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A NOTE ON TWO OF VUKMAN'S CONJECTURES

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Abstract. In this paper we prove, under certain condition, when R is a semiprime ring with suitable characteristic restrictions, that every nonzero (m,n)-Jordan triple centralizer (resp., (m,n)-Jordan triple derivation) is a two-sided centralizer (resp., a derivation which maps R into $Z(R)$). This give partial affirmative answers to two conjectures of Vukman.

1. Introduction

Throughout this paper, R will represent an associative ring with centre $Z(R)$. We denote by $char(R)$ the characteristic of ring R. Let $n \geq 2$ be an integer. A ring R is said to be n-torsion free if, for all $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if, for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$. A ring R is called semiprime if, for any $a \in R$, $aRa = \{0\}$ implies $a = 0$.

An additive mapping $T : R \to R$ is called a left (resp., a right) centralizer if $T(xy) = T(x)y$ (resp., $T(xy) = xT(y)$) is fulfilled for all $x, y \in R$, and it is called a left (resp., a right) Jordan centralizer if $T(x^2) = T(x)x$ (resp., $T(x^2) = xT(x)$) is fulfilled for all $x \in R$. We call an additive mapping $T : R \to R$ a two-sided centralizer (resp., a two-sided Jordan centralizer) if T is both a left and a right centralizer (resp., a left and a right Jordan centralizer).

The study of relations between various sorts of Jordan centralizers and centralizers has attracted several authors. In [20], Zalar proved that any left (resp., right) Jordan centralizer on a 2-torsion free semiprime ring is a left (resp., a right) centralizer. In [18], Vukman proved that, for a 2-torsion free semiprime ring R, every additive mapping $T : R \to R$ satisfying the relation $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$ is a two-sided centralizer. Motivated by these results and inspired by his work [18], Vukman introduced in [19] the notion of an (m,n)-Jordan centralizer as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $T : R \to R$ is

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called an (m, n)-Jordan centralizer if $(m+n)T(x^2) = mT(x)x + nxT(x)$ holds for all $x, y \in R$.

Obviously, a $(1, 0)$-Jordan centralizer (resp., a $(0, 1)$-Jordan centralizer) is a left (resp., a right) Jordan centralizer. When $n = m = 1$, we recover the map studied in [18].

In [19], Vukman defined an (m, n)-Jordan triple centralizer as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $T : R \to R$ is called an (m, n)-Jordan triple centralizer if

$$\begin{aligned} 2(m+n)^2 T(xyx) &= mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2 \\ &\quad + 2mnxT(y)x - mnx^2T(y) + n(m+2n)xyT(x) + mnyxT(x) \end{aligned}$$

holds for all $x, y \in R$.

Based on some observations and inspired by the classical results, Vukman, in [19], made the following conjecture.

Conjecture 1.1. *([19, Conjecture 7]) Let $m, n \geq 1$ be two fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $T : R \to R$ be an (m, n)-Jordan triple centralizer. In this case T is a two-sided centralizer.*

In [14], Peršin and Vukman proved the following result which gives an affirmative answer to the conjecture in the case of prime rings.

Theorem 1.2. *([14, Theorem 2]) Let $m \geq 1$, $n \geq 1$ be some fixed integers and let R be a prime ring with $char(R) = 0$ or $(m+n)^2 < char(R)$ and let $T : R \to R$ be an additive mapping satisfying the relation*

$$2(m+n)^2 T(x^3) = m(2m+n)T(x)x^2 + 2mnxT(x)x + n(2n+m)x^2T(x)$$

for all $x \in R$. In this case T is a two-sided centralizer.

It seems that this conjecture deserves more efforts to be solved completely. Our first main result is the following theorem which gives a partial affirmative answer to this conjecture.

Theorem 1.3. *Let $m, n \geq 1$ be distinct integers, let R be a semiprime ring with $char(R) = 0$ or $(m+n)^2 < char(R) = p'$, where p' is a prime number, and let $T : R \to R$ be an (m, n)-Jordan triple centralizer. If $T(P) \subseteq P$ for every prime ideal P of R, then T is a two-sided centralizer.*

Recall that a ring R is called von Neumann regular if, for each $r \in R$, there is $r' \in R$ with $rr'r = r$. It is known that von Neumann regular rings are semiprime (see for instance [13]). Let I be an ideal of R. For an element $x \in R$, we use $\overline{x}$ to denote the equivalence class of x modulo I.

Remark 1.4. Let $m, n \geq 1$ be distinct integers, let K be a finite field with prime characteristic $p' > (m+n)^2$. Considers the ring $M_n(K)$ of n-by-n square matrices with entries from field K. Hence $M_n(K)$ is a von Neumann regular ring with characteristic $p' > (m+n)^2$.

In the following proposition we present two situations which illustrate our conditions in the first main result.

Proposition 1.5. *Let $m, n \geq 1$ be distinct integers, let R be a ring with $char(R) = 0$ or $(m+n)^2 < char(R) = p'$, where p' is a prime number, and let $T : R \to R$ be an (m,n)-Jordan triple centralizer. Then $T(P) \subseteq P$ for every prime ideal P of R. If one of the following conditions hold:*

(i) R be a von Neumann regular ring;

(ii) R be a semiprime ring with unity.

Proof. (i) Let P be a prime ideal of R and set $\overline{R} = R/P$. Consider an element $p \in P$. By hypothesis, there exist $q \in R$ such that $pqp = p$. Putting $x = p$ and $y = q$ in the equation of the definition of an (m,n)-Jordan triple centralizer, we get

$$2(m+n)^2T(pqp) = mnT(p)pq + m(2m+n)T(p)qp - mnT(q)p^2$$
$$+2mnpT(q)p - mnp^2T(q) + n(m+2n)pqT(p) + mnqpT(p).$$

Thus, $2(m+n)^2\overline{T(pqp)} = 0$. Since $\overline{R}$ is $(m+n)$-torsion free, then $\overline{T(p)} = 0$. Therefore $T(P) \subseteq P$.

(ii) Let P be a prime ideal of R and set $\overline{R} = R/P$. Consider an element $p \in R$. Putting $x = 1$ and $y = p$ in the equation of the definition of an (m,n)-Jordan triple centralizer, we get

$$2(m+n)^2T(p) = mnT(1)p + m(2m+n)T(1)p - mnT(p)$$
$$+ 2mnT(p) - mnT(p) + n(m+2n)pT(1) + mnpT(1).$$

Thus, $2(m+n)^2\overline{T(p)} = 0$. Since $\overline{R}$ is $(m+n)$-torsion free. Then $\overline{T(p)} = 0$. Therefore $T(P) \subseteq P$. □

Using the same arguments we can give a partial answer to another Vukman's conjecture. It concerns the relation between a sort of Jordan maps and derivation. Let us explain this in details. It is worth mentioning that the study of relations between various sorts of derivations goes back to Herstein's classical result [12] which shows that any Jordan derivation on a 2-torsion free prime ring is a derivation (see also [4] for a brief proof of Herstein's result). In [5], Cusack generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). Motivated by these classical results, Vukman [15] proved that any generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation. In the last few years several authors have introduced and studied various sorts of parameterized derivations. In [1], Ali and Fošner defined the notion of (m,n)-derivations as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $d : R \to R$ is called an (m,n)-derivation if $(m+n)d(xy) = 2md(x)y + 2nxd(y)$ holds for all $x, y \in R$.

Obviously, a $(1,1)$-derivation on a 2-torsion free ring is a derivation.

In the same paper [1], a generalized (m,n)-derivation is defined as follows: Let $m, n \geq 0$ be two fixed integers with $m + n \neq 0$. An additive mapping $D : R \to R$ is called a generalized (m,n)-derivation if there exists an (m,n)-derivation $d : R \to R$ such that $(m+n)D(xy) = 2mD(x)y + 2nxd(y)$ holds for all $x, y \in R$.

Obviously, every generalized $(1,1)$-derivation on a 2-torsion free ring is a generalized derivation.

In [17], Vukman defined an (m,n)-Jordan triple derivation as follows: Let $m,n \geq 0$ be two fixed integers with $m+n \neq 0$. An additive mapping $D : R \to R$ is called an (m,n)-Jordan triple derivation if

$$(m+n)^2 D(xyx) = m(n-m)D(x)xy + m(m-n)D(y)x^2 + n(n-m)x^2D(y)$$
$$+n(m-n)yxD(x) + m(3m+n)D(x)yx + 4mnxD(y)x + n(m+3n)xyD(x)$$

holds for all $x,y \in R$.

Based on some observations and inspired by the classical results, Vukman, in [17], made the following conjecture.

Conjecture 1.6. *([17, Conjecture 2]) Let $m,n \geq 1$ be two fixed integers with $m+n \neq 0$, $m \neq n$, and let $D : R \to R$ be an (m,n)-Jordan triple derivation, where R is a semiprime ring with suitable torsion restrictions. In this case D is a derivation which maps R into $Z(R)$.*

In [9], Fošner and Vukman proved the following result which gives an affirmative answer to the conjecture for prime rings.

Theorem 1.7. *([9, Theorem 2]) Let $m \geq 1$, $n \geq 1$ be some fixed integers with $m \neq n$, and let R be a prime ring with $(m+n)^2 < char(R)$ and let $D : R \to R$ be a nonzero additive mapping satisfying the relation*

$$(m+n)^2 D(x^3) = m(3m+n)D(x)x^2 + 4mnxD(x)x + n(3n+m)x^2D(x)$$

for all $x \in R$. In this case D is a derivation and R is commutative.

Our second main result is the following theorem which gives a partial affirmative answer to this conjecture.

Theorem 1.8. *Let $m \geq 1$, $n \geq 1$ be some fixed integers with $m \neq n$, and let R be a semiprime ring with $(m+n)^2 < char(R) = p'$, where p' is a prime number, and let $D : R \to R$ be an (m,n)-Jordan triple derivation. If $D(P) \subseteq P$ for every prime ideal P of R, then D is a derivation which maps R into $Z(R)$.*

Furthermore, the previous remark 1.4 and Proposition 1.5, we give another proposition and remark meant to illustrate our conditions in our second result.

Proposition 1.9. *Let $m,n \geq 1$ be distinct integers, let R be a von Neumann regular ring with $(m+n)^2 < char(R) = p$ which p prime number, and let $D : R \to R$ be an (m,n)-Jordan triple derivation. Then $D(P) \subseteq P$ for every prime ideal P of R.*

Proof. Let P be a prime ideal of R and set $\overline{R} = R/P$. Consider an element $p \in P$. By hypothesis, there exists $q \in R$ such that $pqp = p$. Putting $x = p$ and $y = q$ in the equation of the definition of an (m,n)-Jordan triple derivation, we get

$$(m+n)^2 D(pqp) = m(n-m)D(p)pq + m(m-n)D(q)p^2 + n(n-m)p^2D(q)$$
$$+n(m-n)qpD(p) + m(3m+n)D(p)qp + 4mnpD(q)p + n(m+3n)pqD(p).$$

Thus, $(m+n)^2\overline{D(pqp)} = 0$. Since $\overline{R}$ is $(m+n)$-torsion free, then $\overline{D(p)} = 0$. Therefore $D(P) \subseteq P$. □

Remark 1.10. From [11], we have that any minimal prime ideal of a semiprime algebra over a field of characteristic 0 is always invariant under derivations and from [10], we have that, if P is a minimal prime ideal of a ring R such that R/P has characteristic 0, then P is invariant under any derivations of R. Many authors studied this problem (for instance see [6–8]). Thus the following natural question could be of interest: Are the minimal prime ideals invariant by (m, n)-Jordan triple derivations?

2. Proof of the main theorems

We shall use the relation between semiprime rings and prime ideals. Namely, it is well-known that a ring R is semiprime if and only if the intersection of all prime ideals of R is zero if and only if R has no nonzero nilpotent (left, right) ideals (see for instance Lam's book [13] or the recent book of Brešar [3]).

Proof of Theorem 1.3

Let $x, y \in R$. We prove that $T(xy) = T(x)y$ and $T(xy) = xT(y)$. We may assume that x and y are not 0. Let P be a prime ideal of R and set $\overline{R} = R/P$. Consider an element $p \in P$. By hypothesis, $char(R) = 0$ or $(m+n)^2 < char(R) = p'$, where p' is a prime number and Theorem 1.2, We have

$$\begin{aligned}2(m+n)^2T((y+p)^3) &= m(2m+n)T(y+p)(y+p)^2 \\ &+2mn(y+p)T(y+p)(y+p) + n(2n+m)(y+p)^2T(y+p). \qquad (1)\end{aligned}$$

By hypothesis, $T(P) \subseteq P$ and after a simple simplification of (1), we get

$$2(m+n)^2\overline{T(y^3)} = m(2m+n)\overline{T(y)y^2} + 2mn\overline{yT(y)y} + n(2n+m)\overline{y^2T(y)}.$$

Consider $\overline{T} : \overline{R} \to \overline{R}$ defined by $\overline{T}(\overline{x}) = \overline{T(x)}$ for $x \in R$. Note that $\overline{T}$ is well defined because $T(P) \subseteq P$. Thus,

$$2(m+n)^2\overline{T}(\overline{y}^3) = m(2m+n)\overline{T}(\overline{y})\overline{y}^2 + 2mny\overline{T}(\overline{y})\overline{y} + n(2n+m)\overline{y}^2\overline{T}(\overline{y}).$$

This shows, using the fact that $\overline{R}$ is a prime ring and Theorem 1.2, that $\overline{T}(\overline{xy}) = \overline{T}(\overline{x})\overline{y}$ and $\overline{T}(\overline{xy}) = \overline{x}T(\overline{y})$. Therefore, $T(xy) - T(x)y \in P$ and $T(xy) - xT(y) \in P$. Finally, by the semiprimeness of R, we get the desired result.

Proof of Theorem 1.8

Let $x, y \in R$. We will prove that $D(xy) = D(x)y + xD(y)$. We may assume that x and y are not 0. Let P be a prime ideal of R and set $\overline{R} = R/P$. Consider an element $p \in P$. By hypothesis, $(m+n)^2 < char(R) = p'$, where p' is a prime number and by Theorem 1.7 we have

$$\begin{aligned}(m+n)^2D((x+p)^3) &= m(3m+n)D(x+p)(x+p)^2 \\ &+4mn(x+p)D(x+p)(x+p) + n(3n+m)(x+p)^2D(x+p) \qquad (2)\end{aligned}$$

By hypothesis, $D(P) \subseteq P$, and after a simplification of (2) we get

$$(m+n)^2\overline{D}(\overline{x}^3) = m(3m+n)\overline{D}(\overline{x})\overline{x}^2 + 4mn\overline{x}\overline{D}(\overline{x})\overline{x} + n(3n+m)\overline{x}^2\overline{D}(\overline{x}).$$

Since $\overline{R}$ is a prime ring, using Theorem 1.7, we get that $\overline{D}(\overline{xy}) = \overline{D}(\overline{x})\overline{y} + \overline{x}\overline{D}(\overline{y})$. Therefore, $D(xy) - D(x)y - xD(y) \in P$. Finally, by the semiprimeness of R, we get the desired result.

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