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INTUITIONISTIC UNPROVABILITY

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Abstract. In 1952, S.C. Kleene introduced a Gentzen-type system $G3$ which is designed to be suitable for showing that the given sequents (and consequently the corresponding formulae) are unprovable in the intuitionistic logic. We show that some classes of predicate formulae are unprovable in the intuitionistic predicate calculus, using the system $G3$ and some properties of sequents that remain invariant throughout derivations in this system. The unprovability of certain formulae obtained by Kleene follows from our results as a corollary.

1. Introduction

A. Heyting [4] proved that the formula

$$\neg\neg\forall x(A(x) \vee \neg A(x)) \tag{1}$$

is unprovable in intuitionistic logic. S. C. Kleene [5] and D. Nelson [10], showed unprovability of the formula (1) using the notion of "recursive realisability". Later, Kleene [6] announced a different and completely elementary method of establishing the unprovability of certain formulae within the intuitionistic predicate calculus. Among them is the formula (1), but also some others, e.g.

$$\forall x(A \vee B(x)) \to (A \vee \forall x B(x)). \tag{2}$$

Essentially, the main tool used in this method is the cut-elimination theorem (or Gentzen's *Hauptsatz*, see [2]). Kleene [6] explains:

> In attempting to find a proof in Gentzen's normal form (slightly modified for convenience) for a formula of the predicate calculus, one may actually find a proof, or one may be able to demonstrate some feature of the situation which shows that there cannot be any.

Kleene [7], elaborated this method in detail by introducing a Gentzen-type system $G3$, in which the structural inference figures are not counted as separate inferences. He gave a motivation behind this system:

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> The system $G3$ is designed to minimize the number of choices of premise(s) for a given conclusion, when we are attempting to exhaust the possibilities for proving a given endsequent, especially in showing the endsequent to be unprovable.

Another similar applications of Gentzen's Hauptsatz can be found in the book [1] by H. B. Curry. A. Mostowski [9] demonstrated unprovability of (2) (and many other formulae), by an interpretation of the intuitionistic predicate calculus in terms of "complete Brouwerian" lattices.

In this paper, we generalize the method that Kleene used to show unprovability of the formulae (1) and (2). As a result, we obtain some wider classes of formulae which are unprovable in intuitionistic logic. Then, the unprovability of the formulae (1) and (2) follows from our results as a corollary.

2. Notation and background

Recall that a sequent is a formal expression of the form $A_1, \ldots, A_l \vdash B_1, \ldots, B_m$ where $l, m \geq 0$ and $A_1, \ldots, A_l, B_1, \ldots, B_m$ are formulas. The part $A_1, \ldots, A_l$ is the antecedent, and $B_1, \ldots, B_m$ the succedent of the sequent $A_1, \ldots, A_l \vdash B_1, \ldots, B_m$. In this paper, sequents are restricted to succedents with at most one formula occurrence, what is characterisation of the intuitionistic logic (see [2]).

In order to talk about the system $G3$, let us say a few words about the systems $G1$ and $G2$. The system $G1$ was introduced (under the name LJ) by Gentzen [2]. The system $G2$ is derived from $G1$ by replacing the structural rule "cut" with the rule "mix". The system $G3$, which is suitable for showing the unprovability of some sequents was introduced by Kleene and it was obtained from $G1$ by rejecting the structural rules and introducing some changes in the logical rules. Kleene [7] showed equivalence between the systems $G1$ and $G3$.

Postulates for the intuitionistic formal system $G3$:

Axiom schema: $A, \Gamma \vdash A$

Logical rules:

$$\frac{A, A \wedge B, \Gamma \vdash \Theta}{A \wedge B, \Gamma \vdash \Theta}\wedge\vdash \qquad \frac{B, A \wedge B, \Gamma \vdash \Theta}{A \wedge B, \Gamma \vdash \Theta}\wedge\vdash \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\vdash\wedge$$

$$\frac{A, A \vee B, \Gamma \vdash \Theta \qquad B, A \vee B, \Gamma \vdash \Theta}{A \vee B, \Gamma \vdash \Theta}\vee\vdash \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vdash\vee \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vdash\vee$$

$$\frac{A \to B, \Gamma \vdash A \qquad B, A \to B, \Gamma \vdash \Theta}{A \to B, \Gamma \vdash \Theta}\to\vdash \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B}\vdash\to$$

$$\frac{\neg A, \Gamma \vdash A}{\neg A, \Gamma \vdash \Theta}\neg\vdash \qquad \frac{A, \Gamma \vdash}{\Gamma \vdash \neg A}\vdash\neg$$

$$\frac{F(t), \forall x F(x), \Gamma \vdash \Theta}{\forall x F(x), \Gamma \vdash \Theta} \forall\vdash \qquad \frac{\Gamma \vdash F(a)}{\Gamma \vdash \forall x F(x)} \vdash\forall$$

$$\frac{F(a), \exists x F(x), \Gamma \vdash \Theta}{\exists x F(x), \Gamma \vdash \Theta} \exists\vdash \qquad \frac{\Gamma \vdash F(t)}{\Gamma \vdash \exists x F(x)} \vdash\exists$$

The rules $\vdash \forall$ and $\exists \vdash$ have restriction on variables: The variable a of the postulate shall not occur free in its conclusion.

Logical rules we also call "rules of inference","inference figures" or just "rules". Sequents above the line are the premises, while the sequent bellow the line is the conclusion. If a rule has two premises, we call it a two-premise rule. The logical rules constitute introductions of a logical symbol, but sometimes in the succedent (right column), and sometimes in the antecedent (left column). The formula in which the logical symbol is introduced is called the principal formula.

A proof is a finite sequence of one or more sequents such that each sequent of the sequence is either an axiom or an immediate consequence of preceding sequents of the sequence. A proof is said to be a proof of its last sequent, and this sequent is said to be provable. We will use a proof in a tree form: the premises for each inference are written immediately over the conclusion, and no sequent serves as a premise for more than one inference. A root sequent is a sequent that is not a premise of any inference in the proof.

Definition 2.1. Two sequents, $\Gamma \vdash \Theta$ and $\Gamma' \vdash \Theta'$ are *cognate* if exactly the same formulae occur in Γ (in Θ) as in Γ' (in Θ').

For the system $G3$ any application of a logical rule will remain an application of the same logical rule when any sequent is replaced by a cognate sequent.

Definition 2.2. A proof in $G3$ is *irredundant*, if it contains no pair of cognate sequents in the same branch.

From now on, we consider only irredundant proofs. Hence, if we search for a proof of a sequent, we are searching for an irredundant proof.

Recall that the formulae that may have arisen in the course of the construction of a formula, including that formula itself, are called subformulae. For example, the subformulae of $A \vee \forall x B(x)$ are $A, \forall x B(x), A \vee \forall x B(x)$ as well as all formulae of the form $B(y)$, where y represents any free variable. When A is a subformula of B, we say also that B *contains* A. Now, we define a subformula of the antecedent.

Definition 2.3. Let $\Gamma \vdash D$ be a sequent. We say that A is a subformula of the antecedent Γ if A is a subformula of some formula in Γ.

3. Two results about unprovability

Let F be a propositional formula unprovable in intuitionistic logic, and let $F(x)$ be the formula obtained from F by replacing each propositional letter by an atomic formula with one and the same free variable x. Let $F(a)$ be the result of the replacement of x by a in $F(x)$, and for an arbitraty sequence $\Gamma(x)$ of subformulae of $F(x)$, let $\Gamma(a)$ be the result of the replacement of x by a in every formula of $\Gamma(x)$. For $k \geq 0$, let us call a sequent of the form $\neg\forall x F(x), \Gamma_1(a_1), \ldots, \Gamma_k(a_k) \vdash \forall x F(x)$ an *expanding* sequent, where $a_i \neq a_j$ for $i \neq j$ and $i, j \in \{1 \ldots k\}$.

Lemma 3.1. *Every $G3$ proof of an expanding sequent contains an expanding sequent properly above the root sequent.*

Proof. Suppose that the sequent $\neg\forall x F(x), \Gamma_1(a_1), \ldots, \Gamma_k(a_k) \vdash \forall x F(x)$ has a proof. We want to show that at least one branch in that proof has an expanding sequent properly above the root sequent. Consider the last rule in the proof. If the principal formula of this rule is in $\Gamma_i(a_i)$, for some $1 \leq i \leq k$, then the premises of this rule will be expanding sequents. Note that the formula $\neg\forall x F(x)$ cannot be a principal formula of the last rule, because we will get an irredundant proof – the premise will be the same as the conclusion.

If the last rule is $\vdash \forall$, then the first step of the proof will be:

$$\begin{array}{ll} 2 & \\ 1 & \end{array} \quad \dfrac{\neg\forall x F(x), \Gamma_1(a_1), \ldots, \Gamma_k(a_k) \vdash F(a_{k+1})}{\neg\forall x F(x), \Gamma_1(a_1), \ldots, \Gamma_k(a_k) \vdash \forall x F(x)} \vdash \forall$$

where $a_{k+1} \neq a_i$, for all $1 \leq i \leq k$. The formula $F(a_{k+1})$ is not an atomic formula – otherwise, for all $1 \leq i \leq k$, sequence Γ_i is empty and the root sequent is unprovable.

In order to define a branch with the desired property, we have to choose a premise every time when we pass through a two-premise rule. Let us assume that $\Gamma, \Delta(a) \vdash \Theta(a)$ is the last chosen sequent in our branch, where $\Delta(a) \vdash \Theta(a)$ is unprovable intuitionisticaly, $\Theta(a)$ is either empty or it is a subformula of $F(a)$, $\Delta(a)$ is either empty or it is a sequence of subformulae of $F(a)$ and Γ contains $\neg\forall x F(x)$ and subformulae of $F(a_i)$, $1 \leq i \leq k$, and $a_i \neq a$.

1) If the last rule by which we obtain $\Gamma, \Delta(a) \vdash \Theta(a)$ is $\vdash \wedge$, then $\Theta(a)$ is of the form $F_1(a) \wedge F_2(a)$. We know that either $\Delta(a) \vdash F_1(a)$ or $\Delta(a) \vdash F_2(a)$ is unprovable, otherwise the sequent $\Delta(a) \vdash \Theta(a)$ would be provable. Therefore, in the inference

$$\dfrac{\Gamma, \Delta(a) \vdash F_1(a) \qquad \Gamma, \Delta(a) \vdash F_2(a)}{\Gamma, \Delta(a) \vdash F_1(a) \wedge F_2(a)} \vdash \wedge$$

we choose the left premise if $\Delta(a) \vdash F_1(a)$ is unprovable, otherwise we choose the right premise.

2) If the last rule is $\rightarrow\vdash$, then either Γ contains a formula of the form $F_1(b) \rightarrow F_2(b)$, where $b = a_i$, for some i, $1 \leq i \leq k$, or $\Delta(a)$ contains $F_1(a) \rightarrow F_2(a)$. In the inference:

$$\dfrac{F_1(b) \rightarrow F_2(b), \Gamma', \Delta(a) \vdash F_1(b) \qquad F_2(b), F_1(b) \rightarrow F_2(b), \Gamma', \Delta(a) \vdash \Theta(a)}{F_1(b) \rightarrow F_2(b), \Gamma', \Delta(a) \vdash \Theta(a)} \rightarrow\vdash$$

where Γ' is obtained from Γ by excluding $F_1(b) \rightarrow F_2(b)$, we choose the right premise, while in the inference:

$$\frac{F_1(a) \to F_2(a), \Gamma, \Delta'(a) \vdash F_1(a) \qquad F_2(a), F_1(a) \to F_2(a), \Gamma, \Delta'(a) \vdash \Theta(a)}{F_1(a) \to F_2(a), \Gamma, \Delta'(a) \vdash \Theta(a)} \to\vdash$$

where $\Delta'(a)$ is obtained from $\Delta(a)$ by excluding $F_1(a) \to F_2(a)$, we choose the left premise if $F_1(a) \to F_2(a), \Delta'(a) \vdash F_1(a)$ is unprovable, otherwise we choose the right premise.

3) If the last rule is $\vee\vdash$, then either Γ contains a formula of the form $F_1(b) \vee F_2(b)$, where $b \neq a$, or $\Delta(a)$ contains $F_1(a) \vee F_2(a)$. In the inference:

$$\frac{F_1(b), \Gamma', \Delta(a) \vdash \Theta(a) \qquad F_2(b), \Gamma', \Delta(a) \vdash \Theta(a)}{F_1(b) \vee F_2(b), \Gamma', \Delta(a) \vdash \Theta(a)} \vee\vdash$$

where Γ' is obtained from Γ by excluding $F_1(b) \vee F_2(b)$, we can choose any of the premises, while in the inference:

$$\frac{F_1(a), F_1(a) \vee F_2(a), \Gamma, \Delta'(a) \vdash \Theta(a) \qquad F_2(a), F_1(a) \vee F_2(a), \Gamma, \Delta'(a) \vdash \Theta(a)}{F_1(a) \vee F_2(a), \Gamma, \Delta'(a) \vdash \Theta(a)} \vee\vdash$$

where $\Delta'(a)$ is obtained from $\Delta(a)$ by excluding $F_1(a) \vee F_2(a)$, we choose the left premise if $F_1(a), F_1(a) \vee F_2(a), \Delta'(a) \vdash \Theta(a)$ is unprovable, otherwise we choose the right premise.

Note that every sequent in our branch is not an axiom. In the case of an one-premise rule, whose conclusion is $\Gamma, \Delta(a) \vdash \Theta(a)$ it is easy to show that the premise is not an axiom. For example, if this rule is $\vdash\to$, then $\Theta(a)$ is of the form $F_1(a) \to F_2(a)$. Then the premise is of the form $F_1(a), \Gamma, \Delta(a) \vdash F_2(a)$, where $F_1(a), \Delta(a) \vdash F_2(a)$ is unprovable. The premise is not an axiom, because Γ does not contain $F_2(a)$. In the case of a two-premise rule, our choice of a premise guarantees that it is not an axiom.

Consider our branch after application of $r-2$ rules of the form $\vdash\neg$, $\vdash\vee$, $\vdash\wedge$, $\vdash\to$, $\neg\vdash$, $\vee\vdash$, $\wedge\vdash$, $\to\vdash$, where $\neg\forall x F(x)$ is not a principal formula.

$$\begin{array}{ll}
r & \neg\forall x F(x), \Gamma'_1(a_1), \ldots, \Gamma'_k(a_k), \Gamma(a_{k+1}) \vdash G(a_{k+1}) \\ \hline
\vdots & \Pi \\ \hline
2 & \neg\forall x F(x), \Gamma_1(a_1), \ldots, \Gamma_k(a_k) \vdash F(a_{k+1}) \\ \hline
1 & \neg\forall x F(x), \Gamma_1(a_1), \ldots, \Gamma_k(a_k) \vdash \forall x F(x). \quad \neg\forall
\end{array}$$

The sequence $\Gamma(a_{k+1})$ is the sequence of subformulae of $F(a_{k+1})$ and $G(a_{k+1})$ is subformula of $F(a_{k+1})$.

Since the sequent at the r-th line is not an axiom at some point the rule $\neg\vdash$ tied to $\neg\forall x F(x)$ must be used. Suppose that the rth rule is $\neg\vdash$ whose principal formula is $\neg\forall x F(x)$:

$$
\begin{array}{rcl}
r+1 & \neg\forall x F(x), \Gamma_1'(a_1), \ldots, \Gamma_k'(a_k), \Gamma(a_{k+1}) \vdash \forall x F(x) & \\ \cline{2-2}
r & \neg\forall x F(x), \Gamma_1'(a_1), \ldots, \Gamma_k'(a_k), \Gamma(a_{k+1}) \vdash G(a_{k+1}) & \neg\vdash \\ \cline{2-2}
\vdots & \Pi & \\ \cline{2-2}
2 & \neg\forall x F(x), \Gamma_1(a_1), \ldots, \Gamma_k(a_k) \vdash F(a_{k+1}) & \\ \cline{2-2}
1 & \neg\forall x F(x), \Gamma_1(a_1), \ldots, \Gamma_k(a_k) \vdash \forall x F(x). & \vdash\forall
\end{array}
$$

The sequent in the line $r+1$ is an expanding sequent. □

Corollary 3.2. *Every expanding sequent is unprovable intuitionisticaly.*

Proof. From Lemma 3.1 it follows that every proof of such a sequent contains an infinite branch. □

Corollary 3.3. *The formula* $\neg\neg\forall x F(x)$ *is unprovable intuitionisticaly.*

Proof. We attempt to construct an irredundant proof of $\vdash \neg\neg\forall x F(x)$ in the intuitionistic system G3 as follows:

$$
\begin{array}{rcl}
3 & \neg\forall x F(x) \vdash \forall x F(x) & \\ \cline{2-2}
2 & \neg\forall x F(x) \vdash & \neg\vdash \\ \cline{2-2}
1 & \vdash \neg\neg\forall x F(x). & \vdash\neg
\end{array}
$$

Note that the sequent in line 3 is an expanding sequent. Hence, by Corollary 3.2, the formula $\neg\neg\forall x F(x)$ is unprovable. □

Remark 3.4. Since $p \vee \neg p$ is unprovable in intuitionistic propositional calculus, for propositional letter p, we obtain directly from Corollary 3.3 that the formula (1) is unprovable intuitionisticaly. Similarly, using Corollary 3.3, we can find many other formulae, which are provable in the classical sense, but are unprovable intuitionisticaly, e.g.

$$
\neg\neg\forall x(\neg\neg A(x) \rightarrow A(x)),
$$
$$
\neg\neg\forall x((A(x) \rightarrow B(x)) \vee (B(x) \rightarrow A(x))),
$$

where $A(x)$ and $B(x)$ are atomic formulae. All formulae that are unprovable intuitionisticaly by Corrolary 3.3 are formulae of the monadic fragment of first-order intuitionistic logic, but note that this fragment is undecidable (see [8]).

The rest of this section is devoted to the second unprovability result. We start with some auxiliary notions.

Definition 3.5. We say that a sequent $\Gamma \vdash D$ is *disjunctively balanced* if the following holds:

(D1) there is the same number $k > 0$ of appearances of the connective $\vee$ in Γ and D, and all disjuncts in Γ are atomic formulae;

(D2) if A is a subformula of both Γ and D, then A is a disjunct of a subformula of Γ;

(D3) the multiset of all disjuncts in Γ can be partitioned into two disjoint multisets $\mathcal{A}$ and $\mathcal{B}$ each of which contains exactly one element from each disjunction;

(D4) the multiset of all disjuncts in D can be partitioned into two multisets $\mathcal{C}$ and $\mathcal{D}$ each of which contains exactly one element from each disjunction. Moreover, $\mathcal{A} = \mathcal{C}$ and $\mathcal{B} \cap \mathcal{D} = \emptyset$.

Remark 3.6. Without loss of generality, for a disjunctively balanced sequent $\Gamma \vdash D$, we may assume that $A_1 \vee B_1, \dots, A_n \vee B_n$ are the disjunctions of its antecedent, while $A_1 \vee C_1, \dots, A_n \vee C_n$ are the disjunctions in its succedent. Here, $A_i \neq B_j \neq C_k$ for all $i, j, k \in \{1, \dots, n\}$. In the sequel, we assume that a disjunctively balanced sequent $\Gamma \vdash D$ is of that form.

Definition 3.7. Let $\Gamma \vdash D$ be disjunctively balanced sequent. Let Δ be a sequence of subformulae of Γ, and E be some subformula of D. We say that the sequent $\Delta \vdash E$ has *property* λ when the following holds:
($\boldsymbol{\lambda}$1) if E contains A_i for some $i \in \{1, \dots, n\}$, then A_i does not occur as a formula in Δ;

($\boldsymbol{\lambda}$2) if $E = B_j$ for some $j \in \{1, \dots, n\}$, then B_j does not occur as a formula in Δ.

Definition 3.8. Let $\Gamma \vdash D$ be a disjunctively balanced sequent. Suppose that the following holds:
(S1) for every conjunction that is a subformula of Γ, we have that both its conjuncts are different from A_i, B_i, for all $i \in \{1, \dots, n\}$;

(S2) for every conjunction that is a subformula of E, we have that at least one of its conjuncts is different from B_i, for all $i \in \{1, \dots, n\}$;

(S3) for every subformula of Γ of the form $\forall x F(x)$ and for every term t, formula $F(t)$ is different from A_i, B_i, for all $i \in \{1, \dots, n\}$;

(S4) for every subformula of D of the form $\exists x G(x)$ and for every term t, formula $G(t)$ is different from B_i, for all $i \in \{1, \dots, n\}$.
Then we say that the sequent $\Gamma \vdash D$ is *strongly disjunctively balanced*.

Theorem 3.9. *Let $\Gamma \vdash D$ be strongly disjunctively balanced sequent which has property λ, and Γ, D contain no logical symbols except $\wedge, \vee, \forall$ and $\exists$. Then, the sequent $\Gamma \vdash D$ is unprovable intuitionisticaly.*

Proof. We attempt to construct a proof of the sequent $\Gamma \vdash D$ in the system $G3$. The only rules we can apply in this proof are $\wedge \vdash$, $\vdash \wedge$, $\vee \vdash$, $\vdash \vee$, $\forall \vdash$, $\vdash \forall$, $\exists \vdash$ and $\vdash \exists$. Let $\Delta \vdash E$ be an arbitrary sequent which appears in the proof tree. Note that we have the following *subformula property*: all formulae which occur in Δ must be subformulae of Γ, and E must be a subformula of D.

From **(D2)** we can conclude that the only possible axiomatic sequents in the proof are either of the form $A_i, \Gamma \vdash A_i$ or $B_i, \Gamma \vdash B_i$, $1 \leq i \leq n$. Note that whenever two-premise rules $\vee \vdash$ and $\vdash \wedge$ are applied, the proof tree will branch.

We show that there always exists at least one branch, called a *designated branch*, that cannot be terminated by an axiom. It is defined by specifying the premises of the applications of the rules $\vee \vdash$ and $\vdash \wedge$. Let us consider first $\vee \vdash$. Because of the subformula property, the principal formula for the $\vee\vdash$ must be $A_i \vee B_i$, for some $i \in \{1,\dots,n\}$, so we have the following inference figure

$$\frac{A_i, A_i \vee B_i, \Delta \vdash E \quad B_i, A_i \vee B_i, \Delta \vdash E}{A_i \vee B_i, \Delta \vdash E.}$$

If the formula E contains A_i, then the premise containing B_i (which we shall call *designated premise*) belongs to the designated branch. Otherwise, the premise containing A_i belongs to the designated branch.

For the rule $\vdash\wedge$, let $F_1 \wedge F_2$ be the principal formula, and let

$$\frac{\Delta \vdash F_1 \quad \Delta \vdash F_2}{\Delta \vdash F_1 \wedge F_2}$$

be the corresponding inference figure. From **(S2)** we know that either F_1 or F_2 is different from B_i, for every $1 \le i \le n$. Now, we shall choose for the designated premise the sequent whose succedent is different from B_i.

It remains to prove that every sequent in the designated branch has property λ, which will mean that designated branch cannot be terminated by an axiom (since we can easily see that the axioms $A_i, \Gamma \vdash A_i$ and $B_i, \Gamma \vdash B_i$ for $1 \le i \le n$ do not have the property λ). Note that the sequent $\Gamma \vdash D$ has the property λ according to the conditions of the theorem. Further, we shall show that all the mentioned rules of inference preserve property λ in the sense that if the conlusion of the inference has property λ, so does the premise (or in the case of rules $\vee\vdash$ and $\vdash\wedge$ the designated premise). Now we shall consider eight cases, one for each rule of inference. Below, Δ will always denote some sequence of subformulae of Γ, and E will be some subformula of D.

1) Let the inference figure be

$$\frac{F_1, F_1 \wedge F_2, \Delta \vdash E}{F_1 \wedge F_2, \Delta \vdash E}\wedge\vdash \qquad \text{or} \qquad \frac{F_2, F_1 \wedge F_2, \Delta \vdash E}{F_1 \wedge F_2, \Delta \vdash E.}\wedge\vdash$$

Since the sequent $\Gamma \vdash D$ satisfies the condition **(S1)**, we can easily see that the property λ is preserved in both of previous inferences.

2) Suppose that we have the following inference figure

$$\frac{\Delta \vdash F_1 \quad \Delta \vdash F_2}{\Delta \vdash F_1 \wedge F_2.}\vdash\wedge$$

Since both F_1 and F_2 are subformulae of $F_1 \wedge F_2$, the condition ($\boldsymbol{\lambda}$**1**) is trivially satisfied for the both premises, and ($\boldsymbol{\lambda}$**2**) is satisfied by our previous choice of the designated premise.

3) Consider the following inference figure

$$\frac{A_i, A_i \vee B_i, \Delta \vdash E \quad B_i, A_i \vee B_i, \Delta \vdash E}{A_i \vee B_i, \Delta \vdash E.}\vee\vdash$$

Now, if E does not contain A_i, we choose the left premise for the designated one, and then the condition **($\boldsymbol{\lambda}$2)** is obviously satisfied. The condition **($\boldsymbol{\lambda}$2)** holds because if $E = B_j$ for some $j \in \{1,\dots,n\}$, then $E \ne A_i$, by the initial assumptions on A_i's, B_i's and C_i's. If E contains A_i, we choose the right premise to be the designated and again we see that the condition **($\boldsymbol{\lambda}$1)** is apparently satisfied. Let's prove that the condition ($\boldsymbol{\lambda}$**2**) is satisfied, too. Suppose that $E = B_j$ for some $j \in \{1.\dots,n\}$. It follows that A_i is the subformula of B_j, and since A_i and B_j are atomic formulae, by **(S2)**, we conclude that $A_i = B_j$, which is the contradiction to the starting hypotheses. Thus, $E \neq B_j$ for all $j \in \{1,\dots,n\}$, which means that the condition **($\boldsymbol{\lambda}$2)** is immediately satisfied.

4) Let the inference figure be

$$\frac{\Delta \vdash A_i}{\Delta \vdash A_i \vee C_i}{\vdash\vee} \quad \text{or} \quad \frac{\Delta \vdash C_i}{\Delta \vdash A_i \vee C_i.}{\vdash\vee}$$

We see that in both cases the condition **($\boldsymbol{\lambda}$1)** obviously holds, and **($\boldsymbol{\lambda}$2)** is satisfied because $A_i \neq B_j \neq C_k$ for all $i,j,k \in \{1,\dots,n\}$. Therefore, the property λ is preserved.

5) If the inference figure is

$$\frac{F(t), \forall x F(x), \Delta \vdash E}{\forall x F(x), \Delta \vdash E,}{\forall\vdash}$$

we see, from the condition **(S3)**, that the property λ is preserved.

6) Suppose that we have the following inference figure

$$\frac{\Delta \vdash F(a)}{\Delta \vdash \forall x F(x).}{\vdash\forall}$$

By the restriction on variables for the $\vdash \forall$, the variable a of the side formula $F(a)$ must be a variable not occuring in the antecedent, which assures that the condition **($\boldsymbol{\lambda}$2)** holds, and the property λ is preserved.

7) If the inference figure is

$$\frac{F(a), \exists x F(x), \Delta \vdash E}{\exists x F(x), \Delta \vdash E,}{\exists\vdash}$$

property λ is again preserved, which follows in a similar way like in previous case, because of the restriction on variables.

8) Finally, consider the following inference figure

$$\frac{\Delta \vdash F(t)}{\Delta \vdash \exists x F(x).}{\vdash\exists}$$

Because of **(S4)**, we see that the condition **($\boldsymbol{\lambda}$2)** is satisfied, and consequently, property λ is again preserved. □

REMARK 3.10. It is easy to check that the sequent $\forall x(A \vee B(x)) \vdash A \vee \forall x B(x)$ satisfies the conditions of Theorem 3.9, which instantly means that the formula (2) is unprovable intuitionisticaly. Similarly, using Theorem 3.9, we can show that many other formulae, which are provable classically, are unprovable in intuitionistic logic, e.g.

$$\exists x \forall y((A \vee B(x,y)) \wedge (C \vee D(x,y))) \to ((A \vee \exists x \forall y B(x,y)) \wedge (C \vee \exists x \forall y D(x,y))),$$

where $A, B(x,y), C$ and $D(x,y)$ are atomic formulae.

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