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COARSER COMPACT TOPOLOGIES

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Abstract. The concept of a quasi-king space is introduced, which is a natural generalisation of a king space. In the realm of suborderable spaces, king spaces are precisely the compact spaces, so are the quasi-king spaces. In contrast, quasi-king spaces are more flexible in handling coarser selection topologies. The main purpose of this paper is to show that a weakly orderable space is quasi-king if and only if all of its coarser selection topologies are compact.

1. Introduction

Throughout the paper, all spaces are Hausdorff topological spaces. For a set X, let $\mathscr{F}_2(X) = \{S \subset X : 1 \leq |S| \leq 2\}$. A map $\sigma : \mathscr{F}_2(X) \to X$ is a *weak selection* for X if $\sigma(S) \in S$ for every $S \in \mathscr{F}_2(X)$. Every weak selection σ for X generates an order-like relation $\leq_\sigma$ on X defined by $x \leq_\sigma y$ if $\sigma(\{x, y\}) = x$ [19, Definition 7.1]; and we write $x <_\sigma y$ if $x \leq_\sigma y$ and $x \neq y$. The relation $\leq_\sigma$ is similar to a linear order being both total and antisymmetric, but is not necessarily transitive. If X is a topological space, then σ is *continuous* if it is continuous with respect to the Vietoris topology on $\mathscr{F}_2(X)$. This can be expressed only in terms of $\leq_\sigma$ by the property that for every $x, y \in X$ with $x <_\sigma y$, there are open sets $U, V \subset X$ such that $x \in U$, $y \in V$ and $s <_\sigma t$ for every $s \in U$ and $t \in V$, see [9, Theorem 3.1].

In 1921, studying dominance hierarchy in chickens and other birds, Schjelderup-Ebbe coined the term "pecking order". Subsequently, in 1951, H. G. Landau [17] (see also [18]) used this 'order' to show that any finite flock of chickens has a most dominant one, called a *king*. Landau's mathematical model was based on Graph Theory and became known as "The King Chicken Theorem". The pecking order is rarely linear, in fact it is equivalent to the existence of a weak selection σ on the flock X. In this interpretation, an element $q \in X$ is called a σ*-king* if for every $x \in X$ there exists $y \in X$ with $x \leq_\sigma y \leq_\sigma q$. Thus, Landau actually showed that each weak

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selection σ on a finite set X has a σ-king. Extending Landau's result to the setting of topological spaces, Nagao and Shakhmatov called a space X to be a *king space* [22] if X has a continuous weak selection, and every continuous weak selection σ for X has a σ-king. Next, they showed that every compact space with a continuous weak selection is a king space [22, Theorem 2.3]. In the inverse direction, Nagao and Shakhmatov showed that each linearly ordered king space is compact ([22, Corollary 3.3]); also that each king space which is either pseudocompact, or zero-dimensional, or locally connected, is compact as well ([22, Theorem 3.5]). Subsequently, answering a question of [22], it was shown in [8, Theorem 4.1] that each locally pseudocompact king space is also compact.

On the other hand, there are simple examples of connected king spaces which are not compact. For instance, such a space is the topological sine curve

$$X = \{(0,0)\} \cup \{(t, \sin 1/t) : 0 < t \leq 1\}.$$

However, X has a coarser topology — that of the interval $[0,1]$, which is compact and admits the same compatible pecking orders (i.e. the same continuous weak selections). In this paper, we address such spaces, and study the compactness of coarser topologies induced by weak selections. To state our main result, we briefly recall some related terminology. Any weak selection σ for X generates a natural topology $\mathscr{T}_\sigma$ on X [9], called a *selection topology* and defined following the pattern of the open interval topology, see Section 2. If X is a space and σ is continuous, then $\mathscr{T}_\sigma$ is a coarser topology on X, but σ is not necessarily continuous with respect to $\mathscr{T}_\sigma$ [9] (see also [11, 13]). A weak selection σ for a space X is called *properly continuous* if $\mathscr{T}_\sigma$ is a coarser topology on X and σ is continuous with respect to $\mathscr{T}_\sigma$ [12, Definition 4.4]. Thus, every properly continuous weak selection is continuous, but the converse is not necessarily true. For a weak selection σ for X, we will say that a point $q \in X$ is a *quasi σ-king* if for each $x \in X$ there are finitely many points $y_1, \dots, y_n \in X$ with $x \leq_\sigma y_1 \leq_\sigma \cdots \leq_\sigma y_n \leq q$. Finally, we shall say that X is a *quasi-king space* if X has a weak selection σ such that $\mathscr{T}_\sigma$ is a coarser topology on X, and each such weak selection σ has a quasi σ-king. The following theorem will be proved in this paper.

THEOREM 1.1. *Let X be a quasi-king space with a properly continuous weak selection. Then each coarser selection topology on X is compact.*

Regarding the proper place of Theorem 1.1, let us remark that a quasi-king space is a relaxed version of a king space allowing dominance in several intermediate steps. Using this, in Section 3 we give a simple direct proof that each weak selection σ on a set X, which generates a compact selection topology $\mathscr{T}_\sigma$, has a quasi σ-king (Theorem 3.4). In the same section, we also give an example of a space X with a continuous weak selection σ which admits a quasi σ-king, but has no σ-king (Example 3.3). On the other hand, all mentioned results for king spaces remain valid for quasi-king spaces. Namely, in Section 4 we show that each suborderable quasi-king space is compact (Proposition 4.3), which is an element in the proof of Theorem 1.1. This brings the following natural question.

QUESTION 1. Does there exist a quasi-king space which is not a king space?

Theorem 1.1 also gives a partial solution to a problem in the theory of continuous weak selections. Briefly, a space is called *weakly orderable* if it has a coarser orderable topology, see Section 2. Michael showed that each connected space with a continuous weak selection is weakly orderable [19, Lemma 7.2]. Subsequently, van Mill and Wattel showed that in the realm of compact spaces, Michael's result remains valid without connectedness, namely that each compact space with a continuous weak selection is (weakly) orderable [20, Theorem 1.1]. This led them to pose the question whether a space with a continuous weak selection is weakly orderable; the question itself became known as the *weak orderability problem.* In 2009, Hrušák and Martínez-Ruiz gave a counterexample by constructing a separable, first countable and locally compact space which admits a continuous weak selection but is not weakly orderable [15, Theorem 2.7]; the interested reader is also referred to [12] where the construction was discussed in detail. However, this counterexample is a special Isbell-Mrówka space which is not normal. Thus, the weak orderability problem still remains open in the realm of normal spaces, see [12, Question 5]. Another special case of this problem was proposed in [12], it is based on the fact that each weakly orderable space has a properly continuous weak selection [12, Corollary 4.5]. Namely, the following question was raised in [12, Question 3], also in [6, Problem 4.31].

Question 2. Let X be a space which has a properly continuous weak selection. Then, is it true that X is weakly orderable?

An essential element in the proof of Theorem 1.1 is that in the realm of quasi-king spaces, the answer to Question 2 is in the affirmative, see Corollary 6.5.

The paper is organised as follows. In the next section, we give a brief account on various orderable-like spaces. The idea of quasi-king spaces is discussed in Section 3. In Section 4, we show a special case of Theorem 1.1 that each clopen cover of a weakly orderable quasi-king space has a finite subcover (Theorem 4.1). This is used further in Section 5 to show that each coarser selection topology on a weakly orderable quasi-king space is compact (Theorem 5.1). The proof of Theorem 1.1 is finally accomplished in Section 6 by showing that for a quasi-king space, the selection topology induced by any properly continuous weak selection is pseudocompact, hence compact as well (Theorem 6.1).

2. Selection topologies

Let σ be a weak selection for X, and $\leq_\sigma$ be the order-like relation generated by σ, see the Introduction. For subsets $A, B \subset X$, we write that $A \leq_\sigma B$ ($A <_\sigma B$) if $x \leq_\sigma y$ (respectively, $x <_\sigma y$) for every $x \in A$ and $y \in B$. For a singleton $A = \{x\}$, we will simply write $x \leq_\sigma B$ or $x <_\sigma B$ instead of $\{x\} \leq_\sigma B$ or $\{x\} <_\sigma B$; in the same way, we write $A \leq_\sigma y$ or $A <_\sigma y$ for a singleton $B = \{y\}$. Finally, we will use the standard notation for the intervals generated by $\leq_\sigma$. For instance, $(\leftarrow, x)_{\leq_\sigma}$ will stand for all $y \in X$ with $y <_\sigma x$; $(\leftarrow, x]_{\leq_\sigma}$ for that of all $y \in X$ with $y \leq_\sigma x$; the $\leq_\sigma$-intervals $(x, \rightarrow)_{\leq_\sigma}$ and $[x, \rightarrow)_{\leq_\sigma}$ are similarly defined. However, working with such intervals should be done with caution keeping in mind that the relation $\leq_\sigma$ is not necessarily transitive.

Each weak selection σ for X generates a natural topology $\mathscr{T}_\sigma$ on X, called a *selection topology* [9, 11]. It is patterned after the open interval topology by taking the collection $\{(\leftarrow, x)_{\leq_\sigma}, (x, \rightarrow)_{\leq_\sigma} : x \in X\}$ as a subbase for $\mathscr{T}_\sigma$. Thus, $\mathscr{T}_\sigma = \mathscr{T}_{\leq_\sigma}$ is the usual open interval topology, whenever $\leq_\sigma$ is a linear order on X. Each selection topology $\mathscr{T}_\sigma$ is Tychonoff [16, Theorem 2.7]. On the other hand, $\mathscr{T}_\sigma$ may lack several of the other strong properties of the open interval topology, see [4, 13].

If σ is a continuous weak selection for a topological space $(X, \mathscr{T})$, then $\mathscr{T}_\sigma \subset \mathscr{T}$. The converse is not true, and the inclusion $\mathscr{T}_\sigma \subset \mathscr{T}$ does not imply continuity of σ even in the realm of compact spaces, see [1, Example 1.21], [9, Example 3.6] and [12, Example 4.3]. In particular, a continuous weak selection σ is not necessarily continuous with respect to $\mathscr{T}_\sigma$. Based on this, a weak selection σ for a space $(X, \mathscr{T})$ was called

(i) *separately continuous* if $\mathscr{T}_\sigma \subset \mathscr{T}$ [1, 12]; and
(ii) *properly continuous* if $\mathscr{T}_\sigma \subset \mathscr{T}$ and σ is continuous with respect to $\mathscr{T}_\sigma$ [12].

Thus, each properly continuous weak selection is continuous, and each continuous one is separately continuous, but none of these implications is reversible.

In what follows, for a weak selection σ for X, we will write $\sigma \restriction Z$ to denote the restriction of σ on a subset $Z \subset X$, i.e. $\sigma \restriction Z = \sigma \restriction \mathscr{F}_2(Z)$. Similarly, for a topology $\mathscr{T}$ on X, we will use $\mathscr{T} \restriction Z$ for the subspace topology on Z. The following properties are evident from the definitions, and are left to the reader.

PROPOSITION 2.1. *Let σ be a weak selection for X. Then:*
(i) $\mathscr{T}_{\sigma \restriction Z} \subset \mathscr{T}_\sigma \restriction Z$, whenever $Z \subset X$;
(ii) σ is separately continuous with respect to $\mathscr{T}_\sigma$;
(iii) σ is continuous with respect to $\mathscr{T}_\sigma$, whenever $\leq_\sigma$ is a linear order on X.

For a topology $\mathscr{T}$ on X, we will use the prefix "$\mathscr{T}$-" to express properties of subsets of X with respect to this topology. If σ is a continuous selection for a connected space X, then $\leq_\sigma$ is a linear order on X and $\mathscr{T}_\sigma$ is a coarser topology on X [19, Lemma 7.2], which gives that X is weakly orderable with respect to $\leq_\sigma$. The property remains valid for separately continuous weak selections, and will play an important role in the paper.

PROPOSITION 2.2. *Let σ be a weak selection for X and $Z \subset X$ be a $\mathscr{T}_\sigma$-connected subset of X. Then:*
(i) $x \notin Z$ if and only if $x <_\sigma Z$ or $Z <_\sigma x$;
(ii) $\mathscr{T}_{\sigma \restriction Z} = \mathscr{T}_\sigma \restriction Z$ is the subspace topology on Z;
(iii) $\leq_\sigma$ is a linear order on Z.
In particular, $\sigma \restriction Z$ is a continuous weak selection for $(Z, \mathscr{T}_{\sigma \restriction Z})$.

Proof. The property in (i) is [7, Proposition 2.4], while (ii) is [7, Proposition 2.5]. The property (iii) is [9, Proposition 2.2]. The second part now follows from Proposition 2.1, see also [1, Proposition 1.22]. □

Let $\mathscr{D}$ be a partition of X and γ be a weak selection for $\mathscr{D}$. Following the idea of lexicographical sums of linear orders, to each collection of weak selections $\eta_\Delta : \mathscr{F}_2(\Delta) \to \Delta$, for $\Delta \in \mathscr{D}$, we will associate the weak selection σ for X defined by

$$\begin{cases} \sigma \restriction \Delta = \eta_\Delta, & \text{for every } \Delta \in \mathscr{D}, \\ \Gamma <_\sigma \Delta, & \text{whenever } \Gamma, \Delta \in \mathscr{D} \text{ with } \Gamma <_\gamma \Delta. \end{cases} \tag{1}$$

We will refer to σ as the *lexicographical* γ*-sum* of η_Δ, $\Delta \in \mathscr{D}$, or simply as the *lexicographical sum*, and will denote it by $\sigma = \sum_{(\gamma,\Delta\in\mathscr{D})} \eta_\Delta$. In the case $\eta_\Delta = \eta \restriction \Delta$, $\Delta \in \mathscr{D}$, for some weak selection η for X, the lexicographical sum $\sum_{(\gamma,\Delta\in\mathscr{D})} \eta_\Delta$ was used in [7] under the name of a $(\mathscr{D}, \gamma)$*-clone* of η.

Proposition 2.3. *Let $\mathscr{D}$ be an open partition of a space X, γ be a weak selection for $\mathscr{D}$, and η_Δ be a separately continuous weak selection for Δ, for each $\Delta \in \mathscr{D}$. Then the lexicographical γ-sum $\sigma = \sum_{(\gamma,\Delta\in\mathscr{D})} \eta_\Delta$ is a separately continuous weak selection for X. Moreover, σ is continuous provided so is each η_Δ, $\Delta \in \mathscr{D}$.*

Proof. Let $\Delta \in \mathscr{D}$ and $x \in \Delta$. According to (1), we have that

$$(\leftarrow, x)_{\le_\sigma} = (\leftarrow, x)_{\le_{\eta_\Delta}} \cup \bigcup_{\Gamma <_\gamma \Delta} \Gamma.$$

Hence, $(\leftarrow, x)_{\le_\sigma}$ is open in X because η_Δ is separately continuous and $\mathscr{D}$ consists of open sets. Similarly, $(x, \rightarrow)_{\le_\sigma}$ is also open. Thus, σ is separately continuous.

Suppose that each η_Δ, $\Delta \in \mathscr{D}$, is continuous. To show that σ is also continuous, take $p, q \in X$ with $p <_\sigma q$. It now suffices to find open sets $U, V \subset X$ such that $p \in U$, $q \in V$ and $U <_\sigma V$. To this end, let $\Delta_p, \Delta_q \in \mathscr{D}$ be the unique elements with $p \in \Delta_p$ and $q \in \Delta_q$. If $\Delta_p \ne \Delta_q$, then by (1), $\Delta_p <_\sigma \Delta_q$ and we can take $U = \Delta_p$ and $V = \Delta_q$ because $\mathscr{D}$ consists of open sets. If $\Delta_p = \Delta_q = \Delta$, we can use that $\sigma \restriction \Delta = \eta_\Delta$ is continuous to take open sets $U, V \subset \Delta$ such that $p \in U$, $q \in V$ and $U <_{\eta_\Delta} V$. Evidently, $U <_\sigma V$. □

3. Quasi-king spaces

Let σ be a weak selection for X, and $\lll_\sigma, \ll_\sigma \subset X^2$ be the binary relations defined for $x, y \in X$ by

$$\begin{cases} x \ll_\sigma y & \text{if } x \le_\sigma y_1 \le_\sigma y, \quad \text{for some } y_1 \in X, \quad \text{and} \\ x \lll_\sigma y & \text{if } x \le_\sigma y_1 \le_\sigma \cdots \le_\sigma y_n \le_\sigma y, \quad \text{for some } y_1, \dots, y_n \in X. \end{cases} \tag{2}$$

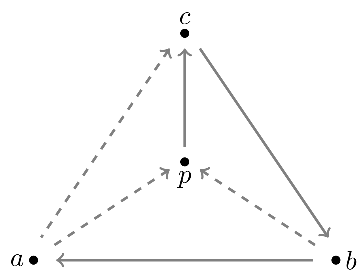
It is evident that $\le_\sigma \subset \ll_\sigma \subset \lll_\sigma$, and that $\ll_\sigma$ and $\lll_\sigma$ are total and reflexive because so is $\le_\sigma$. Furthermore, $\lll_\sigma$ is always transitive. However, in general, $\ll_\sigma$ and $\lll_\sigma$ are not antisymmetric, and may contain properly the relation $\le_\sigma$. In fact, $\le_\sigma$ is equal to one of these relations precisely when $\le_\sigma$ is transitive (i.e. a linear order), which is summarised in the proposition below.

Proposition 3.1. *Let σ be a weak selection for X. Then $\lll_\sigma = \le_\sigma$ if and only if $\ll_\sigma = \le_\sigma$, which is in turn equivalent to $\le_\sigma$ being transitive.*

Proof. Evidently, $\lll_\sigma = \le_\sigma$ implies that $\ll_\sigma = \le_\sigma$ because $\le_\sigma \subset \ll_\sigma \subset \lll_\sigma$. If $\le_\sigma$ is not transitive, then X contains points $x, y, z \in X$ with $x <_\sigma y <_\sigma z <_\sigma x$. In this case, $\ll_\sigma \ne \le_\sigma$ because $x <_\sigma y \ll_\sigma x$. □

Our interest in these binary relations is the interpretation that $p \in X$ is a *σ-king* if $x \ll_\sigma p$ for all $x \in X$; and p is a *quasi σ-king* if $x \lll_\sigma p$ for all $x \in X$, see the Introduction. In other words, the σ-kings of X are the $\ll_\sigma$-maximal elements of X, and the quasi σ-kings are the $\lll_\sigma$-maximal ones. We proceed with some examples about the difference between σ-kings and quasi σ-ones.

Example 3.2. Let $X = \{a, b, c, p\}$ consist of four points, and γ be the weak selection for X defined by $a <_\gamma b <_\gamma c <_\gamma a$ and $c <_\gamma p <_\gamma \{a, b\}$. Graphically, $\le_\gamma$ is represented by the diagram below, where "$<_\gamma$"="$\leftarrow$" and the shortest chain $a \leftarrow \cdots \leftarrow p$ of arrows illustrating the relation $a \lll_\gamma p$ is emphasised.



Then p is a quasi γ-king for X, but not a γ-king. On the other hand, a, b and c are γ-kings for X.

In case of infinite spaces, we have the following similar example where all points of X are quasi σ-kings for some continuous weak selection σ, but X has no σ-king.

Example 3.3. Following Example 3.2, let $X = \Delta_a \uplus \Delta_b \uplus \Delta_c$ be the topological sum of three copies Δ_a, Δ_b and Δ_c of the interval $(0, 1)$, and let γ be the weak selection on the open partition $\mathscr{D} = \{\Delta_a, \Delta_b, \Delta_c\}$ defined by $\Delta_a <_\gamma \Delta_b <_\gamma \Delta_c <_\gamma \Delta_a$. Take the standard selection $\eta(\{x, y\}) = \min\{x, y\}$, $x, y \in (0, 1)$, on each one of the open segments Δ_a, Δ_b and Δ_c. Finally, let σ be the lexicographical γ-sum of these selections. In other words, σ is the weak selection for X which is continuous on each of these open segments, and $\Delta_a <_\sigma \Delta_b <_\sigma \Delta_c <_\sigma \Delta_a$. According to Proposition 2.3, σ is continuous. Moreover, each element of X is a quasi σ-king, but X has no σ-king because none of the open segments contains a last element with respect to $\le_\sigma$.

Regarding the existence of quasi σ-kings, we have the following natural result which is complementary to [22, Theorem 2.3].

Theorem 3.4. *Let σ be a weak selection for X such that $\mathscr{T}_\sigma$ is a compact topology on X. Then X has a quasi σ-king.*

Proof. For every $x \in X$, let

$$K_x = \{p \in X : x \lll_\sigma p\}. \tag{3}$$

Evidently, each K_x is nonempty because $x \in K_x$. Take $x, y \in X$ with $x \leq_\sigma y$, and $p \in K_y$. Then $p \in K_x$ because $x \leq_\sigma y \lll_\sigma p$ implies $x \lll_\sigma p$, see (2). Thus, every two elements of the collection $\{K_x : x \in X\}$ are comparable by inclusion. Hence, it has the finite intersection property. Let $\mathrm{cl}_{\mathscr{T}_\sigma}(A) = \overline{A}^{\mathscr{T}_\sigma}$ be the closure of a subset $A \subset X$ in the topology $\mathscr{T}_\sigma$. Since $\mathscr{T}_\sigma$ is a compact topology, we get that $\bigcap_{x\in X} \mathrm{cl}_{\mathscr{T}_\sigma}(K_x) \neq \emptyset$. Let $p \in \bigcap_{x\in X} \mathrm{cl}_{\mathscr{T}_\sigma}(K_x)$. If $x \leq_\sigma p$ for every $x \in X$, then clearly p is a σ-king for X. If $p <_\sigma q$ for some $q \in X$, then q is a quasi σ-king for X. Indeed, for every $x \in X$ there exists $p_x \in K_x$ with $p_x <_\sigma q$, because $p \in (\leftarrow, q)_{\leq_\sigma} \cap \mathrm{cl}_{\mathscr{T}_\sigma}(K_x)$. According to (2) and (3), $q \in K_x$ for every $x \in X$. □

Recall that a space X is *quasi-king* if it has a separately continuous weak selection, and each separately continuous weak selection σ for X has a quasi σ-king. We now have the following consequence, compare with [22, Theorem 2.3].

Corollary 3.5. *Let X be a space with a separately continuous weak selection. If each coarser selection topology on X is compact, then X is a quasi-king space.*

We conclude with some remarks.

Remark 3.6. The proof of Theorem 3.4 does not follow from that of [22, Theorem 2.3]. In fact, the author is unaware if, in the setting of Theorem 3.4, X has a σ-king.

Remark 3.7. Following the idea of Example 3.3, one can easily characterise the spaces in which each quasi σ-king is a σ-king. Namely, for a quasi-king space X, the following are equivalent:

(a) $\lll_\sigma = \ll_\sigma$, for each separately continuous weak selection σ for X.
(b) X is the topological sum of at most three connected subsets.

Here, the requirement that X is a quasi-king space is important. Indeed, the space in Example 3.3 satisfies (b), but is not quasi-king. So, implicitly, such a partition of a quasi-king space X must be of $\mathscr{T}_\sigma$-compact sets, for each (some) separately continuous weak selection σ for X, see Propositions 2.2 and 4.3. Moreover, (b) implies that each separately continuous weak selection for X is continuous (by Propositions 2.2 and 2.3), therefore such quasi-king spaces are completely identical to king spaces.

Remark 3.8. Let σ be a weak selection for X. Following [18], a point $p \in X$ will be called a *σ-emperor* if it is the $\leq_\sigma$-maximal element of X, namely if $x \leq_\sigma p$ for all $x \in X$. Thus, X may have at most one σ-emperor, and each σ-emperor is a (quasi) σ-king. If X is a finite set, then X has exactly one σ-king if and only if that king is a σ-emperor [18, Theorem 4]. In the setting of infinite sets, this is not necessarily true, and the property defines a special class of topological spaces. To this end, for convenience, let $\mathscr{sel}_2(X)$ be the collection of all weak selections for a set X. Then for a space X with a separately continuous weak selection, the following are equivalent:

(a) X is $\mathscr{T}_\sigma$-compact and $\leq_\sigma$ is a linear order, for each separately continuous $\sigma \in \mathscr{sel}_2(X)$.

(b) Each separately continuous $\sigma \in \mathscr{sel}_2(X)$ has a σ-emperor.
(c) Each separately continuous $\sigma \in \mathscr{sel}_2(X)$ has exactly one quasi σ-king.
(d) Each separately continuous $\sigma \in \mathscr{sel}_2(X)$ has exactly one σ-king.
(e) X is the topological sum of at most two $\mathscr{T}_\sigma$-compact sets, for each separately continuous $\sigma \in \mathscr{sel}_2(X)$.

By Proposition 2.1, the first condition implies that each separately continuous weak selection for X is properly continuous.

4. Clopen compactness

Here, we show that every weakly orderable quasi-king space is compact in the topology generated by its clopen subsets, which furnishes an essential part in the proof of Theorem 1.1.

THEOREM 4.1. *Let X be a weakly orderable quasi-king space. Then each clopen cover of X has a finite subcover.*

The proof of Theorem 4.1 is based on several observations about quasi-king spaces. The next proposition shows that the following property of king spaces is also valid for quasi-king spaces, see [22, Lemma 3.1].

PROPOSITION 4.2. *If X is a quasi-king space, then each clopen subset of X is also a quasi-king space.*

Proof. Let $A \subset X$ be a clopen set, and η be a separately continuous weak selection for A. Since $X \setminus A$ is also clopen and has a separately continuous weak selection, it follows from Proposition 2.3 that X has a separately continuous weak selection σ with $\sigma \restriction A = \eta$ and $X \setminus A <_\sigma A$. Then by hypothesis, X has a quasi σ-king $p \in X$. For a point $x \in A$, this means that $x \leq_\sigma y_1 \leq_\sigma \cdots \leq_\sigma y_n \leq_\sigma p$, for some $y_1, \ldots, y_n \in X$. However, $x \in A$ and $X \setminus A <_\sigma A$, which implies that $y_1, \ldots, y_n, p \in A$. Accordingly, p is a quasi η-king of A because $\sigma \restriction A = \eta$. □

Subspaces of orderable spaces are not necessarily orderable, they are called *suborderable*. Their topology can be also described in terms of "order"-intervals. Briefly, a subset $\Delta \subset X$ of an ordered set $(X, \leq)$ is called a *$\leq$-interval*, or a *$\leq$-convex set*, if $(a,b)_\leq = (a,\rightarrow)_\leq \cap (\leftarrow, b)_\leq \subset \Delta$, for every $a, b \in \Delta$ with $a \leq b$. A topological space $(X, \mathscr{T})$ is called *generalised ordered* if it admits a linear order $\leq$, called *compatible*, such that the corresponding open interval topology $\mathscr{T}_\leq$ is coarser than the topology $\mathscr{T}$, and $\mathscr{T}$ has a base of $\leq$-intervals. According to a result of E. Čech, generalised ordered spaces are precisely the suborderable spaces, see e.g. [2, 23]. We now get with ease that each suborderable quasi-king space is compact, see [22, Lemma 3.2 and Corollary 3.3].

PROPOSITION 4.3. *Each suborderable quasi-king space is compact.*

Proof. Let X be a quasi-king space which is suborderable with respect to a linear order $\leq$. Then $\eta(\{x, y\}) = \min_{\leq}\{x, y\}$, $x, y \in X$, is a continuous weak selection for X with $\leq_\eta = \leq$. Hence, X has a unique quasi η-king, which is the $\leq$-maximal element of X, see Proposition 3.1. Since X is also suborderable with respect to the reverse linear order, it has a $\leq$-minimal element as well. This implies that X is actually orderable with respect to $\leq$. Indeed, let E and D be nonempty clopen subsets of X such that $E < D$ and $X = E \cup D$. By Proposition 4.2, both E and D are quasi-king spaces. Hence, by what has been shown above, E has a maximal element and D has a minimal one. Thus, the pair (E, D) is a clopen jump and, consequently, X is orderable with respect to $\leq$, see e.g. [5, Lemma 6.4]. This also implies that X must be compact. Namely, each nonempty clopen set $A \subset X$ is both a quasi-king space (by Proposition 4.2) and suborderable with respect to $\leq$. So, by the same token, it has maximal and minimal elements. Therefore, X is compact [14], see also [5, Proposition 6.1]. □

Corollary 4.4. *Let X be a quasi-king space which is weakly orderable with respect to a linear order $\leq$. Then the open interval topology $\mathscr{T}_{\leq}$ is a coarser compact topology on X.*

Proof. The topology $\mathscr{T}_{\leq}$ is a coarser topology on X, and, in particular, each separately continuous weak selection for $(X, \mathscr{T}_{\leq})$ is a separately continuous weak selection for X. Therefore, the orderable space $(X, \mathscr{T}_{\leq})$ is also quasi-king. Hence, by Proposition 4.3, $(X, \mathscr{T}_{\leq})$ is compact. □

We are now ready for the proof of Theorem 4.1.

Proof (Proof of Theorem 4.1). Let X be a weakly orderable space with respect to a linear order $\leq$. According to Corollary 4.4, it suffices to show that each clopen subset of X is open in $(X, \mathscr{T}_{\leq})$. So, let $A \subset X$ be clopen in X. Then A is quasi-king (by Proposition 4.2) and suborderable in the subspace topology $\mathscr{T}_{\leq} \restriction A$. In fact, A is a quasi-king space with respect $\mathscr{T}_{\leq} \restriction A$ because $\mathscr{T}_{\leq} \restriction A$ is a coarser topology on A and the weak selection $\min_{\leq}\{x, y\}$, $x, y \in A$, is continuous with respect to this topology (by Proposition 2.1). Thus, by Proposition 4.3, A is a compact subset of $(X, \mathscr{T}_{\leq})$. For the same reason, so is $X \setminus A$. Therefore, $A = X \setminus (X \setminus A)$ is open in $(X, \mathscr{T}_{\leq})$. □

5. Coarser compact selection topologies

Here, we prove the following special case of Theorem 1.1.

Theorem 5.1. *Let X be a weakly orderable quasi-king space, and σ be a separately continuous weak selection for X. Then $\mathscr{T}_\sigma$ is a compact coarser topology on X.*

The proof of Theorem 5.1 is based on properties of components relative to selection topologies. The *components* (called also *connected components*) are the maximal connected subsets of a space X. They form a closed partition $\mathscr{C}[X]$ of X, and each point $x \in X$ is contained in a unique component $\mathscr{C}[x]$ called the *component* of x in X. The *quasi-component* $\mathscr{Q}[x]$ of a point $x \in X$ is the intersection of all clopen subsets of X containing x. The quasi-components also form a partition $\mathscr{Q}[X]$ of X, thus they are simply called *quasi-components* of X. Each component of a point is contained in the quasi-component of that point, but the converse is not necessarily true. However, if X has a continuous weak selection, then $\mathscr{C}[x] = \mathscr{Q}[x]$ for every $x \in X$ [10, Theorem 4.1]. The property remains valid for the components of selection topologies.

Proposition 5.2. *Let σ be a weak selection for X. Then each quasi-component of $(X, \mathscr{T}_\sigma)$ is connected.*

Proof. By Proposition 2.1, σ is a separately continuous weak selection for $(X, \mathscr{T}_\sigma)$. Then the property follows from [7, Corollary 2.3]. □

Regarding Proposition 5.2, let us explicitly remark that if $C \subset X$ is a component of a space X and σ is a separately continuous weak selection for X, then C is also a connected subset of $(X, \mathscr{T}_\sigma)$. However, C is not necessarily a $\mathscr{T}_\sigma$-component, namely a component of the space $(X, \mathscr{T}_\sigma)$. Keeping this in mind, we have the following construction of clopen sets associated to $\mathscr{T}_\sigma$-components.

Proposition 5.3. *Let η be a weak selection for X, and $Z \subset X$ be a $\mathscr{T}_\eta$-component of X which has no $\leq_\eta$-maximal element. Then Z is contained in a $\mathscr{T}_\eta$-clopen set $Y \subset X$ with $Y \setminus Z <_\eta Z$.*

Proof. The set $Y = \bigcup_{z\in Z}(\leftarrow, z)_{\leq_\eta}$ is $\mathscr{T}_\eta$-open. Moreover, $Z \subset Y$ because Z has no last element with respect to $\leq_\eta$. If $y \in X \setminus Z$ and $y \leq_\eta z$ for some $z \in Z$, then $y <_\eta Z$ because Z is $\mathscr{T}_\eta$-connected, see Proposition 2.2. This implies that $Y \setminus Z <_\eta Z$. It also implies that $Y = (\leftarrow, x]_{\leq_\eta} \cup Z$ for some (any) point $x \in Z$. Since both $(\leftarrow, x]_{\leq_\eta}$ and Z are $\mathscr{T}_\eta$-closed, so is Y. □

We now have the following crucial property of selection topologies.

Lemma 5.4. *Let σ be a weak selection for X such that $(X, \mathscr{T}_\sigma)$ is a quasi-king space. Then each $\mathscr{T}_\sigma$-component is $\mathscr{T}_\sigma$-compact.*

Proof. Take a non-degenerate $\mathscr{T}_\sigma$-component $Z \subset X$. Then by Proposition 2.2, $(Z, \mathscr{T}_\sigma \restriction Z)$ is orderable with respect to $\leq_\sigma$ being a connected space. Hence, to show that Z is $\mathscr{T}_\sigma$-compact, it now suffices to show that it has both $\leq_\sigma$-minimal and $\leq_\sigma$-maximal elements. To this end, we will use that σ determines a unique 'complementary' selection $\sigma^* : \mathscr{F}_2(X) \to X$, defined by $S = \{\sigma(S), \sigma^*(S)\}$, $S \in \mathscr{F}_2(X)$. The important property of σ^* is that $\mathscr{T}_{\sigma^*} = \mathscr{T}_\sigma$ because $\leq_{\sigma^*}$ is reverse to $\leq_\sigma$. Thus, given a weak selection η for X with $\mathscr{T}_\eta = \mathscr{T}_\sigma$, it suffices to show that Z has a $\leq_\eta$-maximal element. To see this, assume the contrary that X has a weak selection η with $\mathscr{T}_\eta = \mathscr{T}_\sigma$, but Z has no $\leq_\eta$-maximal element. Then by Proposition 5.3, Z is contained in a $\mathscr{T}_\eta$-clopen set Y with $Y \setminus Z <_\eta Z$. Using that $\mathscr{T}_\eta = \mathscr{T}_\sigma$, it follows from Proposition 4.2 that $(Y, \mathscr{T}_\eta \restriction Y)$ is also a quasi-king space. Moreover, $\gamma = \eta \restriction Y$ is a separately continuous weak selection for $(Y, \mathscr{T}_\eta \restriction Y)$, hence Y has a quasi γ-king $q \in Y$. Since $Y \setminus Z <_\gamma Z$ and Z has no $\leq_\gamma$-maximal element, $q <_\gamma x$ for some $x \in Z$. For the same reason, $q <_\gamma y$, for every $y \in Y$ with $x \leq_\gamma y$, because $\leq_\gamma$ is a linear order on Z. Accordingly, q cannot be a quasi γ-king for Y. A contradiction. □

Finally, we also have that each $\mathscr{T}_\sigma$-component has a base of $\mathscr{T}_\sigma$-clopen sets.

PROPOSITION 5.5. *Let σ be a weak selection for X such that $(X, \mathscr{T}_\sigma)$ is a quasi-king space. Then each $\mathscr{T}_\sigma$-component has a base of clopen sets in $(X, \mathscr{T}_\sigma)$.*

Proof. A space is *rim-finite* if it has a base of open sets whose boundaries are finite. Evidently, $(X, \mathscr{T}_\sigma)$ is rim-finite. Take a $\mathscr{T}_\sigma$-component Z of X, and a $\mathscr{T}_\sigma$-open set $V \subset X$ with $Z \subset V$. Since Z is $\mathscr{T}_\sigma$-compact (by Lemma 5.4) and $(X, \mathscr{T}_\sigma)$ is rim-finite, there exists $W \in \mathscr{T}_\sigma$ such that $Z \subset W \subset V$ and the boundary of W is finite. However, by Proposition 5.2, Z is also a quasi-component of $(X, \mathscr{T}_\sigma)$. Hence, there exists a $\mathscr{T}_\sigma$-clopen set $U \subset X$ with $Z \subset U \subset W \subset V$. □

Proof (Proof of Theorem 5.1). Let σ be a separately continuous weak selection for X. Take an open cover $\mathscr{U} \subset \mathscr{T}_\sigma$ of X, and let $\mathscr{U}^F$ be the cover of X consisting of all finite unions of elements of $\mathscr{U}$. According to Lemma 5.4 and Proposition 5.5, $\mathscr{U}^F$ has a clopen refinement $\mathscr{V}$. Then by Theorem 4.1, $\mathscr{V}$ has a finite subcover. This implies that $\mathscr{U}$ has a finite subcover as well. □

6. Coarser pseudocompact topologies

For simplicity, we shall say that a space $(X, \mathscr{T})$ is *selection-orderable* if it has a continuous weak selection φ with $\mathscr{T} = \mathscr{T}_\varphi$. The main idea behind this convention is that for a space X with a properly continuous weak selection φ, the space $(X, \mathscr{T}_\varphi)$ is selection-orderable.

We now finalise the proof of Theorem 1.1 by showing the following general result involving implicitly pseudocompactness.

THEOREM 6.1. *Each selection-orderable quasi-king space is compact.*

To prepare for the proof of Theorem 6.1, we first extend the following property of king spaces to the case of quasi-king spaces, see [22, Lemma 3.4].

PROPOSITION 6.2. *Let X be a space with a continuous weak selection. If X admits an infinite open partition, then X is not a quasi-king space.*

Proof. Let $\mathscr{U}$ be an infinite open partition of X, and $\leq$ be a linear order on $\mathscr{U}$ such that $\mathscr{U}$ has no last $\leq$-element. Take a weak selection γ for $\mathscr{U}$ with $\leq_\gamma = \leq$. Also, for every $U \in \mathscr{U}$, take a continuous weak selection η_U for U. Finally, let σ be the lexicographical γ-sum of these selections. By Proposition 2.3, σ is continuous. Moreover, σ induces the same linear order on $\mathscr{U}$ as that of γ, see (1). This implies that σ has no quasi σ-king. Indeed, let $q \in V$ for some $V \in \mathscr{U}$. Next, using that $\mathscr{U}$ has no last $\leq_\sigma$-element, take any $U \in \mathscr{U}$ with $V <_\sigma U$. If $x \in U$ and $x \leq_\sigma y$, then y has the same property as x in the sense that $y \in W$ for some $W \in \mathscr{U}$ with $V <_\sigma W$. Hence, for any finite number of points $y_1, \ldots, y_n \in X$ with $x \leq_\sigma y_1 \leq_\sigma \cdots \leq_\sigma y_n$, we have that $q < y_k$ for all $k \leq n$. □

Let $\mathscr{C}[X] = \{\mathscr{C}[x] : x \in X\}$ be the decomposition space determined by the components of X. Recall that a subset $\mathscr{U} \subset \mathscr{C}[X]$ is open in $\mathscr{C}[X]$ if $\bigcup \mathscr{U}$ is open in X. Alternatively, $\mathscr{C}[X]$ is the quotient space obtained by the equivalence relation $x \sim y$ iff $\mathscr{C}[x] = \mathscr{C}[y]$. Since the elements of $\mathscr{C}[X]$ are closed sets, the decomposition space $\mathscr{C}[X]$ is a T_1-space. The following property of the decomposition space was essentially established in [8, Corollary 3.7].

PROPOSITION 6.3. *Let X be a quasi-king space, and φ be a continuous weak selection for X such that $\mathscr{T}_\varphi$ is the topology of X. Then for every $x, y \in X$ with $\mathscr{C}[x] \cap \mathscr{C}[y] = \emptyset$ and $x <_\varphi y$, there are clopen sets $U, V \subset X$ such that $\mathscr{C}[x] \subset U$, $\mathscr{C}[y] \subset V$ and $U <_\varphi V$.*

Proof. Since $x <_\varphi y$, by Proposition 2.2, we get that $\mathscr{C}[x] <_\varphi \mathscr{C}[y]$. Then the existence of such clopen sets $U, V \subset X$ follows by applying Lemma 5.4 and the condition that $\mathscr{T}_\varphi$ is the topology of X. Namely, by Proposition 5.5, it suffices to construct open sets $U, V \subset X$ with $\mathscr{C}[x] \subset U$, $\mathscr{C}[y] \subset V$ and $U <_\varphi V$. Since $\mathscr{C}[y]$ is compact (by Lemma 5.4) and φ is continuous, for each $z \in \mathscr{C}[x]$ there are open sets $U_z, V_z \subset X$ such that $z \in U_z$, $\mathscr{C}[y] \subset V_z$ and $U_z <_\varphi V_z$. Finally, since $\mathscr{C}[x]$ is also compact, there exists a finite set $S \subset \mathscr{C}[x]$ with $\mathscr{C}[x] \subset \bigcup_{z\in S} U_z$. Then $U = \bigcup_{z\in S} U_z$ and $V = \bigcap_{z\in S} V_z$ are as required. □

The crucial final step in the preparation for the proof of Theorem 6.1 is the following result.

LEMMA 6.4. *Let X be a selection-orderable quasi-king space. Then the decomposition space $\mathscr{C}[X]$ is a zero-dimensional sequentially compact space.*

Proof. In this proof, we first show that $\mathscr{C}[X]$ has a continuous weak selection (following [8, Theorem 3.1]), and next that it is pseudocompact (following [22, Theorem 3.5]). To this end, let φ be a continuous weak selection for X such that $\mathscr{T}_\varphi$ is the topology of X. By Proposition 5.5, each element of $\mathscr{C}[X]$ has a base of clopen sets. Hence, the decomposition space $\mathscr{C}[X]$ is zero-dimensional. Moreover, for every $x, y \in X$ with $\mathscr{C}[x] \cap \mathscr{C}[y] = \emptyset$ and $x <_\varphi y$, just as in the previous proof, we have that $\mathscr{C}[x] <_\varphi \mathscr{C}[y]$. Therefore, this defines a weak selection $\mathscr{C}[\varphi]$ for $\mathscr{C}[X]$ such that $\mathscr{C}[x] <_{\mathscr{C}[\varphi]} \mathscr{C}[y]$, whenever $x <_\varphi y$ with $\mathscr{C}[x] \cap \mathscr{C}[y] = \emptyset$. Finally, according to Proposition 6.3, the selection $\mathscr{C}[\varphi]$ is continuous.

To show that X is pseudocompact, take a discrete family $\{\mathscr{V}_n : n \in \mathbb{N}\}$ of nonempty open sets $\mathscr{V}_n \subset \mathscr{C}[X]$. Since $\mathscr{C}[X]$ is zero-dimensional, each $\mathscr{V}_n$, $n \in \mathbb{N}$, contains a nonempty clopen subset $\mathscr{U}_n \subset \mathscr{C}[X]$. Then each $U_n = \bigcup \mathscr{U}_n$, $n \in \mathbb{N}$, is a nonempty clopen subset of X, and the family $\{U_n : n \in \mathbb{N}\}$ is discrete in X. Hence, $U_0 = X \setminus \bigcup_{n\in\mathbb{N}} U_n$ is also a clopen subset of X, and $\{U_n : n < \omega\}$ is an infinite pairwise disjoint open cover of X. According to Proposition 6.2, this is impossible because X is a quasi-king space. Thus, every discrete family of open subsets of $\mathscr{C}[X]$ is finite. Since $\mathscr{C}[X]$ is a Tychonoff space (being zero-dimensional), this implies that it is pseudocompact.

Having already established this, we can use each pseudocompact space with a continuous weak selection is sequentially compact [1, 3, 21], see also [6, Corollary 3.9]. Accordingly, $\mathscr{C}[X]$ is sequentially compact. □

We now have also the proof of Theorem 6.1.

Proof (Proof of Theorem 6.1). Each pseudocompact space X with a continuous weak selection is suborderable, see [1, 3, 21]; also [6, Theorems 3.7 and 3.8]. Moreover, by Proposition 4.3, each suborderable quasi-king space is compact. Hence, it suffices to show that X is pseudocompact. In this, we follow the proof of [8, Theorem 4.1]. Namely, assume to the contrary that X is not pseudocompact. Then it has a continuous unbounded function $g : X \to [0, +\infty)$. Take a point $x_1 \in X$ with $g(x_1) \geq 1$, and let $K_1 = \mathscr{C}[x_1]$ be the component of x_1. Since X is a selection-orderable quasi-king space, by Lemma 5.4, K_1 is compact, and consequently $g \restriction K_1$ is bounded. Hence, there exists a point $x_2 \in X \setminus K_1$ with $g(x_2) \geq 2$. Set $K_2 = \mathscr{C}[x_2]$ and extend the arguments by induction. Thus, there exists a pairwise disjoint sequence $\{K_n : n \in \mathbb{N}\}$ of components of X and points $x_n \in K_n$ with $g(x_n) \geq n$, for every $n \in \mathbb{N}$. We claim that the sequence $\{K_n : n \in \mathbb{N}\}$ is discrete in X. Indeed, suppose that $y \in \overline{\bigcup_{n \geq k} K_n} \setminus \bigcup_{n \geq k} K_n$ for some $k \in \mathbb{N}$. Since $\mathscr{C}[y]$ is compact, $g \restriction \mathscr{C}[y]$ is bounded, and so is $g \restriction U$ for some neighbourhood U of $\mathscr{C}[y]$. By Proposition 5.5, this implies that $g \restriction H$ is bounded for some clopen subset $H \subset X$ with $\mathscr{C}[y] \subset H \subset U$. However, $y \in H \cap \overline{\bigcup_{n \geq k} K_n}$ and, therefore, H meets infinitely many terms of the sequence $\{K_n : n \geq k\}$. In fact, being a clopen set, H must contain infinitely many terms of this sequence because $K_n \subset H$, whenever $H \cap K_n \neq \emptyset$. Hence, $g \restriction H$ must be also unbounded because $g(x_n) \geq n$ for every $n \in \mathbb{N}$. A contradiction. Thus, $\{K_n : n \in \mathbb{N}\}$ is discrete.

We complete the proof as follows. Since $\{K_n : n \in \mathbb{N}\} \subset \mathscr{C}[X]$ is discrete in X, by Proposition 5.5, it defines a discrete sequence of elements in the decomposition space $\mathscr{C}[X]$. However, this is impossible because, by Lemma 6.4, the decomposition space $\mathscr{C}[X]$ is sequentially compact. We have duly arrived at a contradiction, showing that X must be pseudocompact. □

The proof of Theorem 1.1 now follows from Theorem 5.1 and the following consequence of Theorem 6.1.

Corollary 6.5. *Let X be a quasi-king space with a properly continuous weak selection. Then X is weakly orderable.*

Proof. Let φ be a properly continuous weak selection for X. Then φ is continuous with respect to its selection topology $\mathscr{T}_\varphi$. Moreover, $\mathscr{T}_\varphi$ is a coarser topology on X. Hence, $(X, \mathscr{T}_\varphi)$ remains a quasi-king space. Thus, $(X, \mathscr{T}_\varphi)$ is a selection-orderable quasi-king space and by Theorem 6.1, it is compact. Finally, by a result of van Mill and Wattel [20, Theorem 1.1], $(X, \mathscr{T}_\varphi)$ is an orderable space. Therefore, X is weakly orderable. □

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