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LM-FUZZY METRIC SPACES AND CONVERGENCE

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Abstract. We study a category of lattice-valued metric spaces that contains many categories of lattice-valued metric spaces studied before. Furthermore, introducing suitable convergence structures, we are able to show that the category of these lattice-valued metric spaces is a coreflective subcategory of the category of lattice-valued convergence spaces and we even characterize lattice-valued metric spaces by convergence.

1. Introduction

A generalization of metric spaces was given by probabilistic metric spaces [23], first introduced by Menger [16], where the distances between points were not numbers but distance distribution functions. In this sense, the range of the metric was generalized from non-negative real numbers to distance distribution functions. Such spaces can also be interpreted as fuzzy metric spaces, by suitable adaptation of the axioms [7,11]. Going one step further, the range of a metric can be chosen to be a quantale [13]. In this way, one arrives at quantale-valued metric spaces and for different choices of the quantale one obtains e.g. preordered sets, metric spaces or probabilistic metric spaces [6]. Another direction of generalization lies in not quantifying distances between points of a space but distances between fuzzy points or even between fuzzy sets, see e.g. [4, 5, 17, 24, 26]. All these approaches have been developed to a certain extent over the years.

Our paper has two purposes. First, we give a general framework, that encompasses many of the definitions of generalized metric spaces. Our framework has two variables, a complete lattice L and a quantale M with underlying completely distributive lattice. Depending on the choice of these variables, we recover many of the previous definitions of "generalized metric spaces". The second purpose of the paper is to show that such "generalized metric spaces" can be characterized by convergence. To this end, we define suitable convergence tower spaces, the category of which contains the category

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of our generalized metric spaces as a coreflective subcategory and – demanding a further axiom – a subcategory which is even isomorphic to our category of LM-fuzzy metric spaces.

2. Preliminaries

Let L be a complete lattice. We assume that L is non-trivial in the sense that $\top \neq \bot$ for the top element $\top$ and the bottom element $\bot$. In any complete lattice L we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j\in J} \beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is *completely distributive* (sometimes called *constructively completely distributive*) if and only if we have $\alpha = \bigvee\{\beta \ : \ \beta \lhd \alpha\}$ for any $\alpha \in L$, [20]. In a completely distributive lattice, the well-below relation satisfies the interpolation property, i.e. $\alpha \lhd \beta$ implies $\alpha \lhd \gamma \lhd \beta$ for some $\gamma \in L$. An element $a \in L$ is called *prime* if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. In a completely distributive lattice, every element is a meet of primes. More precisely, we have $b = \bigwedge\{a \in L \ : \ a \text{ prime}, b \leq a\}$. An element $a \in L$ is called *coprime* if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of all non-zero coprime elements in L is denoted by $J(L)$. For more results on lattices we refer to [8].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *quantale* [22] if $(L, *)$ is a semigroup, and $*$ is distributive over arbitrary joins, i.e.

$$\left(\bigvee_{i\in J} \alpha_i\right) * \beta = \bigvee_{i\in J} (\alpha_i * \beta) \quad \text{and} \quad \beta * \left(\bigvee_{i\in J} \alpha_i\right) = \bigvee_{i\in J} (\beta * \alpha_i).$$

A quantale $\mathsf{L} = (L, \leq, *)$ is called *commutative* if $(L, *)$ is a commutative semigroup and it is called *integral* if the top element of L acts as the unit, i.e. if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$.

We consider in this paper only commutative and integral quantales $\mathsf{L} = (L, \leq, *)$ with completely distributive lattices L. The following are two important examples for such quantales. A further example is given in the next section.

EXAMPLE 2.1. A *triangular norm* or *t-norm* [9,23] is a binary operation $*$ on the unit interval $[0, 1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. The triple $\mathsf{L} = ([0, 1], \leq, *)$ can be considered as a quantale if the t-norm is left-continuous. The three most commonly used (left-continuous) t-norms are:

(i) the minimum t-norm: $\alpha * \beta = \alpha \wedge \beta$,
(ii) the product t-norm: $\alpha * \beta = \alpha \cdot \beta$,
(iii) the Lukasiewicz t-norm: $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

EXAMPLE 2.2 (*Lawvere's quantale*, [13]). The interval $[0, \infty]$ with the opposite order and addition as the quantale operation $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + \alpha = \infty$) for all $\alpha, \beta \in [0, \infty]$ is a quantale $\mathsf{L} = ([0, \infty], \geq, +)$.

For a set X, we denote its power set by $P(X)$ and the set of all filters $\mathbb{F}, \mathbb{G}, \dots$ on X by $\mathfrak{F}(X)$. The set $\mathfrak{F}(X)$ is ordered by set inclusion and maximal elements of $\mathfrak{F}(X)$ in this order are called *ultrafilters*. The set of all ultrafilters on X is denoted by $\mathfrak{U}(X)$. In particular, for each $x \in X$, the point filter $[x] = \{A \subseteq X \ : \ x \in A\} \in \mathfrak{F}(X)$ is an ultrafilter. If $\mathbb{F} \in \mathfrak{F}(X)$ and $f : X \longrightarrow Y$ is a mapping, we define $f(\mathbb{F}) \in \mathfrak{F}(Y)$ by $f(\mathbb{F}) = \{G \subseteq Y \ : \ f(F) \subseteq G \text{ for some } F \in \mathfrak{F}\}$.

We denote further the set of L-sets of X by $L^X = \{a : X \longrightarrow L\}$. Using the pointwise order inherited from L, the set L^X also becomes a lattice and we call $a \in J(L^X)$ an *L-point*. For $a \in L^X$, $b \in L^{X'}$ and $f : X \longrightarrow X'$ we define $f(a) \in L^{X'}$ by $f(a)(y) = \bigvee_{x: f(x)=y} a(x)$ for $y \in X'$ and $f^{\leftarrow}(b) = b \circ f$. It is not difficult to show that for $a \in J(L^X)$ we have $f(a) \in J(L^{X'})$.

We assume some familiarity with category theory and refer to the textbooks [1,18] for more details and notation.

3. The quantale of M-valued distance distribution functions

Let M be a completely distributive lattice. A function $\varphi : [0, \infty] \longrightarrow M$ with $\varphi(t) = \bigvee_{s<t} \varphi(s)$ for all $t \in [0, \infty)$ is called an *M-valued distance distribution function*. The set of all these M-valued distance distribution functions is denoted by Δ^+_M. In case $M = [0, 1]$ we obtain the distance distribution functions that are used in the theory of probabilistic metric spaces [23], however we omit the *finiteness condition* $\varphi(\infty) = 1$. We note that $\varphi \in \Delta^+_M$ satisfies $\varphi(0) = \perp_M$ (using $\bigvee \emptyset = \perp_M$) and is non-decreasing. For example, for each $0 \le a \le \infty$ the functions

$$\varepsilon_a(x) = \begin{cases} \perp_M & \text{if } 0 \le x \le a \\ \top_M & \text{if } a < x \le \infty \end{cases}$$

are in Δ^+_M. The set Δ^+_M is ordered pointwise, i.e. for $\varphi, \psi \in \Delta^+_M$ we define $\varphi \le \psi$ if for all $s \ge 0$ we have $\varphi(s) \le \psi(s)$. The bottom element of Δ^+_M is ε_∞ and the top element is ε_0. It is not difficult to show that for $\varphi_j \in \Delta^+_M$ ($j \in J$), the pointwise supremum $\bigvee_{j \in J} \varphi_j$ is in Δ^+_M. If $\varphi, \psi \in \Delta^+_M$, also the pointwise minimum $\varphi \wedge \psi$ is in Δ^+_M. From the frame law for M we immediately obtain $(\varphi \wedge \psi)(t) \ge \bigvee_{s<t} (\varphi \wedge \psi)(s)$. For the converse inequality, let $\alpha \lhd \bigvee_{s<t} \bigvee_{u<t} (\varphi(s) \wedge \psi(u))$. Then there are $s < t$ and $u < t$ such that $\alpha \le \varphi(s) \wedge \psi(u)$. As both φ, ψ are non-decreasing, we conclude with $w = s \vee u < t$ that $\alpha \le \varphi(w) \wedge \psi(w) = (\varphi \wedge \psi)(w)$. Hence $\alpha \le \bigvee_{w<t} (\varphi \wedge \psi)(w)$ and from the complete distributivity of M we conclude $(\varphi \wedge \psi)(t) \le \bigvee_{w<t} (\varphi \wedge \psi)(w)$. Hence the set Δ^+_M with the pointwise order becomes a complete lattice. We note that $\bigwedge_{i \in I} \varphi_i$ is in general not the pointwise infimum. We are now going to show that Δ^+_M is completely distributive. To this end, we define for $0 < \delta \le \infty$ and $\epsilon \in M \setminus \{\perp_M\}$ the M-valued distance distribution function $f_{\delta\epsilon}$ by

$$f_{\delta\epsilon}(t) = \begin{cases} \perp & \text{if } t \le \delta \\ \epsilon & \text{if } t > \delta. \end{cases}$$

Note that we have $f_{\delta\epsilon} = \epsilon \wedge 1_{(\delta,\infty]}$ with the characteristic function of the interval $(\delta,\infty]$ defined by

$$1_{(\delta,\infty]}(x) = \begin{cases} \top_M & \text{if } x \in (\delta,\infty] \\ \bot_M & \text{otherwise.} \end{cases}$$

It is not difficult to show that $f_{\delta\epsilon} \le f_{\delta'\epsilon'}$ if and only if $\delta' \le \delta$ and $\epsilon \le \epsilon'$ and we can deduce the following result.

Lemma 3.1. *We have $f_{\delta\epsilon} = \bigvee_{\delta<\alpha,\gamma\lhd\epsilon} f_{\alpha\gamma}$.*

Proof. We use the representation of $f_{\delta\epsilon}$ by characteristic functions. From $(\delta,\infty] = \bigcup_{\delta<\alpha}(\alpha,\infty]$ we immediately see that $f_{\delta\epsilon} = \bigvee_{\delta<\alpha} f_{\alpha\epsilon}$ and from $\epsilon = \bigvee\{\gamma \in M \ : \ \gamma \lhd \epsilon\}$ we obtain $f_{\delta\epsilon} = \bigvee_{\gamma\lhd\epsilon} f_{\delta\gamma}$. □

Lemma 3.2. *We have $f_{\delta\epsilon} \lhd f_{\delta'\epsilon'}$ if and only if $\delta' < \delta$ and $\epsilon \lhd \epsilon'$.*

Proof. By Lemma 3.1, if $f_{\delta\epsilon} \lhd f_{\delta'\epsilon'} = \bigvee_{\delta'<\alpha,\gamma\lhd\epsilon'} f_{\alpha\gamma}$, then there is $\alpha > \delta'$ and $\gamma \lhd \epsilon'$ such that $f_{\delta\epsilon} \le f_{\alpha\gamma}$ and hence $\delta' < \alpha \le \delta$ and $\epsilon \le \gamma \lhd \epsilon'$, from which we get $\delta' < \delta$ and $\epsilon \lhd \epsilon'$.

For the converse, let $\delta' < \delta$ and $\epsilon \lhd \epsilon'$ and assume $f_{\delta\epsilon} \not\lhd f_{\delta'\epsilon'}$. Then there are $\psi_j \in \Delta_M^+$, $j \in J$, such that $f_{\delta'\epsilon'} \le \bigvee_{j\in J} \psi_j$ but $f_{\delta\epsilon} \not\le \psi_j$ for all $j \in J$. Hence, for all $j \in J$ there is t_j such that $\epsilon = f_{\delta\epsilon}(t_j) \not\le \psi(j)(t_j)$. We conclude $\delta' < \delta \le t_0 = \inf\{s_j : j \in J\}$ and hence $\epsilon \lhd \epsilon' = f_{\delta'\epsilon'}(t_0) \le \bigvee_{j\in J} \psi_j(t_0)$. Thus there is j_0 such that $\epsilon \le \psi_{j_0}(t_0) \le \psi_{j_0}(t_{j_0})$, a contradiction. □

Lemma 3.3. *For $\varphi \in \Delta_M^+$ we have $\varphi = \bigvee\{f_{\delta\epsilon} \ : \ f_{\delta\epsilon} \le \varphi\}$.*

Proof. Clearly $\bigvee\{f_{\delta\epsilon} \ : \ f_{\delta\epsilon} \le \varphi\} \le \varphi$. For the converse inequality, we fix $t_0 \in (0,\infty]$. We further note that for any $s \in (0,\infty]$ we have $f_{s\varphi(s)} \le \varphi$. If $t < t_0$ then $\varphi(t) = f_{t\varphi(t)}(t_0) \le \varphi(t_0)$ and hence $\varphi(t_0) = \bigvee_{t<t_0} \varphi(t) = \bigvee_{t<t_0} f_{t\varphi(t)}(t_0) \le \varphi(t_0)$. We conclude $\varphi(t_0) = \bigvee_{t<t_0} f_{t\varphi(t)}(t_0) \le \bigvee\{f_{\delta\epsilon}(t_0) \ : \ f_{\delta\epsilon} \le \varphi\}$. □

Lemma 3.4. *We have $f_{\delta\epsilon} \lhd \varphi$ if and only if $\epsilon \lhd \varphi(\delta)$.*

Proof. Let first $f_{\delta\epsilon} \lhd \varphi = \bigvee\{f_{\delta\epsilon} \ : \ f_{\delta\epsilon} \le \varphi\}$. Then there are δ', ϵ' such that $f_{\delta\epsilon} \lhd f_{\delta'\epsilon'}$ and hence $\delta' < \delta$ and $\epsilon \lhd \epsilon'$ and we conclude $\epsilon \lhd \epsilon' = f_{\delta'\epsilon'}(\delta) \le \varphi(\delta)$.

Let now $\epsilon \lhd \varphi(\delta)$ and let $\varphi \le \bigvee_{j\in J} \psi_j$ for $\psi_j \in \Delta_M^+$, $j \in J$. Then $\epsilon \lhd \bigvee_{j\in J} \psi_j(\delta)$ and hence there is j_0 such that $\epsilon \le \psi_{j_0}(\delta)$. We conclude $f_{\delta\epsilon} \le \psi_{j_0}$ and hence $f_{\delta\epsilon} \lhd \varphi$. □

Lemma 3.5. *Let $\varphi \in \Delta_M^+$. Then $\varphi = \bigvee\{f_{\delta\epsilon} \ : \ f_{\delta\epsilon} \lhd \varphi\}$.*

Proof. The inequality $\varphi \ge \bigvee\{f_{\delta\epsilon} \ : \ f_{\delta\epsilon} \lhd \varphi\}$ is obvious. To show the converse inequality, we first note $f_{t\varphi(t)} = \bigvee\{f_{t\epsilon} \ : \ \epsilon \lhd \varphi(t)\}$. Hence for any t_0 we get

$$\begin{aligned} \varphi(t_0) = \bigvee_{t<t_0} f_{t\varphi(t)}(t_0) &= \bigvee_{t<t_0} \bigvee\{f_{t\epsilon}(t_0) \ : \ \epsilon \lhd \varphi(t)\} \\ &= \bigvee_{t<t_0} \bigvee\{f_{t\epsilon}(t_0) \ : \ f_{t\epsilon \lhd \varphi}\} \ \le \ \bigvee\{f_{\delta\epsilon}(t_0) \ : \ f_{\delta\epsilon} \lhd \varphi\}. \end{aligned}$$

□

As a consequence, we obtain the main result.

THEOREM 3.6. $(\Delta^+_M, \leq)$ *is completely distributive.*

A binary operation, $* : \Delta^+_M \times \Delta^+_M \longrightarrow \Delta^+_M$, which is commutative, associative, non-decreasing in each variable and that satisfies the boundary condition $\varphi * \varepsilon_0 = \varphi$ for all $\varphi \in \Delta^+_M$, is called a *triangle function* [23]. A triangle function is called *sup-continuous* [23], if $(\bigvee_{i\in I} \varphi_i) * \psi = \bigvee_{i\in I}(\varphi_i * \psi)$ for all $\varphi_i, \psi \in \Delta^+_M$, $(i \in I)$, i.e. if $\mathsf{L} = (\Delta^+_M, \leq, *)$ is a quantale. Two typical sup-continuous triangle functions are defined as follows. Let $(M, \leq)$ be completely distributive and let $*$ be a quantale operation on M. We define, for $\varphi, \psi \in \Delta^+_M$ and $0 \leq t \leq \infty$,

$$\varphi \boxast \psi(t) = \varphi(t) * \psi(t) \quad \text{and} \quad \varphi \circledast \psi(t) = \bigvee_{u+v=t} \varphi(u) * \psi(v).$$

It is not difficult to show that $(\Delta^+_M, \leq, \boxast)$ and $(\Delta^+_M, \leq, \circledast)$ are quantales.

4. LM-fuzzy metric spaces

For a complete lattice $\mathsf{L} = (L, \leq)$ and a quantale $\mathsf{M} = (M, \leq, *)$, an LM-*fuzzy metric space* is a pair (X, d) of a set X and a mapping $d : J(L^X) \times J(L^X) \longrightarrow M$ such that, for all $a, b, c \in J(L^X)$,

(LMD1) $d(a, a) = \top_M$;
(LMD2) $d(a, b) * d(b, c) \leq d(a, c)$;
(LMD3) $d(a, b) = \bigvee_{c \lhd b} d(a, c)$.

A mapping between two LM-fuzzy metric spaces, $f : (X, d) \longrightarrow (X', d')$ is called an LM-*fuzzy metric morphism* if $d(a, b) \leq d'(f(a), f(b))$ for all $a, b \in J(L^X)$. We denote the category of LM-fuzzy metric spaces with LM-fuzzy metric morphisms by LM-FMET.

Note that an M-*metric space* (Z, d) (also called an M-*preordered set*, see e.g. [12] and the literature cited there) is a set Z together with a mapping $d : Z \times Z \longrightarrow M$ which is *reflexive*, $d(z, z) = \top_M$ for all $z \in Z$, and *transitive*, $d(x, y) * d(y, z) \leq d(x, z)$ for all $x, y, z \in Z$. Special instances of M-metric spaces are metric spaces, with Lawevere's quantale $\mathsf{M} = ([0, \infty], \geq, +)$, [13], or probabilistic metric spaces with $\mathsf{M} = (\Delta^+_{[0,1]}, \leq, *)$, see [6]. The major difference between an LM-fuzzy metric space (X, d) and an M-metric space defined on $Z = J(L^X)$ is the axiom (LMD3). This axiom has an interesting consequence, that we state in the next result.

LEMMA 4.1. *Let $d : J(L^X) \times J(L^X) \longrightarrow M$ satisfy (LMD3). Then (LMD1) is equivalent to (LMD1') $d(a, b) = \top_M$ whenever $a \leq b$.*

Proof. It is clear that (LMD1') implies (LMD1). For the converse, let $a \leq b$, $a, b \in J(L^X)$ and let $\alpha \lhd \top_M$. By (LMD1) and (LMD3), $\alpha \lhd d(a, a) = \bigvee_{c \lhd a} d(a, c)$. Hence there is $c \lhd a$ such that $\alpha \leq d(a, c)$ and as also $c \lhd b$ we conclude, again using (LMD3), that $\alpha \leq \bigvee_{c \lhd b} d(a, c) = d(a, b)$. This is true for all $\alpha \lhd \top_M$ and by complete distributivity $\top_M = d(a, b)$. □

We note further, that in an LM-metric space, the property (LMD1') is equivalent to (LMD4) $d(b,c) \le d(a,c)$ whenever $a \le b$. In fact, if (LMD1') is valid and $a \le b$, by (LMD2) $d(a,c) \ge d(a,b) * d(b,c) = d(b,c)$, i.e. (LMD4) is valid and conversely, from (LMD4) we conclude with (LMD1), that $a \le b = c$ implies $d(a,b) \ge d(b,b) = \top_M$, i.e. (LMD1') is satisfied. Similarly, we can show that (LMD1') is equivalent to (LMD4') $d(c,a) \le d(c,b)$ whenever $a \le b$.

REMARK 4.2. A mapping $d : J(L^X) \times J(L^X) \longrightarrow M$ can be identified with its "exponential mate" $\ulcorner d \urcorner : X \times X \longrightarrow M^{J(L)\times J(L)}$, defined by $\ulcorner d \urcorner(x,y)(\alpha,\beta) = d(x_\alpha, y_\beta)$. Here we use the representation $J(L^X) = \{x_\alpha \ : \ x \in X, \alpha \in J(L)\}$ where $x_\alpha \in J(L^X)$ is defined by $x_\alpha(z) = \alpha$ if $z = x$ and $x_\alpha(z) = \bot_M$ for $z \ne x$. If we use on $M^{J(L)\times J(L)}$ the pointwise quantale structure, i.e. if we put for $\varphi, \psi : J(L) \times J(L) \longrightarrow M$, $\varphi * \psi(\alpha,\beta) = \varphi(\alpha,\beta) * \psi(\alpha,\beta)$, then we can demand that $\ulcorner d \urcorner$ is an $M^{J(L)\times J(L)}$-metric space. However, we cannot identify this quantale-metric space with an LM-fuzzy metric space in general. For instance, the axiom $\ulcorner d \urcorner(x,x) = \top_{M^{J(L)\times J(L)}}$ means here that $\ulcorner d \urcorner(x,x)(\alpha,\beta) = \top_M$ for all $\alpha, \beta \in J(L)$, which would entail $d(x_\alpha, x_\beta) = \top_M$ for all $x_\alpha, x_\beta \in J(L^X)$. This is in general not satisfied by an LM-fuzzy metric space and counterexamples are readily available, e.g. in the Example 4.5 below. Also the triangle inequality $\ulcorner d \urcorner(x,y) * \ulcorner d \urcorner(y,z) \le \ulcorner d \urcorner(x,z)$ entails for all $\alpha, \beta \in J(L)$ that $d(x_\alpha, y_\beta) * d(y_\alpha, z_\beta) \le d(x_\alpha, z_\beta)$, which again is not true in general for an LM-fuzzy metric space. Again, examples are easily constructed in Example 4.5 below. We note however, that for an LM-fuzzy metric space (X,d), if $y_\beta \le y_\alpha$, i.e. if $\alpha \le \beta$ for $\alpha, \beta \in J(L)$, by (LMD1') we do have $d(x_\alpha, y_\beta) * d(y_\alpha, z_\beta) = d(x_\alpha, y_\beta) * \underbrace{d(y_\beta, y_\alpha)}_{=\top_M} * d(y_\alpha, z_\beta) \le d(x_\alpha, y_\alpha) * d(y_\alpha, z_\beta) \le d(x_\alpha, z_\beta)$.

We will show with Remark 4.8 below, that an LM-fuzzy metric space can induce an LM-fuzzy topological space [27]. It is unclear at present how this can be achieved with quantale-valued metric spaces.

We give some examples for LM-fuzzy metric spaces.

EXAMPLE 4.3. Let $\mathsf{L} = (\{0,1\}, \le)$ and $\mathsf{M} = ([0,\infty], \ge, +)$. For a set X, a coprime element of L^X can be identified with a point $x \in X$. Hence the axiom (LMD1) becomes $d(a,a) = 0$, the axiom (LMD2) turns into the well-known triangle inequality $d(a,c) \le d(a,b) + d(b,c)$ and the axiom (LMD3) is void. So an LM-fuzzy metric space is a pseudo-quasimetric space and a LM-fuzzy metric morphism is a non-expansive map.

EXAMPLE 4.4. Let $\mathsf{L} = (\{0,1\}, \le)$ and $\mathsf{M} = (\Delta^+_{[0,1]}, \le, *)$ with a sup-continuous triangle function. An LM-fuzzy metric space is a *probabilistic pseudo-quasimetric space* [23]. These spaces can be identified with the *fuzzy pseudo-quasimetric spaces* in the sense of Kramosil and Michalek [11], where a *fuzzy pseudo-quasimetric on X* is a mapping $M : X \times X \times [0,\infty] \longrightarrow [0,1]$ such that $M(x,y,\cdot)$ is left-continous, $M(x,y,0) = 0$, $M(x,x,t) = 1$ for all $t > 0$ and $M(x,z,t+s) \ge M(x,y,t) * M(y,z,s)$ for all $x,y,z \in X, s,t \in [0,\infty]$. The identification is obtained if we define $d : X \times X \longrightarrow \Delta^+_{[0,1]}$ by $d(x,y)(t) = M(x,y,t)$ and use the triangle function $\circledast$ on $\Delta^+_{[0,1]}$.

Example 4.5. Let $\mathsf{L} = \mathsf{M} = ([0,1], \leq, *)$ with the Lukasiewicz t-norm $\alpha * \beta = (\alpha + \beta - 1) \vee 0$ and let $X = \mathbb{R}$. If we denote the L-points by x_λ defined by $x_\lambda(y) = \lambda$ if $y = x$ and $x_\lambda(y) = 0$ if $y \neq x$, where $\lambda > 0$, then $d(x_\lambda, y_\mu) = 1 - (|x - y| + ((\lambda - \mu) \vee 0))$ defines an LM-fuzzy metric on $\mathbb{R}$. With regard to Remark 4.2, we mention that $d(x_{0.5}, x_{0.25}) = \frac{3}{4} \neq 1$.

Example 4.6. Let $\mathsf{L} = (L, \leq)$ be completely distributive and let $\mathsf{M} = ([0,\infty], \geq, +)$. An LM-fuzzy metric $d : J(L^X) \times J(L^X) \longrightarrow M$ satisfies the axioms (M1) $d(a,a) = 0$ for all $a \in J(L^X)$, (M2) $d(a,c) \leq (a,b) + d(b,c)$ for all $a,b,c \in J(L^X)$, (M3) $d(a,b) = \bigvee_{c \lhd b} d(a,c)$ for all $a,b \in J(L^X)$ and (M4) $a \leq b$ implies $d(a,c) \leq d(b,c)$ for all $a,b,c \in J(L^X)$. In this sense an LM-fuzzy metric space can be identified with a *pointwise pseudo-quasimetric* in the sense of Shi [24]. In contrast to Shi's definition in [24], we allow the value ∞ for the distance and do not demand a symmetry axiom.

Example 4.7. Let $\mathsf{L} = (L, \leq, \wedge)$ and $\mathsf{N} = (N, \leq, *)$ where the lattices L, N are completely distributive and let $\mathsf{M} = (\Delta_N^+, \leq, \circledast)$. An LM-fuzzy metric $d : J(L^X) \times J(L^X) \longrightarrow M$ can be identified with a mapping $D : J(L^X) \times J(L^X) \times [0,\infty] \longrightarrow M$ via $D(a,b,t) = d(a,b)(t)$ and satisfies the axioms (LMD1) $D(a,b,0) = \bot_M$, (LMD2) $D(a,b,t) = \bigvee_{s<t} D(a,b,s)$, (LMD4) $a \leq b$ implies $\bigwedge_{t>0} D(a,b,t) = \top_M$, (LMD5) $D(a,b,s) * D(b,c,t) \leq D(a,c,s+t)$, (LMD6) $D(a,b,t) = \bigvee_{c \lhd b} D(a,c,t)$. In this way, for $* = \wedge$, an LM-fuzzy metric space can be identified with an (L,M)-fuzzy pseudo-quasimetric space in the definition of Shi [26]. Note that our axioms are slightly altered, as we allow again the value ∞ and we do not demand a *finiteness-condition* (LMD3) $\bigvee_{t>0} D(a,b,t) = \top_M$.

Remark 4.8 (LM-approach distances and LM-fuzzy topological spaces). One motivation for considering LM-fuzzy metric spaces is the possibility of defining distances between L-points and L-sets. For notational convenience, we denote in this remark L-sets by capital letters $A, B, \dots \in L^X$. Let (X,d) be an LM-fuzzy metric space, let $a \in J(L^X)$ and let $A \in L^X$. We define $\delta^d : J(L^X) \times L^X \longrightarrow M$ by

$$\delta^d(a,A) = \bigvee_{b \leq A, b \in J(L^X)} d(a,b).$$

We then have the following properties.

(LMAD1) $\delta^d(a,a) = \top_M$;
(LMAD2) $\delta^d(a,\bot_L) = \bot_M$;
(LMAD3) $\delta^d(a, A \vee B) = \delta^d(a,A) \vee \delta^d(a,B)$;
(LMAD4) $\delta^d(a,A) \geq \delta^d(a, A^{(\epsilon)}) * \epsilon$, where $A^{(\epsilon)} = \bigvee\{b \in J(L^X) \,:\, \delta^d(b,A) \geq \epsilon\}$.

We only prove (LMAD4). First we note that $\delta^d(a,A) = \bigvee_{b \lhd A} d(a,b)$. Let $\alpha \lhd \delta^d(a,A)$. Then there is $c \leq A$ such that $\alpha \lhd d(a,c)$ and hence, by (LMD3), there is $b \lhd c \leq A$ such that $\alpha \leq d(a,b)$. It follows $\alpha \leq \bigvee_{b \lhd A} d(a,b)$ and by the complete distributivity $\delta^d(a,A) \leq \bigvee_{b \lhd A} d(a,b)$. The converse inequality is clear. In order to show (LMAD4), let $c \lhd A^{(\epsilon)}$. Then there is $c \leq b$ such that $\delta^d(b,A) \geq \epsilon$. Let $\delta \lhd \epsilon$. Then there is $e \leq A$ such that $d(b,e) \geq \delta$. From (LMD2) we infer $d(a,e) \geq d(a,b) * d(b,e) \geq d(a,b) * \delta$ and hence $\delta^d(a,A) \geq d(a,b) * \delta$. This is true for all $\delta \lhd \epsilon$

and hence we conclude from the distributivity of the quantale operation over joins, the complete distributivity of M and (LMD1) and (LMD2)

$$\delta^d(a, A) \geq d(a, b) * \epsilon \geq d(a, c) * d(c, b) * \epsilon = d(a, c) * \epsilon.$$

This implies $\delta^d(a, A) \geq \bigvee_{c \lhd A^{(\epsilon)}} d(a, c) * \epsilon = \delta^d(a, A^{(\epsilon)}) * \epsilon$.

The axioms (LMAD1)–(LMAD4) are suitable generalizations of the axioms of an *approach distance* [14] to the lattice-valued case.

We now show that an LM-fuzzy metric space can induce an LM-fuzzy topological space in a natural way. An *LM-fuzzy topological space* [27] is a set X together with a mapping $\mathcal{T} : L^X \longrightarrow M$ such that

(LMT1) $\mathcal{T}(\top_X) = \top_M$,
(LMT2) $\mathcal{T}(A) \wedge \mathcal{T}(B) \leq \mathcal{T}(A \wedge B)$ and
(LMT3) $\bigwedge_{j \in J} \mathcal{T}(A_j) \leq \mathcal{T}(\bigvee_{j \in J} A_j)$

for all $A, B, A_j \in L^X$. An *LM-closure operator* [25] is a mapping $Cl : L^X \longrightarrow M^{J(L^X)}$ such that

(LMC1) $Cl(A)(a) = \bigwedge_{b \lhd a} Cl(A)(b)$,
(LMC2) $Cl(\bot_X)(a) = \bot_M$ for all $a \in J(L^X)$,
(LMC3) $Cl(A)(a) = \top_M$ whenever $a \leq A$,
(LMC4) $Cl(A \vee B) = Cl(A) \vee Cl(B)$ and
(LMC5) $Cl(\bigvee (Cl(A))_{[\lambda]})(a) \geq \lambda$ implies $Cl(A)(a) \geq \lambda$, where $(Cl(A))_{[\lambda]} = \{b \in J(L^X) : Cl(A)(b) \geq \lambda\}$.

It is shown in [26] together with [25] that if L, M are completely distributive lattices equipped with order-reversing involutions, the categories of LM-closure spaces and of LM-fuzzy topological spaces are isomorphic.

PROPOSITION 4.9. *Let $\mathsf{L} = (L, \leq), \mathsf{M} = (M, \leq)$ be completely distributive lattices and consider the minimum $\wedge$ as the quantale operation on M. Then $Cl_d : L^X \longrightarrow M^{J(L^X)}$ defined by $Cl_d(A)(a) = \delta^d(a, A)$ for $A \in L^X$ and $a \in J(L^X)$ is an LM-closure operator. Hence, in case both L and M are equipped with order-reversing involutions, an LM-fuzzy metric space induces an LM-fuzzy topological space $(X, \mathcal{T}_d)$ via the LM-closure operator Cl_d, i.e. we have [25, 26]*

$$\mathcal{T}_d(A) = \bigwedge_{a \not\leq A', a \in J(L^X)} (Cl_d(A')(a))' = \bigwedge_{a,b \in J(L^X), a \not\leq A', b \leq A} (d(a, b))'$$

for all $A \in L^X$.

Proof. We only need to show (LMC1), (LMC3) and (LMC5). For (LMC1), we note that if $b \lhd a$, we have $b \leq a$ and hence $d(a, c) \leq d(b, c)$ by (LMD4). We conclude $Cl_d(A)(a) = \bigwedge_{c \leq A} d(a, c) \leq \bigwedge_{c \leq a} d(b, c) = Cl_d(A)(b)$ and hence we have $Cl_d(A)(a) \leq \bigwedge_{b \lhd a} Cl_d(A)(b)$. For the converse, assume that $Cl_d(A)(a) \not\geq \bigwedge_{b \lhd a} Cl_d(A)(b)$. Clearly, $Cl_d(A)(a) \neq \top$ and as every element in M is a meet of primes, we conclude that there is a prime element $\lambda \in M \setminus \{\top_M\}$ such that $\lambda \geq Cl_d(A)(a)$ but $\lambda \not\geq \bigwedge_{b \lhd a} Cl_d(A)(b)$. Hence for any $b \lhd a$ there exists $c \leq A$ with $c \in J(L^X)$ such that $d(b, c) \not\leq \lambda$ and $d(a, c) \leq \lambda$. By (LMD2) we have $d(a, b) \wedge d(b, c) \leq d(a, c) \leq \lambda$ and since λ is a prime element, we conclude $d(a, b) \leq \lambda$. Thus we obtain $\top_M = d(a, a) = \bigvee_{b \lhd a} d(a, b) \leq \lambda$, a contradicition.

(LMC3) is a consequence of (LMAD1), as for $a \leq A$ we have $Cl_d(A)(a) = \bigvee_{b \leq a} d(a,b) \geq d(a,a) = \top_M$.

For (LMC5) we note that $\bigvee (Cl(A))_{[\lambda]} = A^{(\lambda)}$ and hence with (LMAD4) from above we get $\delta^d(a,A) \geq \delta^d(a, A^{(\lambda)}) \wedge \lambda$. So if $Cl_d(\bigvee (Cl(A))_{[\lambda]})(a) = \delta^d(a, A^{(\lambda)}) \geq \lambda$, also $Cl_d(A)(a) = \delta^d(a,A) \geq \lambda$. $\square$

5. LM-fuzzy metric spaces as LM-fuzzy convergence tower spaces

Let X be a set. We consider filters in $J(L^X)$ and denote the set of all such filters by $\mathfrak{F}(J(L^X))$. For instance, a sequence of L-points generates such a filter in the usual way by considering the filter base consisting of the end pieces of the sequence.

A family of mappings $\overline{c} = (c_\alpha : \mathfrak{F}(J(L^X)) \longrightarrow P(J(L^X)))_{\alpha \in M}$ which satisfies the axioms, for all $a, b \in J(L^X), \alpha, \beta \in M$, $\mathbb{F}, \mathbb{G} \in \mathfrak{F}(J(L^X))$,

(LMC1) $a \in c_\alpha([b])$ whenever $a \leq b$;
(LMC2) $c_\alpha(\mathbb{F}) \subseteq c_\alpha(\mathbb{G})$ whenever $\mathbb{F} \leq \mathbb{G}$;
(LMC3) $c_\beta(\mathbb{F}) \subseteq c_\alpha(\mathbb{F})$ whenever $\alpha \leq \beta$;
(LMC4) $x \in c_\perp(\mathbb{F})$ for all $x \in X, \mathbb{F} \in \mathfrak{F}(J(L^X))$;

is called an LM-*fuzzy convergence tower on* X and the pair $(X, \overline{c})$ is called an LM-*fuzzy convergence tower space*. A mapping $f : X \longrightarrow X'$ between the LM-fuzzy convergence tower spaces $(X, \overline{c})$ and $(X', \overline{c'})$, is called *continuous* if, for all $a \in J(L^X)$ and all $\mathbb{F} \in \mathfrak{F}(J(L^X))$, $f(a) \in c'_\alpha(f(\mathbb{F}))$ whenever $a \in c_\alpha(\mathbb{F})$. The category of LM-fuzzy convergence tower spaces with continuous mappings as morphisms is denoted by LM-FCTS.

It is not difficult to show that LM-FCTS is a topological category, where initial structures are formed as follows. Let for each $j \in J$, $(X_j, \overline{c^j})$ be an LM-fuzzy convergence tower space and let $f_j : X \longrightarrow X_j$ be a mapping. The initial LM-fuzzy convergence tower on X is defined by

$$a \in c_\alpha(\mathbb{F}) \iff f_j(a) \in c^j_\alpha(f_j(\mathbb{F})) \forall j \in J.$$

We would like to point out, however, that, in general, the category LM-FCTS is not well-fibred, as there may be more than one LM-fuzzy convergence tower on a one-point set. This is due to the fact that for a one-point set $X = \{x\}$ we can identify $J(L^X)$ with $J(L)$ and there is in general more than one filter on this set.

An LM-fuzzy convergence tower space $(X, \overline{c})$ is called

(i) *pretopological* if $\bigcap_{i \in I} c_\alpha(\mathbb{F}_i) \subseteq c_\alpha(\bigwedge_{i \in I} \mathbb{F}_i)$ whenever $\alpha \in M$ and $(\mathbb{F}_i)_{i \in I} \in \mathfrak{F}(J(L^X))^I$;
(ii) *left-continuous* if for all subsets $A \subseteq M$ we have $a \in c_{\bigvee A}(\mathbb{F})$ whenever $a \in c_\alpha(\mathbb{F})$ for all $\alpha \in A$;
(iii) $*$-*transitive* if $a \in c_{\alpha * \beta}([c])$ whenever $a \in c_\alpha([b])$ and $b \in c_\beta([c])$;
(iv) *approximating* if $a \in c_\alpha([b])$ if and only if for all $\beta \lhd \alpha$ there is $c \lhd b$ such that $a \in c_\beta([c])$.

REMARK 5.1. A left-continuous LM-fuzzy convergence tower $\overline{c}$ can be identified with a *limit function* $\lambda : \mathfrak{F}(J(L^X)) \longrightarrow M^X$, where $\lambda(\mathbb{F})(a) = \bigvee\{\alpha \in M \ : \ a \in c_\alpha(\mathbb{F})\}$.

We give some examples for LM-fuzzy convergence tower spaces.

EXAMPLE 5.2. Identifying for $L = \{0,1\}$ the set $J(L^X)$ with X, we see that in this case we have the following examples.

1. *Generalized convergence spaces.* If $M = \{0,1\}$, an LM-fuzzy convergence space is a generalized convergence space in the sense of Preuss [19].
2. *Probabilistic convergence spaces.* If $M = [0,1]$, an LM-fuzzy convergence space is a probabilistic convergence space in the sense of Richardson and Kent [21].
3. *Probabilistic convergence spaces.* If $\mathsf{M} = (\Delta^+_{[0,1]}, \leq, *)$ with a sup-continuous triangle function, an LM-fuzzy convergence space is a probabilistic convergence space in the sense of [10].
4. *Limit tower spaces.* If $\mathsf{M} = ([0,\infty], \geq, +)$, a left-continuous LM-fuzzy convergence space is a limit tower space in the sence of Brock and Kent [2]. These spaces can be identified with convergence approach spaces [15]. We note that in [2] one additional axiom is demanded that we do not use here.

EXAMPLE 5.3. Let $X = [0,1]$ and let $\mathsf{L} = \mathsf{M} = ([0,1], \leq, *)$ with the Lukasiewicz t-norm defined by $\alpha * \beta = (\alpha + \beta - 1) \vee 0$. We define, for $\mathbb{F} \in \mathfrak{F}(J(L^X))$ and $\alpha \in M$, $x_\lambda \in c_\alpha(\mathbb{F})$ if for all $\epsilon > 0$ there is $F \in \mathbb{F}$ such that for all $y_\mu \in F$ we have $|x - y| + ((\lambda - \mu) \vee 0) < 1 - \alpha + \epsilon$. Then $([0,1], \overline{c})$ is an LM-fuzzy convergence space.

A space $(X, \overline{c}) \in |\mathsf{LM\text{-}FCTS}|$ that is $*$-transitive, left-continuous, pretopological and approximating, is called an LM*-prefuzzy-metric fuzzy convergence tower space* and we denote the subcategory of $\mathsf{LM\text{-}FCTS}$ with these spaces as objects by $\mathsf{LM\text{-}PreFMET\text{-}FCTS}$.

Let $(X,d) \in |\mathsf{LM\text{-}FMET}|$. We define a family of mappings $\overline{c^d} = (c^d_\alpha : \mathfrak{F}(J(L^X)) \longrightarrow P(J(L^X)))_{\alpha \in M}$ by

$$a \in c^d_\alpha(\mathbb{F}) \iff \bigvee_{F \in \mathbb{F}} \bigwedge_{b \in F} d(a,b) \geq \alpha.$$

PROPOSITION 5.4. *E : $\mathsf{LM\text{-}FMET} \longrightarrow \mathsf{LM\text{-}PreFMET\text{-}FCTS}$ defined by $E((X,d)) = (X, \overline{c^d})$, $E(f) = f$ is an embedding functor.*

Proof. We first show that for $(X,d) \in \mathsf{LM\text{-}FMET}$, $(X, \overline{c^d}) \in \mathsf{LM\text{-}FCTS}$.

We first note that $a \in c^d_\alpha([b])$ if and only if $d(a,b) \geq \alpha$. This follows from $\bigvee_{F \in [b]} \bigwedge_{c \in F} d(a,c) \geq d(a,b)$, as $\{b\} \in [b]$, and also $\bigvee_{F \in [b]} \bigwedge_{c \in F} d(a,c) \leq \bigvee_{F \in [b]} d(a,b)$ $= d(a,b)$, as $b \in F$ for all $F \in [b]$.

(LMC1) We have by (LMD1') for $a \leq b$, $d(a,b) = \top_M \geq \alpha$ and hence $a \in c^d_\alpha([b])$. (LMC2), (LMC3) and (LMC4) are obvious.

Let now $f : (X,d) \longrightarrow (X', d')$ be an LM-fuzzy metric morphism. We show that $f : (X, \overline{c^d}) \longrightarrow (X', \overline{c^{d'}})$ is continuous. Let $a \in c^d_\alpha(\mathbb{F})$. Then

$$\alpha \leq \bigvee_{F \in \mathbb{F}} \bigwedge_{b \in F} d(a,b) \leq \bigvee_{F \in \mathbb{F}} \bigwedge_{b \in F} d'(f(a), f(b))$$

$$= \bigvee_{F \in \mathbb{F}} \bigwedge_{b' \in f(F)} d'(f(a), b') \leq \bigvee_{G \in f(\mathbb{F})} \bigwedge_{b' \in G} d'(f(a), b').$$

Hence $f(a) \in c_\alpha^{d'}(f(\mathbb{F}))$.

Next we show that $(X, \overline{c^d})$ is $*$-transitive, left-continuous, pretopological and approximating. For transitivity, let $a \in c_\alpha^d([b])$ and $b \in c_\beta^d([c])$. Then $d(a,b) \geq \alpha$ and $d(b,c) \geq \beta$ and by (LMD2) this implies $d(a,c) \geq d(a,b) * d(b,c) \geq \alpha * \beta$, i.e. $a \in c_{\alpha * \beta}^d([c])$.

For left-continuity, let $a \in c_\alpha^d(\mathbb{F})$ for all $\alpha \in A \subseteq M$. Then for all $\alpha \in A$ we have $\bigvee_{F \in \mathbb{F}} \bigwedge_{b \in F} d(a,b) \geq \alpha$ and therefore also $\bigvee_{F \in \mathbb{F}} \bigwedge_{b \in F} d(a,b) \geq \bigvee A$, i.e. $a \in c_{\bigvee A}^d(\mathbb{F})$.

To show the pretopologicalness, let $a \in \bigcap_{j \in J} c_\alpha^d(\mathbb{F}_j)$. Then, using the interpolation property for the well-below relation, for all $j \in J$ and all $\epsilon \lhd \alpha$ there is $F_j^\epsilon \in \mathbb{F}_j$ such that for all $b_j \in F_j^\epsilon$, $d(a, b_j) \geq \epsilon$. We conclude $F = \bigcup_{j \in J} F_j^\epsilon \in \bigwedge_{j \in J} \mathbb{F}_j$ and for all $b \in F$ we have $d(a,b) \geq \epsilon$. Hence $\bigvee_{F \in \bigwedge_{j \in J} \mathbb{F}_j} \bigwedge_{b \in F} d(a,b) \geq \epsilon$. This is true for all $\epsilon \lhd \alpha$ and hence, using the complete distributivity of M, $\bigvee_{F \in \bigwedge_{j \in J} \mathbb{F}_j} \bigwedge_{b \in F} d(a,b) \geq \alpha$ and we obtain $a \in c_\alpha^d(\bigwedge_{j \in J} \mathbb{F}_j)$.

For the approximation property, let first $a \in c_\alpha^d([b])$ and $\beta \lhd \alpha$. By (LMD3) this implies the existence of $c \lhd b$ such that $d(a,c) \geq \beta$, i.e. such that $a \in c_\beta^d([c])$. For the converse, let for all $\beta \lhd \alpha$ exist a $c \lhd b$ such that $d(a,c) \geq \beta$. Then for all $\beta \lhd \alpha$ we have $d(a,b) = \bigvee_{c \lhd b} d(a,c) \geq \beta$ and hence, by the complete distributivity, $\alpha \leq d(a,b)$ which is equivalent to $a \in c_\alpha^d([b])$.

Finally we show that this functor is injective on objects. If $d \neq d'$, without loss of generality, there are $a, b \in J(L^X)$ such that $d(a,b) \not\leq d'(a,b)$. Then $a \in c_{d(a,b)}^d([b])$ but $a \notin c_{d(a,b)}^{d'}([b])$. □

We are going to show next, that there is a coreflector for E. To this end, for $(X, \overline{c}) \in |\mathsf{LM\text{-}PreFMET\text{-}FCTS}|$ we define $d^c : J(L^X) \times J(L^X) \longrightarrow M$ by

$$d^c(a,b) = \bigvee_{a \in c_\alpha([b])} \alpha.$$

Proposition 5.5. *(1) $H : \mathsf{LM\text{-}PreFMET\text{-}FCTS} \longrightarrow \mathsf{LM\text{-}FMET}$ defined by $H((X, \overline{c})) = (X, d^c)$, $H(f) = f$ is a functor.*
(2) We have $H \circ E = id_{\mathsf{LM\text{-}FMET}}$.
(3) We have $E \circ H \leq id_{\mathsf{LM\text{-}PreFMET\text{-}FCTS}}$.

Proof. (1) We first show that for $(X, \overline{c}) \in |\mathsf{LM\text{-}PreFMET\text{-}FCTS}|$ we have $(X, d^c) \in |\mathsf{LM\text{-}FMET}|$.

(LMD1) If $a \leq b$, we have $a \in c_\top([b])$ and hence $d^c(a,b) = \top$. Hence (LMD1') is satisfied and therefore also (LMD1). (LMD2) follows from the distributivity of the quantale operation over arbitrary joins and the transitivity as

$$d^c(a,b) * d^c(b,c) = \bigvee_{a \in c_\alpha([b]), b \in c_\beta([c])} \alpha * \beta \leq \bigvee_{a \in c_{\alpha * \beta}([c])} \alpha * \beta \leq d^c(a,c).$$

For (LMD3) we first note that from $c \leq b$ we conclude, using (LMD1') and (LMD2), $d^c(a,c) = d^c(a,c) * d^c(c,b) \leq d^c(a,b)$. Hence $\bigvee_{c \lhd b} d^c(a,c) \leq d^c(a,b)$. For the converse inequality, let $\beta \lhd d^c(a,b)$. Then there is $\alpha \in M$ such that $\beta \lhd \alpha$ and $a \in c_\alpha([b])$. By the approximation property, there is $c \lhd b$ such that $a \in c_\beta([c])$ and hence $\beta \leq \bigvee_{a \in c_\gamma([c])} \gamma = d^c(a,c)$. We conclude $\beta \leq \bigvee_{c \lhd b} d^c(a,c)$ and from the complete distributivity we obtain finally $d^c(a,b) \leq \bigvee_{c \lhd b} d^c(a,c)$.

Next we show that if $f : (X, \overline{c}) \longrightarrow (X', \overline{c'})$ is a morphism in LM-PreFMET-FCTS, then $f : (X, d^c) \longrightarrow (X', d^{c'})$ is an LM-fuzzy metric morphism. We have for $a, b \in J(L^X)$: $d^c(a,b) = \bigvee_{a \in c_\alpha([b])} \alpha \leq \bigvee_{f(a) \in c'_\alpha([f(b)])} \alpha = d^{c'}(f(a), f(b))$. Hence H is a functor.

(2) We show that for $(X,d) \in |\text{LM-FMET}|$ we have $d^{(c^d)} = d$. We note again that $a \in c^d_\alpha([b])$ is equivalent to $d(a,b) \geq \alpha$. Hence $d^{(c^d)}(a,b) = \bigvee_{a \in c^d_\alpha([b])} \alpha = \bigvee_{d(a,b) \geq \alpha} \alpha = d(a,b)$.

(3) We show that for $(X, \overline{c}) \in |\text{LM-PreFMET-FCTS}|$, we have $c^{(d^c)}_\alpha(\mathbb{F}) \subseteq c_\alpha(\mathbb{F})$. Let $a \in c^{(d^c)}_\alpha(\mathbb{F})$ and let $\epsilon \lhd \alpha$. Then there is $F_\epsilon \in \mathbb{F}$ such that for all $b \in F_\epsilon$ there is $\beta \geq \epsilon$ such that $a \in c_\beta([b])$ and, consequently, also $a \in c_\epsilon([b])$. Hence we have for all $\epsilon \lhd \alpha$ a set $F_\epsilon \in \mathbb{F}$ such that $a \in \bigcap_{b \in F_\epsilon} c_\epsilon([b])$. From $F_\epsilon \in \mathbb{F}$ we conclude $[F_\epsilon] = \bigwedge_{b \in F_\epsilon} [b] \leq \mathbb{F}$ and from pretopologicalness we conclude that for all $\epsilon \lhd \alpha$ there is $F_\epsilon \in \mathbb{F}$ such that $a \in c_\epsilon([F_\epsilon]) \subseteq c_\epsilon(\mathbb{F})$. This is true for all $\epsilon \lhd \alpha$ and from the left-continuity we conclude $a \in c_\alpha(\mathbb{F})$. □

Theorem 5.6. *The category* LM-FMET *can be coreflectively embedded into the category* LM-PreFMET-FCTS.

We are now going to identify a subcategory of LM-PreFMET-FCTS that is isomorphic to LM-FMET. To this end, we introduce the following axiom for an LM-fuzzy convergence tower space. A space $(X, \overline{c}) \in |\text{LM-FCTS}|$ satisfies the axiom (FM) if for all $\mathbb{U} \in \mathfrak{U}(J(L^X))$ and all $\alpha \in M$ we have

(FM) $a \in c_\alpha(\mathbb{U}) \iff \forall U \in \mathbb{U}, \beta \lhd \alpha \exists b \in U$ s.t. $a \in c_\beta([b])$.

This axiom was first introduced for the case $L = \{0,1\}$ and $M = ([0,1], \leq, \wedge)$ in [3] for probabilistic convergence spaces in the sense of Richardson and Kent [21].

We denote the subcategory of LM-PreFMET-FCTS with objects the LM-fuzzy convergence tower spaces that satisfy the axiom (FM) by LM-FMET-FCTS. The next result shows that the embedding functor E actually has its range in LM-FMET-FCTS.

Proposition 5.7. *Let* $(X,d) \in |\text{LM-FMET}|$. *Then* $(X, \overline{c^d})$ *satisfies (FM).*

Proof. Let $\mathbb{U} \in \mathfrak{U}(J(L^X))$ and let $\alpha \in L$. Let first $a \in c^d_\alpha(\mathbb{U})$ and let $U \in \mathbb{U}$ and $\beta \lhd \alpha$. Then there is $F_\beta \in \mathbb{U}$ such that for all $b \in F_\beta$ we have $d(a,b) \geq \beta$. Choose $b \in U \cap F_\beta$. Then $\bigvee_{F \in [b]} \bigwedge_{c \in F} d(a,c) \geq \bigwedge_{c \in U \cap F_\beta} d(a,c) \geq \beta$, i.e. $a \in c^d_\beta([b])$.

Conversely, let for all $U \in \mathbb{U}$, $\beta \lhd \alpha$ there is $b = b_\beta \in U$ such that $a \in c^d_\beta([b])$, i.e. such that $\bigvee_{F \in [b]} \bigwedge_{c \in F} d(a,c) \geq \beta$. Let further $F \in \mathbb{U}^a = \bigwedge_{a \in c^d_\beta(\mathbb{F})} \mathbb{F}$. Then, for $U \in \mathbb{U}$ in particular $F \in [b]$, i.e. $b \in F \cap U$. Hence $\mathbb{U} \vee \mathbb{U}^a$ exists and because $\mathbb{U}$

is an ultrafilter, we get $\mathbb{U} \geq \bigwedge_{a \in c^d_\beta(\mathbb{F})} \mathbb{F}$. As $(X, \overline{c^d})$ is pretopological, we conclude $c^d_\beta(\mathbb{U}) \geq \bigcap_{a \in c^d_\beta(\mathbb{F})} c^d_\beta(\mathbb{F})$ and we have $a \in c^d_\beta(\mathbb{U})$. This is true for any $\beta \lhd \alpha$ and by left-continuity we obtain $a \in c^d_\alpha(\mathbb{U})$. □

PROPOSITION 5.8. *Let $(X, \overline{c}) \in |\mathsf{LM\text{-}PreFMET\text{-}FCTS}|$ satisfy the axiom (FM). Then, for all $\alpha \in M$ and all $\mathbb{F} \in \mathfrak{F}(J(L^X))$ we have $c^{(d^c)}_\alpha(\mathbb{F}) = c_\alpha(\mathbb{F})$.*

Proof. Let $\mathbb{U} \in \mathfrak{U}(J(L^X))$ be an ultrafilter and let $a \in c_\alpha(\mathbb{U})$. By the axiom (FM) we obtain, for $\beta \lhd \alpha$ that $N^a_\beta = \{b \in J(L^X) \,:\, a \in c_\beta([b])\}$ satisfies $N^a_\beta \cap U \neq \emptyset$ for all $U \in \mathbb{U}$ and hence $N^a_\beta \in \mathbb{U}$. Furthermore, for $a \in c_\beta([b])$ we have $d^c(a, b) \geq \beta$. Hence

$$\bigvee_{U \in \mathbb{U}} \bigwedge_{b \in U} d^c(a, b) \geq \bigwedge_{b \in N^a_\beta} d^c(a, b) = \bigwedge_{a \in c_\beta([b])} d^c(a, b) \geq \beta.$$

This is true for all $\beta \lhd \alpha$ and hence also $\bigwedge_{b \in N^a_\beta} d^c(a, b) = \bigwedge_{a \in c_\beta([b])} d^c(a, b) \geq \alpha$, which is equivalent to $a \in c^{(d^c)}_\alpha(\mathbb{U})$. Hence we have shown $c_\alpha(\mathbb{U}) \subseteq c^{(d^c)}_\alpha(\mathbb{U})$ for all $\mathbb{U} \in \mathfrak{U}(J(L^X))$ and because both $(X, \overline{c})$ and $(X, \overline{c^{(d^c)}})$ are pretopological, we have for $\mathbb{F} \in \mathfrak{F}(J(L^X))$ that $c_\alpha(\mathbb{F}) \subseteq c^{(d^c)}_\alpha(\mathbb{F})$. The converse implication is always true and so we have the equality. □

If we denote the restriction of the functor H on $\mathsf{LM\text{-}FMET\text{-}FCTS}$ for simplicity again by H, we have $E \circ H = id_{\mathsf{LM\text{-}FMET\text{-}FCTS}}$ and $H \circ E = id_{\mathsf{LM\text{-}FMET}}$ and we obtain the following main result.

THEOREM 5.9. *The categories* LM-FMET-FCTS *and* LM-FMET *are isomorphic.*

REFERENCES

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, Wiley, New York, 1989.

[2] P. Brock, D.C. Kent, *Approach spaces, limit tower spaces, and probabilistic convergence spaces*, Appl. Cat. Struct. **5** (1997), 99–110.

[3] P. Brock, *Probabilistic convergence spaces and generalized metric spaces*, Int. J. Math. Math. Sci. **21** (1998), 439–452.

[4] Z. Deng, *Fuzzy pseudo-metric spaces*, J. Math. Anal. Appl. **86** (1982), 74–95.

[5] M. A. Erceg, *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl. **69** (1979), 205–230.

[6] R.C. Flagg, *Quantales and continuity spaces*, Algebra Univers. **37** (1997), 257–276.

[7] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets & Systems **64** (1994), 395–399.

[8] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, D.S. Scott, *Continuous Lattices and Domains*, Cambridge University Press 2003.

[9] U. Höhle, *Commutative, residuated l-monoids*, in: Non-classical logics and their applications to fuzzy subsets (U. Höhle, E.P. Klement, eds.), Kluwer, Dordrecht 1995, 53–106.

[10] G. Jäger, *A convergence theory for probabilistic metric spaces*, Quaest. Math. **38** (2015), 587–599.

[11] I. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.

[12] H. Lai, D. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets & Systems **157** (2006), 1865–1885.

[13] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Seminario Mat. Fis. Milano, **43** (1973), 135–166. Reprinted in: Reprints in Theory and Applications of Categories **1** (2002), 1–37.

[14] R. Lowen, *Approach Spaces. The Missing Link in the Topology-Uniformity-Metric Triad*, Claredon Press, Oxford 1997.

[15] E. Lowen,R. Lowen, *A quasitopos containing CONV and MET as full subcategories*, Internat. J. Math. Math. Sci. **11** (1988), 417–438.

[16] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 535–537.

[17] N. N. Morsi, *On fuzzy pseudo-normed vector spaces*, Fuzzy Sets and Systems **27** (1988), 351–372.

[18] G. Preuss, *Theory of Topological Structures*, D. Reidel Publishing Company, Dordrecht/Boston/Lancaster/Tokyo 1988.

[19] G. Preuss, *Foundations of Topology. An Approach to Convenient Topology*, Kluwer Academic Publishers, Dordrecht 2002.

[20] G. N. Raney, *A subdirect-union representation for completely distributive complete lattices*, Proc. Amer. Math. Soc. **4** (1953), 518–512.

[21] G. D. Richardson, D. C. Kent, *Probabilistic convergence spaces*, J. Austral. Math. Soc. (Series A) **61** (1996), 400–420.

[22] K. I. Rosenthal, *Quantales and Their Applications*, Pitman Research Notes in Mathematics **234**, Longman, Burnt Mill, Harlow 1990.

[23] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North Holland, New York, 1983.

[24] Fu-Gui Shi, *Pointwise pseudo-metrics in L-fuzzy set theory*, Fuzzy Sets & Systems **121** (2001), 209–216.

[25] Fu-Gui Shi, *L-fuzzy interiors and L-fuzzy closures*, Fuzzy Sets & Systems **160** (2009), 1218–1232.

[26] Fu-Gui Shi, *(L,M)-fuzzy metric spaces*, Indian J. Math. **52** (2010), 231–250.

[27] A. P. Šostak, *Basic structures of fuzzy topology*, J. Math. Sci., **78** (1996), 662–701.

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