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MICHAĬLICHENKO GROUP OF MATRICES OVER SKEW-FIELDS

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Abstract. In this paper we generalize the Mikhaĭlichenko group for matrices over skew-fields.

1. Introduction

In [2] Bardakov and Simonov introduced the Mikhaĭlichenko group of square matrices over a field with respect to nonstandard product of matrices. This group was introduced by Mikhaĭlichenko in studying a classification problem in the theory of physical structures (see [2] and the references therein).

Since quaternions are widely used in physics, it motivates to generalize the Mikhaĭlichenko group for matrices over skew-field of quaternions. In this paper we generalize the Mikhaĭlichenko group for matrices over an arbitrary skew-field.

2. Dieudonne' determinant

For the convenience of the reader we recall here the definition and main properties of Dieudonne' determinant.[1] Let $\mathbb{K}$ be a skew-field. Denote by $\mathbb{K}^*$ its multiplicative group of nonzero elements and by $\mathbb{K}^{ab}$ the factor group $\mathbb{K}^*/[\mathbb{K}^*,\mathbb{K}^*]$. Write π for the homomorphism $\mathbb{K}^* \to \mathbb{K}^{ab}$. We extend the homomorphism π to homomorphism of monoids $\mathbb{K} \to \mathbb{K}^{ab} \cup \{0\}$ by setting $\pi(0) = 0$. Denote by $M_{m,n}(\mathbb{K})$ the set of all $m \times n$-matrices with elements in $\mathbb{K}$. To shorten notation we write $M_n(\mathbb{K})$ instead of $M_{n,n}(\mathbb{K})$.

Definition 2.1. Dieudonne' determinant is the map $\det_D \colon M_n(\mathbb{K}) \to \mathbb{K}^{ab} \cup \{0\}$ satisfying the following properties:
(DD1) $\det_D(AB) = \det_D(A) \cdot \det_D(B)$ for all $A, B \in M_n(\mathbb{K})$,

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[1]For more details about Dieudonne' determinant we refer reader to [1, 3].

(DD2) if $A' \in M_n(\mathbb{K})$ is obtained from $A \in M_n(\mathbb{K})$ by multiplying one row by $\lambda \in \mathbb{K}$ from the left, then $\det_D(A') = \pi(\lambda)\det_D(A)$,

(DD3) if $A' \in M_n(\mathbb{K})$ is obtained from $A \in M_n(\mathbb{K})$ by replacing a row r_i by sum of two different rows $r_i + r_j$, $i \neq j$, then $\det_D(EA) = \det_D(A)$;

For a matrix A we denote its ith row by A_i and ith column by A^i. Next we recall main properties of Dieudonne' determinant.

Theorem 2.2. *Dieudonne' determinant has the following properties:*
(DD4) *matrix A is invertible iff $\det_D(A) \neq 0$;*

(DD5) *if matrix A' is obtained from matrix A by replacing a row A_i by $A_i + \lambda A_j$ for some $\lambda \in \mathbb{K}$ and $j \neq i$, then $\det_D(A') = \det_D(A)$;*

(DD6) *if matrix A' is obtained from matrix A by replacing a column A^i by $A^i + A^j\lambda$ for some $\lambda \in \mathbb{K}$ and $j \neq i$, then $\det_D(A') = \det_D(A)$;*

(DD7) *if we exchange two rows in matrix A, then $\det_D(A)$ is multiplied by $\pi(-1)$;*

(DD8) *if we exchange two columns in matrix A, then $\det_D(A)$ is multiplied by $\pi(-1)$;*

(DD9)
$$\det_D \begin{pmatrix} I_{n-1} & 0 \\ 0 & \lambda \end{pmatrix} = \pi(\lambda);$$

(DD10) *if $A \in M_n(\mathbb{K})$ and $B \in M_m(\mathbb{K})$ then* $\det_D \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det_D(A) \cdot \det_D(B)$.

See [1] for the proof.

3. Mikhaĭlichenko group over a skew-field

Let P be a field. In [2], Bardakov and Simonov studied nonstandard matrix operation
$$X \circledast Y = XVY + XU + U^tY, \qquad X, Y \in M_n(P), \tag{1}$$
where
$$U = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \\ 1 & \dots & 1 \end{pmatrix} \in M_n(P) \quad \text{and} \quad V = (I_n - U)(I_n - U^t).$$
They proved that the set $G_n(P) = \{Y \in M_n(P) \mid \det(VY + U) \neq 0\}$ is a group under $\circledast$ for each $n \geqslant 2$. They also proved that the group $G_n(\mathbb{R})$ is isomorphic to the Mikhaĭlichenko group (see [2]).

Let $\mathbb{K}$ be a skew-field. Write $G_n(\mathbb{K}) = \{Y \in M_n(\mathbb{K}) \mid \det_D(VY + U) \neq 0\}$. To show that the set $G_n(\mathbb{K})$ is a group with respect to the binary operation $\circledast$ we need the following lemma.

LEMMA 3.1. *For each $Y \in M_n(\mathbb{K})$ we have*

$$\det_D(VY+U) = \det_D(YV+U^t).$$

Proof. Direct calculations show that

$$YV+U^t = \begin{pmatrix} y_{11}-y_{1n} & y_{12}-y_{1n} & \dots & y_{1,n-1}-y_{1n} & 1+ny_{1n}-\sum_{i=1}^n y_{1i} \\ y_{21}-y_{2n} & y_{22}-y_{2n} & \dots & y_{2,n-1}-y_{2n} & 1+ny_{2n}-\sum_{i=1}^n y_{2i} \\ \vdots & \vdots & & \vdots & \vdots \\ y_{n1}-y_{nn} & y_{n2}-y_{nn} & \dots & y_{n,n-1}-y_{nn} & 1+ny_{nn}-\sum_{i=1}^n y_{ni} \end{pmatrix}.$$

By **(DD10)** we have

$$\det_D(YV+U^t) = \det_D \begin{pmatrix} 1 & 0 \\ 0 & YV+U^t \end{pmatrix}$$

i.e. we add one row and one column to the matrix $YV+U^t$. Next we add second, third, …, nth column to the last column. By **(DD6)**, the value of determinant remains unchanged. Therefore

$$\det_D(YV+U^t) = \det_D \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & y_{11}-y_{1n} & y_{12}-y_{1n} & \dots & y_{1,n-1}-y_{1n} & 1 \\ 0 & y_{21}-y_{2n} & y_{22}-y_{2n} & \dots & y_{2,n-1}-y_{2n} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & y_{n1}-y_{nn} & y_{n2}-y_{nn} & \dots & y_{n,n-1}-y_{nn} & 1 \end{pmatrix}.$$

In the matrix we replace i-th row by the sum i-th row plus first row multiplied by $y_{i-1,n}$ from the left ($i = 2, \dots, n+1$). By **(DD6)**, we have

$$\det_D(YV+U^t) = \det_D \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ y_{1n} & y_{11}-y_{1n} & y_{12}-y_{1n} & \dots & y_{1,n-1}-y_{1n} & 1 \\ y_{2n} & y_{21}-y_{2n} & y_{22}-y_{2n} & \dots & y_{2,n-1}-y_{2n} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ y_{nn} & y_{n1}-y_{nn} & y_{n2}-y_{nn} & \dots & y_{n,n-1}-y_{nn} & 1 \end{pmatrix}.$$

Adding first column to columns $2, 3, \dots, n$ we get

$$\det_D(YV+U^t) = \det_D \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 0 \\ y_{1n} & y_{11} & y_{12} & \dots & y_{1,n-1} & 1 \\ y_{2n} & y_{21} & y_{22} & \dots & y_{2,n-1} & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ y_{nn} & y_{n1} & y_{n2} & \dots & y_{n,n-1} & 1 \end{pmatrix}$$

by **(DD6)**. Finally we exchange first and last column. Then, by **(DD8)**, we have

$$\det_D(YV+U^t) = \pi(-1)\det_D \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & y_{11} & y_{12} & \dots & y_{1n} \\ 1 & y_{21} & y_{22} & \dots & y_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & y_{n1} & y_{n2} & \dots & y_{nn} \end{pmatrix}.$$

By applying same technique to $\det_D(VY+U)$ (but that the operations for row and columns are interchanged), it can be shown that

$$\det_D(VY+U)=\det_D\begin{pmatrix}1 & 0\\ 0 & VY+U\end{pmatrix}=\pi(-1)\det_D\begin{pmatrix}0 & 1 & 1 & \dots & 1\\ 1 & y_{11} & y_{12} & \dots & y_{1n}\\ 1 & y_{21} & y_{22} & \dots & y_{2n}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & y_{n1} & y_{n2} & \dots & y_{nn}\end{pmatrix}. \qquad \square$$

THEOREM 3.2. *The set $G_n(\mathbb{K})$ is a group under $\circledast$.*

Proof. We divide the proof into four steps.
1. We start the proof by showing that $G_n(\mathbb{K})$ is closed under $\circledast$.

$$\begin{aligned}V(X\circledast Y)+U =& V(XVY+XU+U^tY)+U = VXVY+VXU+\underbrace{VU^t}_{0}Y+U\\ =& VXVY+VXU+\underbrace{UV}_{0}Y+U^2=(VX+U)(VY+U)\end{aligned}$$

If $X,Y\in G_n(\mathbb{K})$ then by the multiplicativity of Dieudonne' determinant we have $\det_D(V(X\circledast Y)+U)=\det_D((VX+U)(VY+U))=\det_D(VX+U)\cdot\det_D(VY+U)\neq 0$, i.e. $X\circledast Y\in G_n(\mathbb{K})$.

2. Identity element. Next we demonstrate that the matrix

$$E=\begin{pmatrix}I_{n-1} & 0\\ 0 & 0\end{pmatrix}$$

is the identity element with respect to $\circledast$. Direct computation yields

$$VE+U=\begin{pmatrix} & & -1\\ & I_{n-1} & \vdots\\ & & -1\\ -1 & \dots\ -1 & n-1\end{pmatrix}\begin{pmatrix}I_{n-1} & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0 & \dots & 0\\ \vdots & \dots & \vdots\\ 0 & \dots & 0\\ 1 & \dots & 1\end{pmatrix}=I_n.$$

Hence $\det_D(VE+U)=\pi(1)\neq 0$ i.e. $E\in G_n(\mathbb{K})$.

Since V, E and I_n are symmetric, we also have $I_n=EV+U^t$. From this and from $EU=0=U^tE$ we get

$$\begin{aligned}E\circledast X&=EVX+EU+U^tX=(EV+U^t)X=I_nX=X,\\ X\circledast E&=XVE+XU+U^tE=X(VE+U)=XI_n=X,\end{aligned}$$

as desired.

3. Associativity. Expanding $(X\circledast Y)\circledast Z$ and $X\circledast(Y\circledast Z)$ and using the equalities $UV=0=VU^t$ we obtain the associativity identity.

4. Inverse elements. Let $X\in G_n(\mathbb{K})$ and we are looking for an element X_r such that $X\circledast X_r=XVX_r+XU+U^tX_r=E$, i.e. $(XV+U^t)X_r=E-XU$. If we can show that $XV+U^t$ is an invertible matrix then we have $X_r=(XV+U^t)^{-1}(E-XU)$. We use Dieudonne' determinant to show that $XV+U^t$ is regular i.e. $\det_D(XV+U^t)\neq 0$.

Since $X \in G_n(\mathbb{K})$, then by Lemma 3.1 we have $\det_D(XV+U^t) = \det_D(VX+U) \neq 0$, i.e. $XV+U^t$ is invertible. Analogously, we are looking for the matrix X_l such that $X_l \circledast X = X_l VX + X_l U + U^t X = E$. From this we get $X_l(VX+U) = E - U^t X$. Since $\det_D(VX+U) \neq 0$ we have that $VX+U$ is invertible and $X_l = (VX+U)^{-1}(E-U^t X)$. To prove the equality $X_l = X_r$ we compute

$$X_l = X_l \circledast E = X_l \circledast (X \circledast X_r) = (X_l \circledast X) \circledast X_r = E \circledast X_r = X_r. \qquad \square$$

4. Embedding the group $G_n(\mathbb{K})$ into $\mathrm{GL}_{n+1}(\mathbb{K})$

Obviously, the mapping $\varphi(X) = VX + U$ is a homomorphism of the group $G_n(\mathbb{K})$ into the general linear group $\mathrm{GL}_n(\mathbb{K})$.

Bardakov and Simonov showed that, if P is a field, then the Mikhaĭlichenko group $G_n(P)$ can be embedded into the general linear group $\mathrm{GL}_{n+1}(P)$ (see [2]). We show next that if $\mathbb{K}$ is a skew-field, then $G_n(\mathbb{K})$ is isomorphic to a subgroup of $\mathrm{GL}_{n+1}(\mathbb{K})$.

As in [2], we consider the mapping $\phi\colon G_n(\mathbb{K}) \to \mathrm{GL}_{n+1}(\mathbb{K})$

$$\phi(X) = \left(\begin{array}{ccc|c} & & & 0 \\ & VX+U & & \vdots \\ & & & 0 \\ \hline x_{n1} & \cdots & x_{nn} & 1 \end{array}\right). \tag{2}$$

Theorem 4.1. *The group $G_n(\mathbb{K})$ can be embedded into $\mathrm{GL}_{n+1}(\mathbb{K})$ for any $n \geqslant 2$.*

Proof. We divide the proof into two lemmas.

Lemma 4.2. *The mapping ϕ is a homomorphism of groups.*

Proof (of Lemma 4.2). Suppose $X, Y \in G_n(\mathbb{K})$. Then, by Theorem 3.2, $(VX+U)(VY+U) = V(X \circledast Y) + U$. From immediate computations we have

$$\left(\begin{array}{ccc|c} & & & 0 \\ & VX+U & & \vdots \\ & & & 0 \\ \hline x_{n1} & \cdots & x_{nn} & 1 \end{array}\right)\left(\begin{array}{ccc|c} & & & 0 \\ & VY+U & & \vdots \\ & & & 0 \\ \hline y_{n1} & \cdots & y_{nn} & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} & & & 0 \\ & (VX+U)(VY+U) & & \vdots \\ & & & 0 \\ \hline z_{n1} & \cdots & z_{nn} & 1 \end{array}\right) = \left(\begin{array}{ccc|c} & & & 0 \\ & V(X \circledast Y)+U & & \vdots \\ & & & 0 \\ \hline z_{n1} & \cdots & z_{nn} & 1 \end{array}\right),$$

where $(z_{n1}, \ldots, z_{nn}) = (x_{n1}, \ldots, x_{nn})(VY+U) + (y_{n1}, \ldots, y_{nn})$. Since

$$VY + U = \begin{pmatrix} & & & -1 \\ & I_{n-1} & & \vdots \\ & & & -1 \\ -1 & \ldots & -1 & n-1 \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \ddots & & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix} + \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ldots & \vdots \\ 0 & \ldots & 0 \\ 1 & \ldots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} y_{11} - y_{n1} & \dots & y_{1n} - y_{nn} \\ y_{21} - y_{n1} & \dots & y_{2n} - y_{nn} \\ \vdots & \ddots & \vdots \\ y_{n-1,1} - y_{n1} & \dots & y_{n-1,n} - y_{nn} \\ 1 + (n-1)y_{n1} - \sum_{k=1}^{n-1} y_{k1} & \dots & 1 + (n-1)y_{nn} - \sum_{k=1}^{n-1} y_{kn} \end{pmatrix} \qquad (3)$$

we obtain

$$\begin{aligned} z_{nj} &= \sum_{k=1}^{n-1} x_{nk}(y_{kj} - y_{nj}) + x_{nn}\left(1 + (n-1)y_{nj} - \sum_{k=1}^{n-1} y_{kj}\right) + y_{nj} \\ &= \sum_{k=1}^{n-1} x_{nk}(y_{kj} - y_{nj}) + \sum_{k=1}^{n-1} x_{nn}(y_{nj} - y_{kj}) + x_{nn} + y_{nj} \\ &= \sum_{k=1}^{n-1} (x_{nk} - x_{nn})(y_{kj} - y_{nj}) + x_{nn} + y_{nj} \end{aligned}$$

for each $1 \leqslant j \leqslant n$. Put $W = X \circledast Y$. Then we have (see also [2])

$$w_{ij} = \sum_{k=1}^{n-1} (x_{ik} - x_{in})(y_{kj} - y_{nj}) + x_{in} + y_{nj}.$$

From this we get $w_{nj} = z_{nj}$ for each j. Hence $\phi(X)\phi(Y) = \phi(X \circledast Y)$ and the lemma is proved. □

Lemma 4.3. *The homomorphism ϕ is injective.*

Proof (of Lemma 4.3). Suppose $\phi(X) = \phi(Y)$ for some $X, Y \in G_n(P)$. Then $x_{nj} = y_{nj}$ for each j by (2). Moreover, from $x_{ij} - x_{nj} = y_{ij} - y_{nj}$ we get $x_{ij} = y_{ij}$ for any $1 \leqslant i \leqslant n-1$ and for any $1 \leqslant j \leqslant n$. Hence $X = Y$ as desired. □

By previous, the theorem is proved. □

Denote by $H_{n+1}(\mathbb{K})$ the subgroup of $\mathrm{GL}_{n+1}(\mathbb{K})$ consisting of all matrices of the form

$$\left(\begin{array}{ccc|c} & & & 0 \\ & Y & & \vdots \\ & & & 0 \\ \hline a_1 & \cdots & a_n & 1 \end{array}\right),$$

where $Y \in \mathrm{GL}_n(\mathbb{K})$ is of the form

$$\begin{pmatrix} y_{11} & y_{12} & \dots & y_{1n} \\ y_{21} & y_{22} & \dots & y_{2n} \\ \vdots & \vdots & \dots & \vdots \\ y_{n-1,1} & y_{n-1,2} & \dots & y_{n-1,n} \\ 1 - \sum_{i=1}^{n-1} y_{i1} & 1 - \sum_{i=1}^{n-1} y_{i2} & \dots & 1 - \sum_{i=1}^{n-1} y_{in} \end{pmatrix}.$$

Obviously $\mathrm{Im}\phi \subseteq H_{n+1}(\mathbb{K})$, by (3). Suppose now

$$\left(\begin{array}{ccc|c} & & & 0 \\ & Y & & \vdots \\ & & & 0 \\ \hline a_1 & \cdots & a_n & 1 \end{array}\right) \in H_{n+1}.$$

It is easy to check that the equation

$$\left(\begin{array}{ccc|c} & & & 0 \\ & VX+U & & \vdots \\ & & & 0 \\ \hline x_{n1} & \cdots & x_{nn} & 1 \end{array}\right) = \left(\begin{array}{ccc|c} & & & 0 \\ & Y & & \vdots \\ & & & 0 \\ \hline a_1 & \cdots & a_n & 1 \end{array}\right)$$

has a unique solution $X \in M_n(\mathbb{K})$. Thus we have the following.

Corollary 4.4. *The group $G_n(\mathbb{K})$ is isomorphic to the subgroup $H_{n+1}(\mathbb{K})$ of $\mathrm{GL}_{n+1}(\mathbb{K})$.*

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