $0 \leq \{\alpha\} < 1$. We thus have $\alpha = [\alpha] + \{\alpha\}$.

2. Generalized Bessel-Maitland function $J^{\mu,q,p}_{\nu,\gamma,\delta}(z)$

DEFINITION 2.1. Let $\mu, \nu, \gamma, \delta \in \mathbb{C}$ with $\Re(\mu) > 0, \Re(\nu) > -1, \Re(\gamma) > 0, \Re(\delta) > 0$. Also, let $p,\ q \in \mathbb{R}^+$ with $q < \Re(\mu) + p$. The generalized Bessel-Maitland function is defined as

$$J^{\mu,q,p}_{\nu,\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}}. \tag{8}$$

Here and in the following, $\mathbb{C}$, $\mathbb{R}^+$ and $\mathbb{N}$ are the sets of complex numbers, positive real numbers, and positive integers, respectively.

REMARK 2.2. Setting $\delta = p = 1$, (8) reduces to the known result (see, [14]), the particular case $q = 0$ of which gives the Bessel-Maitland function in (1).

Relation with Mittag-Leffler functions:

(i) On replacing ν by $\nu - 1$ in (8), we get the following interesting relation:

$$J^{\mu,\gamma,\delta}_{\nu-1,q,p}(-z) = E^{\mu,\gamma,\delta}_{\nu,q,p}(z),$$

where $E^{\mu,\gamma,\delta}_{\nu,q,p}(z)$ is the Mittag-Leffler function defined by Salim and Faraj [12].

(ii) On setting $p = \delta = 1$ and replacing ν by $\nu - 1$ in (8), we get

$$J^{\mu,\gamma,1}_{\nu-1,q,1}(-z) = E^{\mu,\gamma}_{\nu,q}(z),$$

where $E^{\mu,\gamma}_{\nu,q}(z)$ is the Mittag-Leffler function defined by Shukla and Prajapati [13].

(iii) On setting $p = q = \delta = 1$ and replacing ν by $\nu - 1$, we get

$$J^{\mu,\gamma,1}_{\nu-1,1,1}(-z) = E^{\gamma}_{\mu,\nu}(z),$$

where $E^{\gamma}_{\mu,\nu}(z)$ is the Mittag-Leffler function defined by Prabhakar [9].

(iv) On setting $p = q = \delta = \gamma = 1$ and replacing ν by $\nu - 1$, we get

$$J^{\mu,1,1}_{\nu-1,1,1}(-z) = E_{\mu,\nu}(z),$$

where $E_{\mu,\nu}(z)$ is the Mittag-Leffler function defined by Wiman [16].

(v) On setting $p = q = \delta = \gamma = 1$ and replacing ν by $\nu - 1$, we get

$$J^{\mu,1,1}_{0,1,1}(-z) = E_{\mu}(z),$$

where $E_{\mu}(z)$ is the Mittag-Leffler function defined by Ghosta Mittag-Leffler [7].

THEOREM 2.3. *Let $\mu, \nu, \gamma, \delta \in \mathbb{C}$ with $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$. Also, let $p,\ q \in \mathbb{R}^+$ with $m \in \mathbb{N}$. Then*

$$\left(\frac{d}{dz}\right)^m J^{\mu,\gamma,\delta}_{\nu,q,p}(z) = (-1)^m \frac{(\gamma)_{qm}}{(\delta)_{pm}} \sum_{n=0}^{\infty} \frac{(n+1)_m(\gamma + qm)_{qn}(-z)^n}{\Gamma(\mu(n+m) + \nu + 1)(\delta + pm)_{pn}}; \tag{9}$$

$$(\mu n + 1) J^{\mu,\gamma,\delta}_{\nu,q,p}(z) + \mu z \frac{d}{dz} J^{\mu,\gamma,\delta}_{\nu,q,p}(z) = J^{\mu,\gamma,\delta}_{\nu,q,p}(z); \tag{10}$$

$$J^{\mu,\gamma,\delta}_{\nu,q,p}(z) - J^{\mu,\gamma-1,\delta}_{\nu,q,p}(z) = q \sum_{n=0}^{\infty} \frac{n(-z)^n(\gamma)_{qn-1}}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}}. \tag{11}$$

Proof. By using the definition (8) on the left-hand side of (9), we get

$$\left(\frac{d}{dz}\right)^m J^{\mu,\gamma,\delta}_{\nu,q,p}(z) = \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n$$
$$= \sum_{n=m}^{\infty} \frac{n(n-1)\cdots(n-m+1)(\gamma)_{qn}(-1)^n z^{n-m}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}$$
$$= \sum_{n=0}^{\infty} \frac{(n+m)(n+m-1)\cdots(n+1)(\gamma)_{q(n+m)}(-1)^{(n+m)} z^n}{\Gamma(\mu(n+m)+\nu+1)(\delta)_{p(n+m)}}$$
$$= (-1)^m \frac{(\gamma)_{qm}}{(\delta)_{pm}} \sum_{n=0}^{\infty} \frac{(n+1)_m(\gamma+qm)_{qn}(-z)^n}{\Gamma(\mu(n+m)+\nu+1)(\delta+pm)_{pn}}.$$

Now, by using definition (8) on the left-hand side of (10), we have

$$(\mu n+1)J^{\mu,\gamma,\delta}_{\nu,q,p}(z) + \mu z\frac{d}{dz}J^{\mu,\gamma,\delta}_{\nu,q,p}(z)$$
$$=(\mu n+1)\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n + \mu z\frac{d}{dz}\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n$$
$$=(\mu n+1)\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n + \mu z\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}n(-1)(-z)^{n-1}$$
$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n = J^{\mu,\gamma,\delta}_{\nu,q,p}(z),$$

which is our required result (10).

Further, by using definition (8) on the left-hand side of (11), we get

$$J^{\mu,\gamma,\delta}_{\nu,q,p}(z) - J^{\mu,\gamma-1,\delta}_{\nu,q,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n - \sum_{n=0}^{\infty} \frac{(\gamma-1)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n$$
$$=\sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n\left[(\gamma)_{qn}-(\gamma-1)_{qn}\right] = q\sum_{n=0}^{\infty} \frac{n(\gamma)_{qn-1}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-z)^n,$$

which gives the result (11). □

Theorem 2.4. *Let $\mu,\nu,\gamma,\delta \in \mathbb{C}$ with $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$. Also, let p, $q \in \mathbb{R}^+$ with $q \le \Re(\mu)+p$. Then*

$$\int_0^1 t^{\nu}(1-t)^{\beta-1} J^{\mu,\gamma,\delta}_{\nu,q,p}(zt^{\mu})\,dt = J^{\mu,\gamma,\delta}_{\nu+\beta,q,p}(z); \tag{12}$$

$$\frac{1}{\Gamma(\beta)}\int_t^x (x-s)^{\beta-1}(s-t)^{\nu} J^{\mu,\gamma,\delta}_{\nu,q,p}[z(s-t)^{\mu}]\,ds = (x-t)^{\beta+\nu} J^{\mu,\gamma,\delta}_{\nu+\beta,q,p}[z(s-t)^{\mu}]; \tag{13}$$

$$\int_0^x t^{\lambda}(x-t)^{\nu} J^{\mu,\gamma,\delta}_{\nu,q,p}[w(x-t)^{\mu}] J^{\mu,\eta,\delta}_{\lambda,q,p}[wt^{\mu}]\,dt$$

$$= (x)^{\lambda+\nu+1} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\gamma)_{q(n-k)}(\eta)_{qk}(-wx^{\mu})^n}{\Gamma(\mu n + \lambda + \nu + 2)(\delta)_{p(n-k)}(\delta)_{pk}}; \tag{14}$$

$$\int_0^z t^{\nu} J_{\nu,q,p}^{\mu,\gamma,\delta}(wt^{\mu})\, dt = z^{\nu+1} J_{\nu+1,q,p}^{\mu,\gamma,\delta}(wz^{\mu}). \tag{15}$$

Proof. By using (8) on the left-hand side of (12) and changing the order of summation and integration (which is guaranteed under the given condition), we have

$$\int_0^1 t^{\nu}(1-t)^{\beta-1} J_{\nu,q,p}^{\mu,\gamma,\delta}(zt^{\mu})\, dt = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}} (-z)^n \int_0^1 t^{\nu+\mu n}(1-t)^{\beta-1}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}} (-z)^n B(\mu n + \nu + 1, \beta) = J_{\nu+\beta,q,p}^{\mu,\gamma,\delta}(z),$$

which is our desired result (12).

Changing the variable $s = t + w(x-t)$ on the left-hand side of (13) and using (8) results with

$$\int_t^x (x-s)^{\beta-1}(s-t)^{\nu} J_{\nu,q,p}^{\mu,\gamma,\delta}[z(s-t)^{\mu}]\, ds$$

$$=(x-t)^{\beta+\nu} \int_0^1 w^{\mu n+\nu}(1-w)^{\beta-1} J_{\nu,q,p}^{\mu,\gamma,\delta}[z(w(x-t))^{\mu}]\, dw$$

$$=(x-t)^{\beta+\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}[z(x-t)^{\mu}]^n}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}} \int_0^1 w^{\mu n+\nu}(1-w)^{\beta-1}\, dw$$

$$=(x-t)^{\beta+\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}[z(x-t)^{\mu}]^n}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}} B(\mu n + \nu + 1, \beta)$$

$$\frac{1}{\Gamma(\beta)} \int_t^x (x-s)^{\beta-1}(s-t)^{\nu} J_{\nu,q,p}^{\mu,\gamma,\delta}[z(s-t)^{\mu}]\, ds = (x-t)^{\beta+\nu} J_{\nu+\beta,q,p}^{\mu,\gamma,\delta}[z(x-t)^{\mu}],$$

which is our desired result (13).

Now, by using (8) on the left-hand side of (14), we get

$$\int_0^x t^{\lambda}(x-t)^{\nu} J_{\nu,q,p}^{\mu,\gamma,\delta}[w(x-t)^{\mu}] J_{\lambda,q,p}^{\mu,\eta,\delta}[wt^{\mu}]\, dt$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_{qn}(\eta)_{qk}(-w)^{n+k}}{\Gamma(\mu n + \nu + 1)\Gamma(\mu k + \lambda + 1)(\delta)_{pn}(\delta)_{pk}} \int_0^x t^{\mu k+\lambda}(x-t)^{\mu n+\nu}\, dt$$

$$=(x)^{\lambda+\nu+1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma)_{qn}(\eta)_{qk}(-wx^{\mu})^{n+k}}{\Gamma(\mu n + \nu + 1)\Gamma(\mu k + \lambda + 1)(\delta)_{pn}(\delta)_{pk}} B(\mu k + \lambda + 1, \mu n + \nu + 1),$$

which is our desired result (14).

Further, by using (8) on the left-hand side of (15), we have

$$\int_0^z t^\nu J^{\mu,\gamma,\delta}_{\nu,q,p}(wt^\mu)\,dt = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}(-w)^n \int_0^z t^{\nu+\mu n}\,dt = z^{\nu+1} J^{\mu,\gamma,\delta}_{\nu+1,q,p}(wz^\mu),$$

which is the desired result (15). □

Remark 2.5. For $p=\delta=1$, Theorem 2.4 reduces to the known result (see [14, p.48]).

3. Integral transforms of the generalized Bessel-Maitland function $J^{\mu,\gamma,\delta}_{\nu,q,p}(z)$

In this section, we derive several integral transforms for the newly introduced function (8) such as Beta transform, Laplace transform, Whittaker transform and K-transform, which are asserted by the following theorems.

Theorem 3.1 (Beta Transform). *Let $\mu,\nu,\alpha,\beta,\gamma,\delta,\eta \in \mathbb{C}$ with $\Re(\mu)>0$, $\Re(\nu)>-1$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\eta)>0$. Also, let $p,\ q\in\mathbb{R}^+$ with $q\le\Re(\mu)+p$. Then*

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1} J^{\mu,\gamma,\delta}_{\nu,q,p}(xz^\eta)\,dz = \frac{\Gamma(\beta)\Gamma(\delta)}{\Gamma(\gamma)}\, {}_3\Psi_3\left[\begin{array}{ll}(\gamma,q),(\alpha,\eta),(1,1); & \\ & -x \\ (\delta,p),(\nu+1,\mu),(\alpha+\beta,\eta); & \end{array}\right].$$

Proof. From (8) and (3), we have

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1} J^{\mu,\gamma,\delta}_{\nu,q,p}(xz^\eta)\,dz = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-x)^n}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}\int_0^1 z^{\alpha+\eta n-1}(1-z)^{\beta-1}\,dz$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-x)^n}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}B(\alpha+\eta n,\beta) = \frac{\Gamma(\beta)\Gamma(\delta)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)(-x)^n}{\Gamma(\mu n+\nu+1)\Gamma(\delta+pn)}\frac{\Gamma(\alpha+\eta n)}{\Gamma(\alpha+\beta+\eta n)}.$$

In view of the definition (2), we get the required the result. □

Theorem 3.2 (Laplace Transform). *Let $\mu,\nu,\alpha,\gamma,\delta,\eta\in\mathbb{C}$ with $\Re(\mu)>0$, $\Re(\nu)>-1$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\alpha)>0$, $\Re(\eta)>0$. Also, let $p,\ q\in\mathbb{R}^+$ with $q\le\Re(\mu)+p$. Then*

$$\int_0^\infty z^{\alpha-1}e^{-sz} J^{\mu,\gamma,\delta}_{\nu,q,p}(xz^\eta)\,dz = \frac{\Gamma(\delta)}{s^\alpha\Gamma(\gamma)}\,{}_3\Psi_2\left[\begin{array}{ll}(\gamma,q),(\alpha,\eta),(1,1); & \\ & \frac{-x}{s^\eta} \\ (\delta,p),(\nu+1,\mu); & \end{array}\right]. \tag{16}$$

Proof. By using (8) and (4), and putting $(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$, we get:

$$\int_0^\infty z^{\alpha-1}e^{-sz} J^{\mu,\gamma,\delta}_{\nu,q,p}(xz^\eta)\,dz = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-x)^n}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}\int_0^\infty e^{-sz}z^{\alpha+\eta n-1}\,dz$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-x)^n}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}\frac{\Gamma(\alpha+\eta n)}{s^{\alpha+\eta n}} = \frac{\Gamma(\delta)}{s^\alpha\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)(-x)^n}{\Gamma(\mu n+\nu+1)\Gamma(\delta+pn)}\frac{\Gamma(\alpha+\eta n)}{s^{\eta n}}.$$

Summing up the above the series with the help of (2), we easily get (16). □

THEOREM 3.3 (K-Transform). *Let* $\mu, \nu, \alpha, \lambda, \gamma, \delta, \rho \in \mathbb{C}$ *with* $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\alpha \pm \lambda) > 0$, $\Re(\rho) > 0$. *Let* $p,\ q \in \mathbb{R}^+$ *with* $q \leq \Re(\mu) + p$. *Then*

$$\int_0^\infty t^{\alpha-1} K_\lambda(st) J_{\nu,q,p}^{\mu,\gamma,\delta}(wt^\rho)\, dt = \frac{2^{\alpha-2}\Gamma(\delta)}{s^\alpha \Gamma(\gamma)} {}_3\Psi_2 \left[\begin{array}{ll} (\gamma, q), (\frac{\alpha\pm\lambda}{2}, \frac{\rho}{2}), (1,1); & \\ & -w\left(\frac{2}{s}\right)^\rho \\ (\delta, p), (\nu+1, \mu); & \end{array} \right]. \tag{17}$$

Proof. Applying (8) to the left-hand side of (17) and by setting $st = z$, we get

$$\int_0^\infty \left(\frac{z}{s}\right)^{\alpha-1} K_\lambda(z) J_{\nu,q,p}^{\mu,\gamma,\delta}(w(\frac{z}{s})^\rho)\frac{dz}{s} = s^{-\alpha} \sum_{n=0}^\infty \frac{(\gamma)_{qn}(\frac{-w}{s^\rho})^n}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}} \int_0^\infty z^{\alpha+\rho n-1} K_\lambda(z)\, dz.$$

By using the following known integral (see [5])

$$\int_0^\infty x^{\rho-1} K_\nu(x) dx = 2^{\rho-2}\Gamma\left(\frac{\rho\pm\nu}{2}\right),$$

we obtain

$$\int_0^\infty \left(\frac{z}{s}\right)^{\alpha-1} K_\lambda(z) J_{\nu,q,p}^{\mu,\gamma,\delta}(w(\frac{z}{s})^\rho)\frac{dz}{s} = \frac{2^{\alpha-2}\Gamma(\delta)}{s^\alpha\Gamma(\gamma)} \sum_{n=0}^\infty \frac{(\gamma+qn)\left(-\frac{w}{s^\rho}\right)^n}{\Gamma(\mu n+\nu+1)(\delta+pn)} \Gamma\left(\frac{\alpha+\rho n\pm\lambda}{2}\right).$$

In view of the definition (2), we get the required the result. □

THEOREM 3.4 (Whittaker Transform). *Let* $\mu, \nu, \alpha, \lambda, \gamma, \delta, \rho \in \mathbb{C}$ *with* $\Re(\mu) > 0, \Re(\nu) > -1, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\alpha \pm m) > 0, \Re(\rho) > 0, \Re(\alpha - \lambda) > 0$. *Also, let* $p,\ q \in \mathbb{R}^+$ *with* $q \leq \Re(\mu) + p$. *Then*

$$\int_0^\infty t^{\alpha-1} e^{-\frac{st}{2}} W_{\lambda,m}(st) J_{\nu,q,p}^{\mu,\gamma,\delta}(wt^\rho)\, dt = \frac{\Gamma(\delta)}{s^\alpha\Gamma(\gamma)} {}_4\Psi_3 \left[\begin{array}{ll} (\gamma, q), (\frac{1}{2} \pm m + \alpha, \rho), (1,1); & \\ & -\frac{w}{s^\rho} \\ (\delta, p), (\nu+1, \mu), (1-\lambda+\alpha, \rho); & \end{array} \right]. \tag{18}$$

Proof. Applying (8) on the left-hand side of (18) and by setting $st = z$, we get

$$\int_0^\infty e^{-\frac{z}{2}} \left(\frac{z}{s}\right)^{\alpha-1} W_{\lambda,m}(z) J_{\nu,q,p}^{\mu,\gamma,\delta}(w(\frac{z}{s})^\rho)\frac{dz}{s}$$
$$= s^{-\alpha} \sum_{n=0}^\infty \frac{(\gamma)_{qn}(\frac{-w}{s^\rho})^n}{\Gamma(\mu n+\nu+1)(\delta)_{pn}} \int_0^\infty e^{-\frac{z}{2}} z^{\alpha+\rho n-1} W_{\lambda,m}(z)\, dz.$$

Using the formula given in [5]:

$$\int_0^\infty e^{-\frac{x}{2}} x^{\nu-1} W_{\lambda,m}(x)\, dx = \frac{\Gamma(\frac{1}{2}\pm m+\nu)}{\Gamma(1-\lambda+\nu)},$$

we get

$$\int_0^\infty \left(\frac{z}{s}\right)^{\alpha-1} W_{\lambda,m}(z) J_{\nu,q,p}^{\mu,\gamma,\delta}(w(\frac{z}{s})^\rho)\frac{dz}{s}$$

$$=\frac{\Gamma(\delta)}{s^{\alpha}\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\left(-\frac{w}{s^{\rho}}\right)^{n}}{\Gamma(\mu n+\nu+1)\Gamma(\delta+pn}\frac{\Gamma(\frac{1}{2}\pm m+\alpha+\rho n)}{\Gamma(1-\lambda+\alpha+\rho n)}.$$

Now by using (2), we get the desired result. □

THEOREM 3.5. *Let* $\mu,\nu,\alpha,\lambda,\gamma,\delta,\rho\in\mathbb{C}$ *with* $\Re(\mu)>0$, $\Re(\nu)>-1$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\lambda-\alpha)>0$, $\Re(\rho)>0$, $\Re(m\pm\alpha)>-\frac{1}{2}$. *Also, let* $p,\ q\in\mathbb{R}^{+}$ *with* $q\leq\Re(\mu)+p$. *Then*

$$\int_{0}^{\infty}t^{\alpha-1}e^{\frac{-st}{2}}M_{\lambda,m}(st)J_{\nu,q,p}^{\mu,\gamma,\delta}(wt^{\rho})\,dt=$$

$$\frac{\Gamma(2m+1)\Gamma(\delta)}{s^{\alpha}\Gamma(\gamma)\Gamma(m+\lambda+\frac{1}{2})}{}_{4}\Psi_{3}\left[\begin{array}{l}(\gamma,q),(m+\alpha+\frac{1}{2},\rho),(\lambda-\alpha,-\rho),(1,1);\\(\delta,p),(\nu+1,\mu),(m-\alpha+\frac{1}{2},-\rho);\end{array}\ -\frac{w}{s^{\rho}}\right].\quad(19)$$

Proof. Following the proof of the previous theorem, we apply (8) on the left-hand side of (19), set $st=z$ and use the integral given in [5]:

$$\int_{0}^{\infty}e^{-\frac{x}{2}}x^{\nu-1}M_{\lambda,m}(x)dx=\frac{\Gamma(2m+1)\Gamma(m+\nu+\frac{1}{2})\Gamma(\lambda-\nu)}{\Gamma(m-\nu+\frac{1}{2})\Gamma(m+\lambda+\frac{1}{2})}.$$

Thus, we obtain

$$\int_{0}^{\infty}\left(\frac{z}{s}\right)^{\alpha-1}M_{\lambda,m}(z)J_{\nu,q,p}^{\mu,\gamma,\delta}(w(\frac{z}{s})^{\rho})\frac{dz}{s}$$

$$=s^{-\alpha}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\left(-\frac{w}{s^{\rho}}\right)^{n}}{\Gamma(\mu n+\nu+1)(\delta)_{pn}}\frac{\Gamma(2m+1)\Gamma(m+\alpha+\rho n+\frac{1}{2})\Gamma(\lambda-\alpha-\rho n))}{\Gamma(m-\alpha-\rho n+\frac{1}{2})\Gamma(m+\lambda+\frac{1}{2})}$$

$$=\frac{\Gamma(2m+1)\Gamma(\delta)}{s^{\alpha}\Gamma(\gamma)\Gamma(m+\lambda+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\left(-\frac{w}{s^{\rho}}\right)^{n}}{\Gamma(\mu n+\nu+1)\Gamma(\delta+pn)}\frac{\Gamma(m+\alpha+\rho n+\frac{1}{2})\Gamma(\lambda-\alpha-\rho n))}{\Gamma(m-\alpha-\rho n+\frac{1}{2})\Gamma(m+\lambda+\frac{1}{2})}.$$

Further, summing up the above series with the help of (2), we get the result. □

THEOREM 3.6. *Let* $\mu,\nu,\alpha,\lambda,\gamma,\delta,\rho\in\mathbb{C}$ *with* $\Re(\mu)>0$, $\Re(\nu)>-1$, $\Re(\alpha)>-1$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\rho)>0$, $\Re(\frac{\alpha}{2}\pm\lambda)>-1$, $\Re(\frac{\alpha}{2}\pm m)>0$. *Also, let* $p,\ q\in\mathbb{R}^{+}$ *with* $q\leq\Re(\mu)+p$. *Then*

$$\int_{0}^{\infty}t^{\alpha-1}W_{\lambda,m}(st)W_{-\lambda,m}(st)J_{\nu,q,p}^{\mu,\gamma,\delta}(wt^{\rho})\,dt=$$

$$\frac{\Gamma(\delta)}{s^{\alpha}\Gamma(\gamma)}{}_{4}\Psi_{3}\left[\begin{array}{l}(\gamma,q),(\frac{\alpha}{2}\pm m,\frac{\rho}{2}),(\alpha+1,\rho),(1,1);\\(\delta,p),(\nu+1,\mu),(1+\frac{\alpha}{2}\pm\lambda,\frac{\rho}{2});\end{array}\ -\frac{w}{s^{\rho}}\right].\quad(20)$$

Proof. Similar to the proofs of the previous theorems, we apply (8) on the left-hand side of (20), set $st=z$ and use the following integral (see [5]):

$$\int_{0}^{\infty}x^{\nu-1}W_{\lambda,m}(x)W_{-\lambda,m}(x)dx=\frac{\Gamma(\frac{\nu+1}{2}\pm m)\Gamma(\nu+1)}{\Gamma(1+\frac{\nu}{2}\pm\lambda)}.$$

Using (2) for series summation, we get the result. □

4. Fractional integration and fractional differentiation of the generalized Bessel-Maitland function $J^{\mu,\gamma,\delta}_{\nu,q,p}(z)$

In this section, we derive several interesting properties of the function $J^{\mu,\gamma,\delta}_{\nu,q,p}(z)$ defined by (8) associated with the operator of Riemann-Lioville fractional integrals and derivatives.

THEOREM 4.1. *Let* $\mu,\nu,\alpha,\beta,\lambda,\gamma,\delta \in \mathbb{C}$ *with* $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\lambda) > -1$. *Also, let* $p,\ q \in \mathbb{R}^+$ *with* $q \leq \Re(\mu) + p$. *Then*

$$\left(I^{(\alpha)}_{0+}[t^{\nu} J^{\mu,\gamma,\delta}_{\lambda,q,p}(wt^{\beta})]\right)(x) = \frac{x^{\alpha+\nu}\Gamma(\delta)}{\Gamma(\gamma)}\, {}_3\Psi_3\left[\begin{array}{ll}(\gamma,q),(\nu+1,\beta),(1,1); & \\ & -wx^{\beta}\\ (\delta,p),(\lambda+1,\mu),(\alpha+\nu+1,\beta); & \end{array}\right]. \tag{21}$$

Proof. Using (8) and (5) on the left-hand side of (21), we get

$$\begin{aligned}\left(I^{(\alpha)}_{0+}[t^{\nu} J^{\mu,\gamma,\delta}_{\lambda,q,p}(wt^{\beta})]\right)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x t^{\nu}(x-t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-wt^{\beta})^n}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}}\,dt\\ &= \frac{x^{\alpha+\nu}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-wx^{\beta})^n}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}}\int_0^1 z^{\nu+\beta n-1}(1-z)^{\alpha-1}\,dz\\ &= \frac{x^{\alpha+\nu}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-wx^{\beta})^n}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}}B(\alpha,\beta n+\nu+1)\\ &= \frac{x^{\alpha+\nu}\Gamma(\delta)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(\beta n+\nu+1)}{\Gamma(\mu n+\lambda+1)\Gamma(\delta+pn)\Gamma(\beta n+\alpha+\nu+1)}.\end{aligned}$$

In view of the definition (2), we get the required result (21). □

THEOREM 4.2. *Let* $\mu,\nu,\alpha,\beta,\lambda,\gamma,\delta \in \mathbb{C}$ *with* $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\lambda) > -1$. *Also, let* $p,\ q \in \mathbb{R}^+$ *with* $q \leq \Re(\mu) + p$. *Then*

$$\left(I^{(\alpha)}_{0-}[t^{\alpha-\nu-1} J^{\mu,\gamma,\delta}_{\lambda,q,p}(wt^{-\beta})]\right)(x) = \frac{x^{-\nu-1}\Gamma(\delta)}{\Gamma(\gamma)}\, {}_3\Psi_3\left[\begin{array}{ll}(\gamma,q),(\nu+1,\beta),(1,1); & \\ & -wx^{-\beta}\\ (\delta,p),(\lambda+1,\mu),(\alpha+\nu+1,\beta); & \end{array}\right]. \tag{22}$$

Proof. Using the definition (8) and (6) on the left-hand side of (22), we obtain

$$\begin{aligned}&\left(I^{(\alpha)}_{0-}[t^{\alpha-\nu-1} J^{\mu,\gamma,\delta}_{\lambda,q,p}(wt^{-\beta})]\right)(x)\\ =&\frac{1}{\Gamma(\alpha)}\int_x^{\infty} t^{-\alpha-\nu-1}(t-x)^{\alpha-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-wt^{-\beta})^n}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}}\,dt\end{aligned}$$

$$=\frac{x^{-\nu-1}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-wx^{-\beta})^n}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}}\int_0^1 z^{(\beta n+\nu+1)-1}(1-z)^{\alpha-1}\,dz$$
$$=\frac{x^{-\nu-1}}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-wx^{-\beta})^n}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}}B(\beta n+\nu+1,\alpha)$$
$$=\frac{x^{-\nu-1}\Gamma(\delta)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(\beta n+\nu+1)(-wx^{-\beta})^n}{\Gamma(\mu n+\lambda+1)\Gamma(\delta+pn)\Gamma(\beta n+\alpha+\nu+1)}.$$

In view of the definition (2), we get the required result. □

Theorem 4.3. *Let $\mu,\nu,\alpha,\beta,\lambda,\gamma,\delta\in\mathbb{C}$ with $\Re(\mu)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$, $\Re(\lambda)>-1$, $\Re(\nu+1)>\Re(\alpha)>0$. Also, let $p,\ q\in\mathbb{R}^+$ with $q\le\Re(\mu)+p$. Then*

$$\left(D_{0^+}^{(\alpha)}[t^\nu J_{\lambda,q,p}^{\mu,\gamma,\delta}(wt^\beta)]\right)(x)=$$
$$\frac{x^{\nu-\alpha}\Gamma(\delta)}{\Gamma(\gamma)}{}_3\Psi_3\left[\begin{array}{ll}(\gamma,q),(\nu+1,\beta),(1,1); & \\ & -wx^\beta\\ (\delta,p),(\lambda+1,\mu),(\nu-\alpha+1,\beta); & \end{array}\right]. \tag{23}$$

Proof. Applying the definition (8) and (7) on the left-hand side of (23), we find

$$\left(D_{0^+}^{(\alpha)}[t^\nu J_{\lambda,q,p}^{\mu,\gamma,\delta}(wt^\beta)]\right)(x)=\left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{0^+}^{1-\{\alpha\}}[t^\nu J_{\lambda,q,p}^{\mu,\gamma,\delta}(wt^\beta)]\right)(x)$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-w)^n}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^x t^{\beta n+\nu}(x-t)^{-\{\alpha\}}\,dt$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-w)^n\Gamma(\beta n+\nu+1)}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}\Gamma(\beta n+\nu-\{\alpha\}+2)}\left(\frac{d}{dx}\right)^{[\alpha]+1}(x)^{\beta n+\nu-\{\alpha\}+1}$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-w)^n(x)^{\beta n+\nu-\alpha}\Gamma(\beta n+\nu+1)}{\Gamma(\mu n+\lambda+1)(\delta)_{pn}\Gamma(\beta n+\nu-\alpha+1)}$$
$$=\frac{x^{\nu-\alpha}\Gamma(\delta)}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(\beta n+\nu+1)(-wx^\beta)^n}{\Gamma(\mu n+\lambda+1)\Gamma(\beta n+\nu-\alpha+1)\Gamma(\delta+pn)}.$$

In view of the definition (2), we get the required result. □

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