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EXISTENCE AND UNIQUENESS RESULTS FOR THREE-POINT NONLINEAR FRACTIONAL (ARBITRARY ORDER) BOUNDARY VALUE PROBLEM

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Abstract. We present here a new type of three-point nonlinear fractional boundary value problem of arbitrary order of the form

$$\begin{aligned}&{}^cD^q u(t) = f(t, u(t)), \quad t \in [0,1],\\ &u(\eta) = u'(0) = u''(0) = \cdots = u^{n-2}(0) = 0,\ I^p u(1) = 0, \qquad 0 < \eta < 1,\end{aligned}$$

where $n-1 < q \leq n$, $n \in \mathbb{N}$, $n \geq 3$ and ${}^cD^q$ denotes the Caputo fractional derivative of order q, I^p is the Riemann-Liouville fractional integral of order p, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\eta^{n-1} \neq \frac{\Gamma(n)}{(p+n-1)(p+n-2)\ldots(p+1)}$. We give new existence and uniqueness results using Banach contraction principle, Krasnoselskii, Scheafer's fixed point theorem and Leray-Schauder degree theory. To justify the results, we give some illustrative examples.

1. Introduction

Boundary value problems for differential equations of integer and non-integer order have been addressed by several researchers. Differential equations of non-integer order play important role to describe physical phenomena more accurately than the classical integer order differential equation. The need for fractional order differential equations stems in part from the fact that many phenomena cannot be modelled by differential equations with integer derivatives. Some of the areas of present-day applications of fractional calculus include Fluid Flow, Rheology, Dynamical Processes in self-similar and porous structures, Diffusive Transport akin to diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on (see [4–7] and references therein).

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Existence of solutions to fractional differential equations have received considerable interest in recent years. There are several papers dealing with the existence and uniqueness of solution to initial and boundary value problems of fractional order differential equation. Using well known fixed point theorems, like Schauder's fixed point theorem, Leray-Schauder Theorem, and the Banach contraction mapping principle, several results for linear and nonlinear equations have been obtained in the literature on fractional differential and integral equations (see, e.g., [1–7] and references therein).

In 2011, Ahmed et al. [1] studied the following boundary value problem of fractional order differential equations with three-point integral boundary conditions

$$
\begin{gathered}
{}^cD^q x(t) = f(t, x(t)),\ 0 < t < 1,\ 1 < q \leq 2,\\
x(0) = 0,\ x(1) = \alpha \int_0^\eta x(s)\, ds,\ 0 < \eta < 1,
\end{gathered}
$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, and existence and uniqueness results were proved.

In 2012, motivated by [1], Sudsutad et al. [5] discussed the existence and uniqueness of the following boundary value problem with three-point fractional integral boundary conditions

$$
\begin{gathered}
{}^cD^q x(t) = f(t, x(t)),\ t \in [0, 1],\ q \in (1, 2],\\
x(0) = 0,\ \alpha[I^p x](\eta) = x(1),\ 0 < \eta < 1.
\end{gathered}
$$

The results of [5] were complemented in [3] and extended to cover the multivalued case.

Further, Tariboon et al. [6] considered the value $x(\eta)$ for some $\eta \in (0, T)$, instead of the value $x(0)$, which appeared in all the above mentioned boundary value problems. They proved existence and uniqueness results using Leray-Schauder's nonlinear alternative for the following boundary value problem with fractional integral boundary conditions

$$
\begin{aligned}
&{}^cD^q x(t) = f(t, x(t)),\ 0 < t < T,\ q \in (1, 2],\\
&x(\eta) = 0,\ I^p x(T) = 0.
\end{aligned}
$$

Motivated by above mentioned works, this paper deals with the existence and uniqueness of solutions for the following three-point nonlinear fractional boundary value problem of arbitrary order

$$
\begin{cases}
{}^cD^q u(t) = f(t, u(t)), \quad 0 < t < 1;\\
u(\eta) = u'(0) = u''(0) = \cdots = u^{n-2}(0) = 0,\ I^p u(1) = 0,
\end{cases}
\tag{1}
$$

where $n-1 < q \leq n$, $n \in \mathbb{N}$, $n \geq 3$ and ${}^cD^q$ denotes the Caputo fractional derivative of order q, I^p is the Riemann-Liouville fractional integral of order p, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\eta^{n-1} \neq \frac{\Gamma(n)}{(p+n-1)(p+n-2)\ldots(p+1)}$. The novelty of this boundary value problem lies in the fact that we have the arbitrary order of Caputo fractional derivative. We denote $X = C([0, 1], \mathbb{R})$ as the Banach space of all continuous functions from $[0, 1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)| : t \in [0, 1]\}$.

This paper is organized as follows: Section 2 is preliminary while Section 3 contains an auxiliary result that is used to get some existence and uniqueness results in Section 4. In the final Section 5, we illustrate some examples for validation of our results.

2. Preliminaries

In this section, we introduce notation, definitions of fractional calculus and prove a lemma before stating our main results.

DEFINITION 2.1. For a continuous function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)\, ds,\ n = [q]+1$$

provided that $f^{(n)}(t)$ exists, where $[q]$ denotes the integer part of the real number q.

DEFINITION 2.2. The Riemann-Liouville fractional integral of order q for a continuous function $f(t)$ is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds,\ q > 0,$$

provided that such integral exists.

LEMMA 2.3. *([2]) Let $q > 0$, then $I^q\ {}^cD^q u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$, for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, where n is the smallest integer greater than or equal to q.*

3. Auxiliary result

Here we present a supporting result for the existence result of the next section.

LEMMA 3.1. *Let $\eta^{n-1} \neq \frac{\Gamma(n)}{(p+n-1)(p+n-2)\dots(p+1)}$, $n-1 < q \leq n$, $0 < \eta < 1$. Then for $y \in C([0,1], \mathbb{R})$, the problem*

$$\begin{cases} ^cD^q u(t) = y(t), \quad 0 < t < 1; \\ u(\eta) = u'(0) = u''(0) = \dots = u^{n-2}(0) = 0,\ I^p u(1) = 0, \end{cases} \tag{2}$$

is equivalent to the integral equation

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds - \frac{1}{\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} y(s)\, ds \\ &+ \frac{(\eta^{n-1} - t^{n-1})Q}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} y(s)\, ds - \frac{Q(\eta^{n-1} - t^{n-1})}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} y(s)\, ds \end{aligned} \tag{3}$$

where

$$Q = \frac{\Gamma(p+n)}{\Gamma(n) - \eta^{n-1}(p+n-1)(p+n-2)\dots(p+1)}. \tag{4}$$

Proof. In view of Lemma 2.3, we may reduce (2) to an equivalent integral equation

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds - c_0 - c_1 t - c_2 t^2 - \cdots - c_{n-1} t^{n-1} \tag{5}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$.

From $u'(0) = 0$, it follows $c_1 = 0$. Also, $u''(0) = 0 \Rightarrow c_2 = 0$. Continuing in this way, we have $u^{n-2}(0) = 0 \Rightarrow c_{n-2} = 0$. Thus (5) becomes

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s)\, ds - c_0 - c_{n-1} t^{n-1}.$$

$$u(\eta) = 0 \Rightarrow \frac{1}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} y(s)\, ds - c_0 - c_{n-1} \eta^{n-1} = 0. \tag{6}$$

Now

$$\begin{aligned} I^p[u(t)] &= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} [I^q y(s) - c_0 - c_{n-1} s^{n-1}]\, ds \\ &= I^p I^q y(t) - \frac{c_0 t^p}{\Gamma(p+1)} - \frac{c_{n-1} t^{p+n-1} \Gamma(n)}{\Gamma(p+n)} \\ &= \frac{1}{\Gamma(p+q)} \int_0^t (t-s)^{p+q-1} y(s)\, ds - \frac{c_0 t^p}{\Gamma(p+1)} - \frac{c_{n-1} t^{p+n-1} \Gamma(n)}{\Gamma(p+n)}. \end{aligned}$$

$$I^p[u(1)] = 0 \Rightarrow \frac{1}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} y(s)\, ds - \frac{c_0}{\Gamma(p+1)} - \frac{c_{n-1}\Gamma(n)}{\Gamma(p+n)} = 0. \tag{7}$$

On solving (6) and (7) for $c_0,\ c_{n-1}$, we have:

$$c_{n-1} = \frac{Q}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} y(s)\, ds - \frac{Q}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} y(s)\, ds$$

and

$$\begin{aligned} c_0 = &\frac{1}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} y(s)\, ds - \frac{\eta^{n-1} Q}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} y(s)\, ds \\ &+ \frac{\eta^{n-1} Q}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} y(s)\, ds \end{aligned}$$

where Q is defined in (4). On putting the values of c_i in (5), we obtain the solution (3). □

The converse of the above lemma follows from Definition 2.2 and Lemma 2.3.

In view of Lemma 3.1, the BVP (1) can be written as a fixed point problem. For this, we consider the operator $P : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ that is defined by

$$\begin{aligned} P(u)(t) = &\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds - \frac{1}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} f(s, u(s))\, ds \\ &+ \frac{(\eta^{n-1} - t^{n-1})Q}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} f(s, u(s))\, ds \\ &- \frac{Q(\eta^{n-1} - t^{n-1})}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} f(s, u(s))\, ds. \end{aligned} \tag{8}$$

To simplify and to our convenience, we put

$$\Lambda = \frac{2}{\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+1)\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+q+1)}. \tag{9}$$

4. Main results

In this section, we develop four different types of results for existence and uniqueness of the proposed nonlinear fractional differential equation (1). To complete the results, we use Banach contraction principle, Krasnoselskii, Scheafer's fixed point theorem and Leray-Schauder degree theorem.

The first result is based on Banach contraction principle.

THEOREM 4.1. *(A) Assume that there exists a constant $L > 0$ such that $|f(t,u) - f(t,v)| \leq L|u - v|$ for each $t \in [0,1]$, and all $u, v \in \mathbb{R}$, and*

(B) $L\Lambda < 1$, where Λ is defined by (9), then the BVP (1) has a unique solution on $[0,1]$.

Proof. Obviously, fixed points of the operator P defined by (8) are solutions of the problem (1). We shall prove that P is a contraction.

Let $u, v \in C([0,1], \mathbb{R})$, then for each $t \in [0,1]$, we have

$$\begin{aligned}
|P(u)(t) - P(v)(t)| \leq{}& \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,u(s)) - f(s,v(s))|\, ds \\
&+ \frac{1}{\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} |f(s,u(s)) - f(s,v(s))|\, ds \\
&+ \frac{|Q|}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} |f(s,u(s)) - f(s,v(s))|\, ds \\
&+ \frac{|Q|}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} |f(s,u(s)) - f(s,v(s))|\, ds \\
\leq{}& L\|u - v\| \Bigg[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds + \frac{1}{\Gamma(q)} \int_0^\eta (\eta - s)^{q-1}\, ds \\
&+ \frac{|Q|}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta - s)^{q-1}\, ds + \frac{|Q|}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1}\, ds \Bigg] \\
\leq{}& L\|u - v\| \left[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+1)\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+q+1)} \right].
\end{aligned}$$

Thus $\|P(u) - P(v)\| \leq L\Lambda \|u - v\|$. As $L\Lambda < 1$, P is a contraction, which satisfies all the conditions of Banach Contraction Principle. Hence, P has a fixed point which is a solution of the problem (1). ☐

We prove the following result by using Krasnoselskii's fixed point theorem

THEOREM 4.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that the hypothesis (A) of Theorem 4.1 and the following hypotheses hold*

(C) $|f(t,u)| \leq \phi(t)$ *for all* $(t,u) \in [0,1] \times \mathbb{R}$ *and* $\phi \in C([0,1], \mathbb{R}^+)$*;*

(D)
$$L\left[\frac{|Q|}{\Gamma(p+1)\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+q+1)}\right] < 1. \tag{10}$$

Then the boundary value problem (1) *has at least one solution defined on* $[0,1]$*.*

Proof. Suppose $\sup_{t\in[0,1]} |\phi(t)| = \|\phi\|$; we fix $\epsilon \geq \|\phi\|\Lambda$ and consider $B_\epsilon = \{u \in C([0,1], \mathbb{R}) : \|u\| \leq \epsilon\}$. We define the operators F_1 and F_2 on B_ϵ as:

$$(F_1u)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))\,ds - \frac{1}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} f(s,u(s))\,ds, \quad t \in [0,1].$$

$$\begin{aligned}(F_2u)(t) =& \frac{(\eta^{n-1} - t^{n-1})Q}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} f(s,u(s))\,ds \\ &- \frac{Q(\eta^{n-1} - t^{n-1})}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} f(s,u(s))\,ds, \quad t \in [0,1].\end{aligned}$$

For $u, v \in B_\epsilon$, we have:

$$\begin{aligned}\|F_1u + F_2v\| \leq& \frac{\|\phi\|}{\Gamma(q)} \int_0^t (t-s)^{q-1}\,ds + \frac{\|\phi\|}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1}\,ds \\ &+ \frac{\|\phi\|\ |\eta^{n-1} - t^{n-1}|\ |Q|}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1}\,ds \\ &+ \frac{\|\phi\|\ |Q|\ |\eta^{n-1} - t^{n-1}|}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta-s)^{q-1}\,ds \\ \leq& \frac{2\ \|\phi\|}{\Gamma(q+1)} + \frac{\|\phi\|\ |Q|}{\Gamma(p+1)\Gamma(q)} + \frac{\|\phi\|\ |Q|}{\Gamma(p+q)} \leq \epsilon.\end{aligned}$$

Thus $u, v \in B_\epsilon \Rightarrow F_1u + F_2v \in B_\epsilon$. From (A) and (10), F_2 is a contraction mapping. From the continuity of f, one obtains that the operator F_1 is continuous. Also F_1 is uniformly bounded on B_ϵ as

$$\|F_1u\| \leq \frac{2\ \|\phi\|}{\Gamma(q+1)}.$$

Now, we will prove that the operator F_1 is compact. Define

$$\sup_{(t,u)\in[0,1]\times B_\epsilon} |f(t,u)| = f_s.$$

Then, we have

$$\begin{aligned}|(F_1u)(t_1) - (F_1u)(t_2)| =& \left|\frac{1}{\Gamma(q)} \int_0^{t_1} \left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] f(s,u(s))\,ds \right.\\ &\left. + \int_{t_1}^{t_2} (t_2-s)^{q-1} f(s,u(s))\,ds\right| \\ \leq& \frac{f_s}{\Gamma(q+1)}\ |2(t_2-t_1)^q + t_1^q - t_2^q|\end{aligned}$$

which is independent of u and tends to zero as $t_1 \to t_2$. Thus F_1 is equicontinuous.

Hence by Arzelá-Ascoli theorem, F_1 is compact on B_ϵ. Thus all the assumption of Krasnoselskii's fixed point theorem are satisfied. So the conclusion of Krasnoselskii's fixed point theorem implies that the boundary value problem (1) has at least one solution defined on $[0,1]$. $\square$

The next result is based on Schaefer's fixed point theorem.

THEOREM 4.3. *Assume that the function* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is continuous and the following assumption holds.*

(E) There exists a constant $\mu > 0$ *such that* $|f(t,u)| \le \mu$ *for each* $t \in [0,1]$ *and* $u \in \mathbb{R}$.

Then the BVP (1) *has at least one solution on* $[0,1]$.

Proof. We divide the proof into four steps.

Step I. Continuity of P.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C([0,1],\mathbb{R})$. Then for each $t \in [0,1]$

$$\begin{aligned}
|P(u_n)(t) - P(u)(t)| \le{}& \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,u_n(s)) - f(s,u(s))|\, ds \\
&+ \frac{1}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} |f(s,u_n(s)) - f(s,u(s))|\, ds \\
&+ \frac{|Q|}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} |f(s,u_n(s)) - f(s,u(s))|\, ds \\
&+ \frac{|Q|}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} |f(s,u_n(s)) - f(s,u(s))|\, ds \\
\le{}& \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \sup_{s\in[0,1]} |f(s,u_n(s)) - f(s,u(s))|\, ds \\
&+ \frac{1}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} \sup_{s\in[0,1]} |f(s,u_n(s)) - f(s,u(s))|\, ds \\
&+ \frac{|Q|}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} \sup_{s\in[0,1]} |f(s,u_n(s)) - f(s,u(s))|\, ds \\
&+ \frac{|Q|}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} \sup_{s\in[0,1]} |f(s,u_n(s)) - f(s,u(s))|\, ds.
\end{aligned}$$

Since f is continuous function, then $\|P(u_n) - P(u)\| \to 0$ as $n \to \infty$. This means that P is continuous.

Step II. P maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

So, let us prove that for any $\epsilon > 0$, there exists a positive constant M such that for each $u \in B_\epsilon = \{u \in C([0,1],\mathbb{R}) : \|u\| \le \epsilon\}$, we have $\|P(u)\| \le M$. Now, for any $u \in B_\epsilon$, by using (8) and (E), we have:

$$|P(u)(t)| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,u(s))|\, ds + \frac{1}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} |f(s,u(s))|\, ds$$

$$+\frac{|Q|}{\Gamma(p+q)}\int_0^1(1-s)^{p+q-1}|f(s,u(s))|\,ds+\frac{|Q|}{\Gamma(p+1)\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}|f(s,u(s))|\,ds$$
$$\leq\frac{\mu}{\Gamma(q)}\int_0^t(t-s)^{q-1}\,ds+\frac{\mu}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}\,ds+\frac{\mu|Q|}{\Gamma(p+q)}\int_0^1(1-s)^{p+q-1}\,ds$$
$$+\frac{\mu|Q|}{\Gamma(p+1)\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}\,ds\leq\Lambda\mu.$$

Thus, $\|P(u)\|\leq\Lambda\mu:=M$ which implies that P maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

Step III. $P(B_\epsilon)$ is equicontinuous with B_ε defined as in step II.
Let $0\leq t_1<t_2\leq 1$ and $u\in B_\epsilon$. Using (8) and (E), we have:

$$\begin{aligned}|P(u)(t_2)-P(u)(t_1)|\leq&\frac{\mu}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\,ds\\&+\frac{\mu}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\,ds+\frac{\mu|Q|\;|t_1^{n-1}-t_2^{n-1}|}{\Gamma(p+1)\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}\,ds\\&+\frac{\mu|t_1^{n-1}-t_2^{n-1}|\;|Q|}{\Gamma(p+q)}\int_0^1(1-s)^{p+q-1}\,ds\\\leq&\frac{\mu}{\Gamma(q+1)}\left[(t_2-t_1)^q+(t_2^q-t_1^q)\right]+\frac{\mu(t_2-t_1)^q}{\Gamma(q+1)}\\&+\frac{\mu|Q|\;|t_1^{n-1}-t_2^{n-1}|}{\Gamma(p+q+1)}+\frac{\mu|Q|\;|t_1^{n-1}-t_2^{n-1}|}{\Gamma(p+1)\Gamma(q+1)}.\end{aligned}$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps I to III together with the Arzelá-Ascoli theorem, we get that $P:C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is completely continuous.

Step IV. We show that the set

$$\Theta=\{u\in C([0,1],\mathbb{R}):u=\theta P(u)\text{ for some }0<\theta<1\}$$

is bounded.

Let $u\in\Theta$. Then $u(t)=\theta P(u)(t)$ for some $0<\theta<1$. Now, for each $t\in[0,1]$, using (8) and (E), we have

$$\begin{aligned}|u(t)|=&|\theta P(u)(t)|\leq\frac{\theta\mu}{\Gamma(q)}\int_0^t(t-s)^{q-1}\,ds+\frac{\theta\mu}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}\,ds\\&+\frac{\theta\mu|Q|\;|\eta^{n-1}-t^{n-1}|}{\Gamma(p+q)}\int_0^1(1-s)^{p+q-1}\,ds\\&+\frac{\theta\mu|Q|\;|\eta^{n-1}-t^{n-1}|}{\Gamma(p+1)\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}\,ds\\\leq&\frac{\mu}{\Gamma(q+1)}+\frac{\mu}{\Gamma(q+1)}+\frac{\mu|Q|}{\Gamma(p+q+1)}+\frac{\mu|Q|}{\Gamma(p+1)\Gamma(q+1)}.\end{aligned}$$

Hence, we get $\|u\|\leq\Lambda\mu:=N$.
This implies that the set Θ is bounded. By Schaefer's fixed point theorem, P has a fixed point which is a solution of the problem (1). □

Our final existence result is based on Leray-Schauder degree theory.

THEOREM 4.4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(F) there exist constant $0 < k < \frac{1}{\Lambda}$ where Λ is given by (9) and $M > 0$ such that $|f(t,u)| \le k|u| + M$, $\forall t \in [0,1]$, $u \in C[0,1]$.

Then the boundary value problem (1) has at least one solution.

Proof. Let us consider the fixed point problem

$$Pu = u \tag{11}$$

where P is defined by (8). In view of the fixed point problem (11) we just need to prove the existence of at least one solution $u \in C[0,1]$ satisfying (11). Define a ball $B_r \subset C[0,1]$ with radius $r > 0$ as

$$B_r = \Big\{ u \in C[0,1] : \max_{t\in[0,1]} |u(t)| < r \Big\},$$

where r will be fixed later. Then, it is sufficient to show that $F : \overline{B_r} \to C[0,1]$ satisfies

$$u \ne \lambda P u, \quad \forall u \in \partial B_r, \quad \forall \lambda \in [0,1]. \tag{12}$$

Let us define

$$G(\lambda, u) = \lambda P u, \quad \forall u \in C(\mathbb{R}), \quad \lambda \in [0,1].$$

Then, by Arzelá-Ascoli Theorem, $g_\lambda(u) = u - G(\lambda,u) = u - \lambda P u$ is completely continuous. If (12) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} deg(g_\lambda, B_r, 0) &= deg(I - \lambda P, B_r, 0) = deg(g_1, B_r, 0) \\ &= deg(g_0, B_r, 0) = deg(I, B_r, 0) = 1 \ne 0, \ 0 \in B_r \end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $g_1(t) = u - \lambda P u = 0$ for at least one $u \in B_r$. In order to prove (12), we assume that $u = \lambda P u$, $\lambda \in [0,1]$. Then for $u \in \partial B_r$ and $t \in [0,1]$ we have

$$\begin{aligned} |u(t)| =& |\lambda (Pu)(t)| \\ \le& \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,u(s))|\, ds + \frac{1}{\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} |f(s,u(s))|\, ds \\ &+ \frac{|Q|}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1} |f(s,u(s))|\, ds \\ &+ \frac{|Q|}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} |f(s,u(s))|\, ds \\ \le& (k\|u\| + M) \Bigg[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\, ds + \frac{1}{\Gamma(q)} \int_0^\eta (\eta - s)^{q-1}\, ds \\ &+ \frac{|Q|}{\Gamma(p+q)} \int_0^1 (1-s)^{p+q-1}\, ds + \frac{|Q|}{\Gamma(p+1)\Gamma(q)} \int_0^\eta (\eta - s)^{q-1}\, ds \Bigg] \\ \le& (k\|u\| + M)\Lambda \end{aligned}$$

which, taking norm ($\sup_{t\in[0,1]}|u(t)| = \|u\|$) and solving for $\|u\|$, yields

$$\|u\| \leq \frac{M\Lambda}{1-k\Lambda}.$$

On taking $r = \frac{M\Lambda}{1-k\Lambda} + 1$, (12) holds. This completes the proof. ☐

5. Examples

In this section, we discuss some examples to illustrate our results (Theorem 4.1–Theorem 4.4).

EXAMPLE 5.1. Consider the first type of fractional differential equation

$$^cD^{\frac{9}{2}}u(t) = \frac{1}{(t+7)^2}\frac{|u(t)|}{1+|u(t)|}, \quad t \in [0,1] \tag{13}$$

with three-point boundary value conditions

$$u(\frac{1}{10}) = 0,\ u'(0) = 0,\ I^{\frac{7}{2}}u(1) = 0. \tag{14}$$

Here $q = \frac{9}{2} \Rightarrow n = 5$, $\eta = \frac{1}{10}$, $p = \frac{7}{2}$, $\eta^{n-1} = \eta^4 = \frac{1}{10000} \neq \frac{\Gamma(n)}{(p+n-1)(p+n-2)\ldots(p+1)} \neq \frac{4}{(p+1)(p+2)(p+3)(p+4)} = \frac{64}{19305} = 0.003315$ and $f(t,u(t)) = \frac{1}{(t+10)^2}\frac{|u(t)|}{1+|u(t)|}$. As $|f(t,u) - f(t,v)| \leq \frac{1}{49}|u-v|$, (A) is satisfied with $L = \frac{1}{49}$. Further,

$$|Q| = \frac{\Gamma(p+5)}{|\Gamma(5) - \eta^4(p+4)(p+3)(p+2)(p+1)|} = \frac{2027025\sqrt{\pi}}{993.112} = 3617.73$$

and thus

$$\begin{aligned} L\Lambda &= L\left[\frac{2}{\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+1)\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+q+1)}\right] \\ &= \frac{1}{49}\left[\frac{64}{945\sqrt{\pi}} + \frac{3617.73 \times 2^9}{99225\pi} + \frac{3617.73}{7!}\right] \\ &= \frac{1}{49}[0.0382 + 5.9420 + 0.7178] \approx 0.1366 < 1. \end{aligned}$$

Hence all the conditions of Theorem 4.1 are satisfied, and therefore the boundary value problem (13)-(14) has a unique solution on $[0,1]$.

EXAMPLE 5.2. Consider the following fractional differential equation

$$^cD^{\frac{5}{2}}u(t) = \frac{1}{(t+8)^2}\sin u, \quad t \in [0,1] \tag{15}$$

with three-point boundary value conditions

$$u(\frac{1}{4}) = 0,\ u'(0) = 0,\ I^{\frac{1}{2}}u(1) = 0. \tag{16}$$

Here $\eta = \frac{1}{4}$, $q = \frac{5}{2}$, $p = \frac{1}{2}$, $\eta^{n-1} = \eta^2 = \frac{1}{16} \neq \frac{\Gamma(n)}{(p+n-1)(p+n-2)\ldots(p+1)} = \frac{2}{(p+1)(p+2)} = \frac{8}{15}$, $f(t,u) = \frac{1}{(t+8)^2}\sin u$. Clearly, $|f(t,u) - f(t,v)| \leq \frac{1}{64}|\sin u - \sin v| \leq \frac{1}{64}|u-v|$. Thus (A) is satisfied with $L = \frac{1}{64} > 0$. Also, $|f(t,u)| \leq \frac{17}{16} = \phi(t)$,i.e., (C) is satisfied.

Here

$$|Q| = \frac{\Gamma(p+3)}{|\Gamma(3) - \eta^2(p+2)(p+1)|} = \frac{120\sqrt{\pi}}{113}$$

and thus

$$L\left\{\frac{|Q|}{\Gamma(p+1)\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+q+1)}\right\} = \frac{1}{64}\left[\frac{120\times 16}{113\times 15\times\sqrt{\pi}} + \frac{120\sqrt{\pi}}{113\times 6}\right] = 0.0148 < 1.$$

Hence, all the conditions of Theorem 4.2 are satisfied and consequently the boundary value problem (15)-(16) has at least one solution on $[0,1]$.

EXAMPLE 5.3. Consider the following three-point fractional integral boundary value problem

$$^{c}D^{\frac{9}{4}}u(t) = \frac{8t^2}{9} + \frac{e^{-2t}\cos t}{10+\sin u}, \quad t\in[0,1] \tag{17}$$

with

$$u(\frac{1}{3}) = 0, \ u'(0) = 0, \ I^{\frac{4}{5}}u(1) = 0. \tag{18}$$

Here $\eta = \frac{1}{3}$, $q = \frac{9}{4}$, $p = \frac{4}{5}$, $\eta^{n-1} = \eta^2 = \frac{1}{9} \neq \frac{\Gamma(n)}{(p+n-1)(p+n-2)...(p+1)} = \frac{2}{(p+1)(p+2)} = \frac{25}{43}$, $f(t,u) = \frac{8t^2}{9} + \frac{e^{-2t}\cos t}{10+\sin u}$. Clearly, $|f(t,u)| \leq \frac{8}{9} + \frac{1}{9} = 1 = \mu$.

Thus, all the conditions of Theorem 4.3 are satisfied and hence the boundary value problem (17)-(18) has at least one solution on $[0,1]$.

EXAMPLE 5.4. Consider the following fractional differential equation

$$^{c}D^{\frac{5}{2}}u(t) = \frac{1}{100\pi}\sin(2\pi u)\frac{|u(t)|}{4(1+|u(t)|)} + \frac{3}{4}, \quad t\in[0,1] \tag{19}$$

with three-point boundary value conditions

$$u(\frac{1}{2}) = 0, \ u'(0) = 0, \ I^{\frac{3}{2}}u(1) = 0. \tag{20}$$

Here $q = \frac{5}{2} \Rightarrow n = 3$, $\eta = \frac{1}{2}$, $p = \frac{3}{2}$, $\eta^{n-1} = \eta^2 = \frac{1}{4} \neq \frac{\Gamma(n)}{(p+n-1)(p+n-2)...(p+1)} = \frac{2}{(p+2)(p+1)} = \frac{8}{35}$ and $f(t,u(t)) = \frac{1}{100\pi}\sin(2\pi u)\frac{|u(t)|}{4(1+|u(t)|)} + \frac{3}{4}$. As $|f(t,u)| \leq \frac{1}{50}|u| + 1$, therefore (F) is satisfied with $k = \frac{1}{50}$ and $M = 1$. Further,

$$\begin{aligned}
|Q| &= \frac{\Gamma(p+3)}{|\Gamma(3) - \eta^2(p+2)(p+1)|} = \frac{105\sqrt{\pi}}{3} \\
k &= \frac{1}{50} \leq \frac{1}{\Lambda} = L\left[\frac{2}{\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+1)\Gamma(q+1)} + \frac{|Q|}{\Gamma(p+q+1)}\right] \\
&= \left[\frac{16}{15\sqrt{\pi}} + \frac{35\times 2^5}{45\sqrt{\pi}} + \frac{105\sqrt{\pi}}{3\times 24}\right]^{-1} \approx 0.034946.
\end{aligned}$$

Hence all the conditions of Theorem 4.4 are satisfied, therefore the boundary value problem (19)-(20) has at least one solution on $[0,1]$.

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