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ON FIXED POINTS IN THE CONTEXT OF b-METRIC SPACES

Sumit Chandok, Vildan Ozturk and Stojan Radenović

Abstract. In this paper, we obtain sufficient conditions for the existence of common fixed points in the framework of ordered b-metric spaces. Our results generalize some recent results in the literature. Also, to illustrate the usability of the results we give an adequate example in which b-metric is not continuous.

1. Introduction and preliminaries

One of the important generalization of metric spaces are so-called *b-metric spaces* (or *metric type spaces* as called by some authors). This concept was introduced by Bakhtin 1989 [3] and Czerwik 1993 [5].

Consistent with the concepts of [3, 5], the following definitions and results will be needed in the sequel.

Definition 1.1 ([3,5]). Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, +\infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x, y) = 0$ if and only if $x = y$,

(b2) $d(x, y) = d(y, x)$,

(b3) $d(x, z) \leq s(d(x, y) + d(y, z))$.

The pair (X, d) is called a *b-metric space*.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric when $s = 1$ and the following example shows that, in general, a b-metric is not necessarily a metric.

Example 1.2. Let (X, d) be a metric space, and $\rho(x, y) = (d(x, y))^p$, $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$, but ρ is not a metric on X.

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For concepts such as b-convergence, b-completeness, b-Cauchy sequence and b-closed set in b-metric spaces, we refer the reader to [1, 2, 6–14] and the references therein. Also, for the concepts such as partial order, comparability, well ordered, non-decreasing, increasing, dominated, dominating and other, we refer the reader to [1,2].

Definition 1.3 ([4]). Let $\mathcal{F} : \mathbb{R}_+^2 \to \mathbb{R}$ be a continuous mapping; it is called a C-class function if it satisfies the following conditions:
(F1) $\mathcal{F}(s,t) \le s$, for all $(s,t) \in \mathbb{R}_+^2$;

(F2) $\mathcal{F}(s,t) = s$ implies that either $s = 0$ or $t = 0$, for all $(s,t) \in \mathbb{R}_+^2$.

Throughout the paper, we denote the set of C-class functions as $\mathcal{C}$.

Example 1.4 ([4]). The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$:
(1) $F(s,t) = s - t$; (2) $F(s,t) = ms$, $0 < m < 1$;

(3) $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$; (4) $F(s,t) = \log(t + a^s)/(1+t)$, $a > 1$;

(5) $F(s,t) = \frac{\ln(1+a^s)}{2}$, $a > e$; (6) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$;

(7) $F(s,t) = s \log_{t+a} a$, $a > 1$; (8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$;

(9) $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$, and it is continuous; (10) $F(s,t) = s - \frac{t}{k+t}$;

(11) $F(s,t) = s - \varphi(s)$; here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(12) $F(s,t) = sh(s,t)$; here $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$;

(13) $F(s,t) = s - (\frac{2+t}{1+t})t$; (14) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$;

(15) $F(s,t) = \phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$;

(16) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$.

The following results will be used in the proof of our main result.

Lemma 1.5 ([1]). *Let (X,d) be a b-metric space with $s \ge 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have*
$$\frac{1}{s^2} d(x,y) \le \liminf_{n\to\infty} d(x_n,y_n) \le \limsup_{n\to\infty} d(x_n,y_n) \le s^2 d(x,y).$$
In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover for each $z \in X$ we have
$$\frac{1}{s} d(x,z) \le \liminf_{n\to\infty} d(x_n,z) \le \limsup_{n\to\infty} d(x_n,z) \le s d(x,z),$$

Lemma 1.6 ([11, Lemma 3.1]). *Let $\{y_n\}$ be a sequence in a b-metric space (X,d) with $s \ge 1$, such that $d(y_n, y_{n+1}) \le \lambda d(y_{n-1}, y_n)$ for some $\lambda \in [0, \frac{1}{s})$, and each $n = 1, 2, \ldots$ Then $\{y_n\}$ is a b-Cauchy sequence in (X,d).*

In this paper, we obtain sufficient conditions for the existence of common fixed points in the framework of ordered b-metric spaces with $s \geq 1$. The methodology used to obtain the results is shorter than the corresponding results existing in the literature. Our results generalize some recent results in the literature. Also, to illustrate the usability of the results we give an adequate example in which b-metric is not continuous.

2. Main results

We begin this section with the following result generalizing the main results of [1].

THEOREM 2.1. *Let $(X, d, \preceq)$ be an ordered b-complete b-metric space. Let f, g, S and T be self-maps on X, f, g and S, T be dominated and dominating mappings, respectively, with $fX \subseteq TX$ and $gX \subseteq SX$. Suppose that there exist control functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ so that ψ is a continuous monotone non-decreasing function with $\psi(t) = 0$ iff $t = 0$, and φ is a lower semi-continuous function with $\phi(t) = 0$ iff $t = 0$, and for every two comparable elements $x, y \in X$,*

$$\psi(s^4 d(fx, gy)) \leq \mathcal{F}(\psi(M_s(x, y)), \varphi(M_s(x, y))), \tag{1}$$

is satisfied where $\mathcal{F} \in \mathcal{C}$ and

$$M_s(x, y) = \max\left\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2s}\right\}.$$

If for every non-increasing sequence $\{x_n\}$ and a sequence $\{y_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ we have $u \preceq x_n$ and
(a1) $\{f, S\}$ are compatible, f or S is continuous and $\{g, T\}$ is weakly compatible; or
(a2) $\{g, T\}$ are compatible, g or T is continuous and $\{f, S\}$ is weakly compatible,
then f, g, S and T have a common fixed point. Moreover, the set of common fixed points of f, g, S and T is well ordered if and only if f, g, S and T have one and only one common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $fX \subseteq TX$ and $gX \subseteq SX$, we can define inductively sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_{2n+1} = fx_{2n} = Tx_{2n+1}$, $y_{2n+2} = gx_{2n+1} = Sx_{2n+2}$, $n = 0, 1, 2, \ldots$ By the given assumptions, $x_{2n+1} \preceq Tx_{2n+1} = fx_{2n} \preceq x_{2n}$ and $x_{2n} \preceq Sx_{2n} = gx_{2n-1} \preceq x_{2n-1}$. Thus, we have $x_{n+1} \preceq x_n$ for all $n \geq 0$.

Now, we shall show that $d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1})$, where $\lambda \in [0, \frac{1}{s})$, for all $n \in \mathbb{N}$. From (1), we obtain

$$\begin{aligned}\psi(d(y_{2k+1}, y_{2k+2})) &\leq \psi(s^4 d(y_{2k+1}, y_{2k+2})) = \psi(s^4 d(fx_{2k}, gx_{2k+1})) \\ &\leq \mathcal{F}(\psi(M_s(x_{2k}, x_{2k+1})), \varphi(M_s(x_{2k}, x_{2k+1}))),\end{aligned} \tag{2}$$

where

$$M_s(x_{2k}, x_{2k+1}) = \max\Bigg\{d(Sx_{2k}, Tx_{2k+1}), d(fx_{2k}, Sx_{2k}), d(gx_{2k+1}, Tx_{2k+1}), \frac{d(Sx_{2k}, gx_{2k+1}) + d(fx_{2k}, Tx_{2k+1})}{2s}\Bigg\}$$

$$= \max\left\{d(y_{2k},y_{2k+1}),d(y_{2k+1},y_{2k}),d(y_{2k+2},y_{2k+1}),\frac{d(y_{2k},y_{2k+2})+d(y_{2k+1},y_{2k+1})}{2s}\right\}$$
$$= \max\left\{d\left(y_{2k},y_{2k+1}\right),d\left(y_{2k},y_{2k+1}\right),d\left(y_{2k+2},y_{2k+1}\right),\frac{d\left(y_{2k},y_{2k+2}\right)}{2s}\right\}$$
$$\leq \max\left\{d\left(y_{2k},y_{2k+1}\right),d\left(y_{2k+2},y_{2k+1}\right),\frac{d\left(y_{2k},y_{2k+1}\right)+d\left(y_{2k+1},y_{2k+2}\right)}{2}\right\}$$
$$\leq \max\left\{d\left(y_{2k},y_{2k+1}\right),d\left(y_{2k+2},y_{2k+1}\right)\right\} \leq M_s\left(x_{2k},x_{2k+1}\right).$$

Hence, $M_s\left(x_{2k},x_{2k+1}\right) = \max\left\{d\left(y_{2k},y_{2k+1}\right),d\left(y_{2k+2},y_{2k+1}\right)\right\}$.

Suppose that $M_s(x_{2k},x_{2k+1}) = d(y_{2k},y_{2k+1})$. Then, from (2) and using definition of C-class function, we obtain

$$\psi(s^4 d(y_{2k+1},y_{2k+2})) \leq \mathcal{F}(\psi(d(y_{2k},y_{2k+1})),\varphi(d(y_{2k},y_{2k+1}))) \leq \psi(d(y_{2k},y_{2k+1})).$$

This implies that $s^4 d(y_{2k+1},y_{2k+2}) \leq d(y_{2k},y_{2k+1})$.

Now, if $M_s(x_{2k},x_{2k+1}) = d(y_{2k+1},y_{2k+2})$, in the similar manner, we obtain that $s^4 d(y_{2k+1},y_{2k+2}) \leq d(y_{2k+1},y_{2k+2})$, implying that $s^4 \leq 1$, which is a contradiction.

Hence, $d(y_{2k+1},y_{2k+2}) \leq \frac{1}{s^4} d(y_{2k},y_{2k+1})$. Similarly, we have $d(y_{2k+2},y_{2k+3}) \leq \frac{1}{s^4} d(y_{2k+1},y_{2k+2})$. Continuing in this manner, we have $d(y_{n+1},y_{n+2}) \leq \lambda d(y_n,y_{n+1})$, where $\lambda \in [0,\frac{1}{s^4}) \subset [0,\frac{1}{s})$, for all $n \in \mathbb{N}$. Therefore, by using Lemma 1.6, it follows that $\{y_n\}$ is a b-Cauchy sequence. Since X is b-complete, there exists $y \in X$ so that $\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y$. Now, we can show that y is a common fixed point of f, g, S and T. Since S is continuous it follows that $\lim_{n\to\infty} S^2 x_{2n+2} = Sy$, $\lim_{n\to\infty} Sfx_{2n} = Sy$. Using the triangle inequality in the b-metric space, we have $d(fSx_{2n},Sy) \leq s(d(fSx_{2n},Sfx_{2n}) + d(Sfx_{2n},Sy))$. Since the pair $\{f,S\}$ is compatible, $\lim_{n\to\infty} d(fSx_{2n},Sfx_{2n}) = 0$. So taking the upper limit in the above inequality when $n \to \infty$, we have

$$\limsup_{n\to\infty} d(fSx_{2n},Sy) \leq s(\limsup_{n\to\infty} d(fSx_{2n},Sfx_{2n}) + \limsup_{n\to\infty} d(Sfx_{2n},Sy)) = 0.$$

Hence $\lim_{n\to\infty} fSx_{2n} = Sy$.

As $Sx_{2n+2} = gx_{2n+1} \preceq x_{2n+1}$, from (1) we obtain

$$\psi(s^4 d(fSx_{2n+2},gx_{2n+1})) \leq \mathcal{F}(\psi(M_s(Sx_{2n+2},x_{2n+1})),\varphi(M_s(Sx_{2n+2},x_{2n+1}))), \quad (3)$$

where

$$M_s(Sx_{2n+2},x_{2n+1}) = \max\Big\{d(S^2x_{2n+2},Tx_{2n+1}),d(fSx_{2n+2},S^2x_{2n+2}),$$
$$d(gx_{2n+1},Tx_{2n+1}),\frac{d(S^2x_{2n+2},gx_{2n+1})+d(fSx_{2n+2},Tx_{2n+1})}{2s}\Big\}.$$

Now, by using Lemma 1.5 we get

$$\limsup_{n\to\infty} M_s(Sx_{2n+2},x_{2n+1}) \leq \max\left\{s^2 d(Sy,y),0,0,\frac{s^2 d(Sy,y)+s^2 d(Sy,y)}{2s}\right\} = s^2 d(Sy,y).$$

Hence by taking the upper limit in (3) and using Lemma 1.5 and the definition of C-class, we obtain $\psi(s^2 d(Sy,y)) \leq \mathcal{F}(\psi(s^2 d(Sy,y)),\varphi(s^2 d(Sy,y))) \leq \psi(s^2 d(Sy,y))$, which implies that $\mathcal{F}(\psi(s^2 d(Sy,y)),\varphi(s^2 d(Sy,y))) = \psi(s^2 d(Sy,y))$. Hence either $\psi(s^2 d(Sy,y)) = 0$ or $\varphi(s^2 d(Sy,y)) = 0$, i.e., $Sy = y$. Now, since $gx_{2n+1} \preceq x_{2n+1}$ and

$gx_{2n+1} \to y$ as $n \to \infty$, then $y \preceq x_{2n+1}$ and from (1) we have

$$\psi(s^4 d(fy, gx_{2n+1})) \leq \mathcal{F}(\psi(M_s(y, x_{2n+1})), \varphi(M_s(y, x_{2n+1}))), \tag{4}$$

where,

$$M_s(y, x_{2n+1}) = \max\left\{d(Sy, Tx_{2n+1}), d(fy, Sy), d(gx_{2n+1}, Tx_{2n+1}), \frac{d(Sy, gx_{2n+1}) + d(fy, Tx_{2n+1})}{2s}\right\}.$$

Consider

$$\frac{1}{s} d(y, fy) \leq d(y, gx_{2n+1}) + d(fy, gx_{2n+1}). \tag{5}$$

Since, (4) implies $d(fy, gx_{2n+1}) \leq \frac{1}{s^4} M_s(y, x_{2n+1})$, taking the limit $n \to \infty$, in (5), we obtain $\frac{1}{s} d(y, fy) \leq 0 + \frac{1}{s^4} d(y, fy)$, that is, $fy = y$ (because $\dot{s} > 1$). Since $f(X) \subseteq T(X)$, there exists a point $v \in X$ so that $fy = Tv$. Suppose that $gv \neq Tv$. Since $v \preceq Tv = fy \preceq y$, from (1), we have

$$\psi(d(Tv, gv)) = \psi(d(fy, gv)) \leq \mathcal{F}(\psi(M_s(y, v)), \varphi(M_s(y, v))) \tag{6}$$

where

$$M_s(y, v) = \max\left\{d(Sy, Tv), d(fy, Sy), d(gv, Tv), \frac{d(Sy, gv) + d(fy, Tv)}{2s}\right\} = d(gv, Tv).$$

So from (6) we have $\psi(d(Tv, gv)) \leq \mathcal{F}(\psi(d(gv, Tv)), \varphi(d(gv, Tv))) \leq \psi(d(gv, Tv))$. This implies that $gv = Tv$, which is a contradiction. Therefore $gv = Tv$. Since the pair $\{g, T\}$ is weakly compatible, $gy = gfy = gTv = Tgv = Tfy = Ty$ and y is the coincidence point of g and T.

Since $Sx_{2n} \preceq x_{2n}$ and $Sx_{2n} \to y$ as $n \to \infty$, it implies that $y \preceq x_{2n}$ and from (1), we obtain

$$\psi(s^4 d(fx_{2n}, gy)) \leq \mathcal{F}(\psi(M_s(x_{2n}, y)), \varphi(M_s(x_{2n}, y))), \tag{7}$$

where,

$$M_s(x_{2n}, y) = \max\left\{d(Sx_{2n}, Ty), d(fx_{2n}, Sx_{2n}), d(gy, Ty), \frac{d(Sx_{2n}, gy) + d(fx_{2n}, Ty)}{2s}\right\}.$$

From (7), we have $d(fx_{2n}, gy) \leq \frac{1}{s^4} M_s(x_{2n}, y)$. Consider

$$\frac{1}{s} d(y, gy) \leq d(y, fx_{2n}) + d(fx_{2n}, gy) \leq d(y, fx_{2n}) + \frac{1}{s^4} M_s(x_{2n}, y).$$

Taking the limit as $n \to \infty$ in the above inequality, and since $\dot{s} > 1$, we have $gy = y$.

Therefore, $fy = gy = Sy = Ty = y$. The proof is similar when f is continuous.

Similarly, if (a2) holds then the result follows.

Now suppose that the set of common fixed points of f, g, S and T is well ordered. We show that they have a unique common fixed point. Assume on the contrary that, $fu = gu = Su = Tu = u$ and $fv = gv = Sv = Tv = v$ but $u \neq v$. By assumption, we can apply (1) to obtain

$$\psi(d(u, v)) = \psi(d(fu, gv)) \leq \psi(s^4 d(fu, gv)) \leq \mathcal{F}(\psi(M_s(u, v)), \varphi(M_s(u, v))),$$

where

$$M_s(u,v) = \max\left\{d(Su,Tv), d(fu,Su), d(gv,Tv), \frac{d(Su,gv)+d(fu,Tv)}{2s}\right\}$$
$$= \max\left\{d(u,v), 0, 0, \frac{d(u,v)+d(u,v)}{2s}\right\} = d(u,v).$$

Hence $\psi(d(u,v)) \leq \mathcal{F}(\psi(d(u,v)), \varphi(d(u,v))) \leq \psi(d(u,v))$, which is a contradiction. Therefore $u = v$. The converse is obvious. □

Corollary 2.2. *Let $(X, d, \preceq)$ be an ordered b-complete b-metric space. Let f and g be dominated self-maps on X. Suppose that there exist control functions ψ and φ as in Theorem 2.1 so that for every two comparable elements $x, y \in X$, the inequality $\psi(s^4 d(fx, gy)) \leq \mathcal{F}(\psi(M_s(x,y)), \varphi(M_s(x,y)))$, is satisfied for $\mathcal{F} \in \mathcal{C}$ and*

$$M_s(x,y) = \max\left\{d(x,y), d(fx,x), d(gy,y), \frac{d(x,gy)+d(fx,y)}{2s}\right\}.$$

If for every non-increasing sequence $\{x_n\}$ and a sequence $\{y_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ we have $u \preceq x_n$, then f and g have a common fixed point. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Corollary 2.3. *Let $(X, d, \preceq)$ be an ordered b-complete b-metric space. Let f and g be dominated self-maps on X, and suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ iff $t = 0$. Also for every two comparable elements $x, y \in X$, $s^4 d(fx, gy) \leq \mathcal{F}(M_s(x,y), \varphi(M_s(x,y)))$ is satisfied for $\mathcal{F} \in \mathcal{C}$ and*

$$M_s(x,y) = \max\left\{d(x,y), d(fx,x), d(gy,y), \frac{d(x,gy)+d(fx,y)}{2s}\right\}.$$

If for every non-increasing sequence $\{x_n\}$ and a sequence $\{y_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$, it implies that $u \preceq x_n$, then f and g have a common fixed point. Moreover, the set of common fixed points of f and g is well ordered if and only if f and g have one and only one common fixed point.

Putting $\psi(t) = \phi(t) = t$, $\mathcal{F}(s,t) = \frac{s^2 \cdot t}{1+s \cdot t}$, in Theorem 2.1, we get the following result.

Theorem 2.4. *Let $(X, d, \preceq)$ be an ordered b-complete b-metric space. Let f, g, S and T be self-maps on X, $\{f, g\}$ and $\{S, T\}$ be dominated and dominating maps, respectively with $fX \subseteq TX$ and $gX \subseteq SX$. Also, suppose that*

$$s^4 d(fx, gy) \leq \frac{M_s^3(x,y)}{1+M_s^2(x,y)},$$

for every two comparable elements $x, y \in X$, where

$$M_s(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2s}\right\}.$$

If for every non-increasing sequence $\{x_n\}$ and a sequence $\{y_n\}$ with $y_n \preceq x_n$ for all n and $y_n \to u$ we have $u \preceq x_n$ and either

(a1) $\{f, S\}$ are compatible, f or S is continuous and $\{g, T\}$ is weakly compatible; or

(a2) $\{g, T\}$ *are compatible,* g *or* T *is continuous and* $\{f, S\}$ *is weakly compatible, then* f*,* g*,* S *and* T *have a common fixed point. Moreover, the set of common fixed points of* f*,* g*,* S *and* T *is well ordered if and only if* f*,* g*,* S *and* T *have one and only one common fixed point.*

REMARK 2.5. By substituting different examples for $\mathcal{F}(s,t)$, as in Example 1.4, we have many other results. For example, in Theorem 2.1, if we take $\mathcal{F}(s,t) = s - t$, we have the corresponding results of [1] (see [1, Theorem 2.1, Corollary 2.1 and 2.2]).

In all results Theorem 2.1–Theorem 2.4, we have suppose that $s > 1$.

In the case when $s = 1$, we obtain the metric space (X, d) and for the defined sequence $\{y_n\}$ we have that $\{d(y_{n+1}, y_n)\}$ is a decreasing sequence and $d(y_{n+1}, y_n) \to r \geq 0$, as $n \to \infty$. It is easy to prove that $r = 0$. Further, according to [11, Lemma 1.6] it is not hard to see that $\{y_n\}$ is a Cauchy sequence. The rest of the proof is the same as for b-metric spaces.

Therefore, the presented results are the generalization and extension of several other comparable results in the literature.

Since in all assumptions in the results of [1] it is not assumed that b-metric d is continuous, then examples given in [1] are not adequate. In all examples in [1] we see that b-metric d is continuous. Therefore, we give an adequate example in which b-metric d is not continuous.

EXAMPLE 2.6. Let $X = \mathbb{N} \cup \{\infty\}$, and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ |\frac{1}{x} - \frac{1}{y}|, & \text{if one of } x \text{ and } y \text{ is odd and the other is odd or } \infty, \\ 5, & \text{if one of } x \text{ and } y \text{ is even and the other is even or } \infty, \\ 4, & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric with $s = \frac{5}{4}$. Let $x_n = 2n + 1$, for each $n \in \mathbb{N}$. Then $d(2n+1, \infty) = \left|\frac{1}{2n+1}\right| \to \infty$, that is, $x_n \to \infty$, but $d(x_n, 2) = 4 \nrightarrow 5 = d(\infty, 2)$, as $n \to \infty$. Hence b-metric d is not continuous (this is a modification of [9, Example 2]).

Define self mappings f, g, S and T on X by

$$f(x) = \begin{cases} 1, & x \in \mathbb{N} \\ \infty, & x = \infty \end{cases} \qquad T(x) = \begin{cases} 1, & x = 1 \\ \infty, & \text{otherwise} \end{cases}$$

$$g(x) = 1 \qquad S(x) = \begin{cases} 1, & x = 1 \\ 2, & x = 2 \\ 2n+2, & 2 < x < \infty \\ \infty, & x = \infty. \end{cases}$$

Define $\phi, \psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \sqrt{t}$ and $\phi(t) = \frac{t}{100}$. Then the maps f and g are dominated and maps T and S are dominating. Also, the contractive condition (1) is satisfied with $\mathcal{F}(s,t) = s - t$. Therefore, by Theorem 2.1, $1 \in X$ is a unique fixed point of f, g, S and T.

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Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam

Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

E-mail: sumit.chandok@tdtu.edu.vn

Department of Elementary Education, Faculty of Education, Artvin Coruh University, 08000, Artvin, Turkey

E-mail: vildan.ozturk@hotmail.com

Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam

Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

E-mail: stojan.radenovic@tdtu.edu.vn